Relativistic N-body models

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Based on Mackey's theory of induced representations [G. W. Mackey, *Induced Representation of Groups and Quantum Mechanics* (Benjamin, New York, 1968)] a Bakamjian-Thomas [B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953)] procedure for introducing interactions into n-body relativistic systems is formulated. The resulting models have manifest symmetries under exchange of identical particles and space reflection. The two-body interactions in the n-body equations are separately scattering equivalent to the input two-body interactions, but n-body dynamics does not satisfy the necessary cluster separability properties. The nature of the corrections needed to restore cluster properties is discussed. [S0556-2813(96)05709-3]

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I. INTRODUCTION

The principle of special relativity, when applied to quantum mechanical systems, requires the existence of a unitary representation of the Poincaré group [1,2] acting on the system Hilbert space. This representation necessarily involves the dynamics. A generalization of the nonrelativistic procedure of constructing the full Hamiltonian by adding interactions to the free Hamiltonian can be used to construct interacting representations of the Poincaré group. The difficulty is that interactions must be added to the noninteracting representation of the Poincaré group in a manner that preserves the group structure.

Bakamjian-Thomas [3–9] methods provide one way for adding interactions to a noninteracting representation of the Poincaré group. The distinguishing property of all Bakamjian-Thomas methods is that the interactions are block diagonal with respect to the spin of the noninteracting representation and only depend on the noninteracting mass and kinematically invariant degeneracy parameters. Different representations of the noninteracting dynamics lead to distinct realizations of the dynamics [10]. The most useful ones are related to Dirac's forms [11] of the dynamics.

There are difficulties with Bakamjian-Thomas methods when they are applied to many-body systems. The fundamental problem is that Bakamjian-Thomas models fail to satisfy spacelike cluster properties for systems of more than two particles [7,9]. An additional practical complication is that the decomposition into subspaces that are block diagonal in the total kinematic spin is normally done in a stepwise fashion, beginning with subsystem spins. Stepwise coupling is cumbersome and does not treat identical particles symmetrically. The lack of manifest symmetry under the interchange of identical particles is a disadvantage when approximations are used.

An alternative to the stepwise coupling of irreducible representations of the Poincaré group is Mackey's [12–15,9] method of induced representations, which decomposes the n-fold tensor product of one-particle-irreducible representations directly into irreducible representation spaces. The quantum numbers that label degenerate representations can be chosen to have simple transformation properties with respect to permutations of identical particles and space reflections. In this representation it is easy to formulate systematic approximations that preserve both exact Poincaré invariance and particle exchange symmetry at each stage. These approximations can be used to formulate variational calculations for many-body systems.

The use of Mackey's method of induced representations does not correct the failure of spacelike cluster properties. Cluster properties can be restored using the method of Sokolov operators [5,7,9]. The Sokolov operators generate corrections to the naive Bakamjian-Thomas model that restore cluster properties.

The construction presented in this paper is similar to that in [7] with the following differences. First, n-body models are formulated at the outset rather than built up recursively. However, a recursive construction is still necessary to build transformations that systematically restore cluster properties. Second, the representations have manifest symmetries with respect to exchange of identical particles. Third, corrections needed to systematically restore cluster properties are discussed in all three forms of the dynamics. In this construction the instant-form dynamics plays a special role.

Notation and background material are given in the next section. In Sec. III the Clebsch-Gordan coefficients that couple the tensor product of n-positive mass irreducible representations of the Poincaré group to a single simultaneously coupled representation are constructed. The model is formulated in these basis states. The action of the permutation group and parity operator on these basis states is given explicitly.

Cluster properties require that the two-body interactions appearing in the two-body problem be the same as the two-body interactions that appear in an n-body model. Unfortunately the two-body interactions corresponding to different interacting pairs cannot be added in a manner that preserves the commutation relations. In Sec. IV these two-body interactions are transformed to phase-equivalent Bakamjian-Thomas interactions using different unitary transformations. The Bakamjian-Thomas interactions can be combined in a manner that preserves the Poincaré Lie algebra.

In Sec. V the two-body Bakamjian-Thomas interactions are combined to make an n-body Bakamjian-Thomas mass operator. Dynamical equations for the mass eigenstates are written down. The resulting representation of the Poincaré

group does not satisfy cluster separability. Variational approximations for bound states that are exactly Poincaré invariant and preserve the exchange symmetry are formulated.

In Sec. VI the model of Sec. V is shown to be the first approximation to a relativistic model that satisfies cluster properties. Corrections that restore spacelike cluster properties to the interactions and generate additional many-body interactions that are needed to maintain the Poincaré invariance are constructed. The nature of various approximations is discussed in the context of these corrections.

II. NOTATION AND BACKGROUND

A. Poincaré group

The Poincaré group is the group of coordinate transformations that preserve the proper time between space-time points in the Minkowski metric

$$g_{11}=g_{22}=g_{33}=-g_{00}=1. \tag{2.1}$$

A general Poincaré transformation has the form

$$x^\mu \to x'^\mu := \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{2.2}$$

or, equivalently,

$$x \to x' := \Lambda x + a, \tag{2.3}$$

where the matrix Λ is a Lorentz transformation,

$$\Lambda^T g \Lambda = g, \tag{2.4}$$

and a labels a space-time translation. Elements of the Poincaré group are ordered pairs (Λ,a) and the group product is

$$(\Lambda_2,a_2)(\Lambda_1,a_1)=(\Lambda_2\Lambda_1,\Lambda_2 a_1+a_2). \tag{2.5}$$

A Lorentz boost is a special Lorentz transformation $\Lambda=B(p)$ labeled by the four-momentum with the following properties.

(1) It maps a rest four-momentum $p_0:=(m,0,0,0)$ to $p=(\omega_m(\vec{p}),\vec{p})$:

$$B(p)p_0=p \quad \text{where} \quad \omega_m(\vec{p})=\sqrt{m^2+\vec{p}^2}. \tag{2.6}$$

(2) It is the identity for $p=p_0$:

$$B(p_0)=I. \tag{2.7}$$

If $B(p)$ is a Lorentz boost, then so is $B'(p):=B(p)R(p)$, where $R(p)$ is an arbitrary p-dependent rotation that satisfies $R(p_0)=I$. The canonical boost $B_c(p)$ is the unique boost with the property

$$R=B_c^{-1}(Rp)RB_c(p) \tag{2.8}$$

for an arbitrary rotation R. The boost-dependent rotation

$$R_x(\Lambda,p):=B_x^{-1}(\Lambda p)\Lambda B_x(p) \tag{2.9}$$

appearing in Eq. (2.8) is called a Wigner rotation. Equation (2.8) is the statement that the canonical boost is the unique boost with the property that the Wigner rotation of the rotation R is R.

The rotation $R_{cx}(p)$ relating a specific boost to a canonical boost

$$B_x(p)=B_c(p)R_{cx}(p) \tag{2.10}$$

is called a generalized Melosh rotation. Other commonly used boosts are light front boosts and the helicity boosts [9].

The Poincaré group has several important subgroups that are relevant for Dirac's forms of dynamics. These include the Euclidean group which is generated by rotations and space translations, the Lorentz group, and the symmetry group of the light front which is the subgroup of the Poincaré group that leaves the light front, $x^+=x^0+x^3=0$, invariant. We denote the Poincaré group by P and the above subgroups by E, L, and F, respectively.

B. Combinatorics

Combinatorial methods are used to treat counting problems in many-body systems. A partition a of an n-particle system is a grouping of the particles into n_a disjoint nonempty subsystems. Each nonempty subsystem is called a cluster. For example, $a=(135)(24)$ is a two-cluster partition of a five-particle system. It has one three-particle cluster consisting of particles 1, 3, and 5 and a two-particle cluster consisting of particles 2 and 4. The notation n_{a_i} denotes the number of particles in the ith cluster of the partition a.

The notation $a \supseteq b$ is used to indicate that the partition b is a refinement of the partition a, meaning that particles in the same cluster of b are in the same cluster of a. The notation $a \supset b$ indicates that b is a strict refinement of a, i.e., $a \supseteq b$ and $a \neq b$. The relation $\supseteq$ is a partial ordering on the set of partitions. The quantities $a \cup b$ and $a \cap b$ denote the least upper bound and greatest lower bound of the partitions a and b with respect to the partial ordering $\supseteq$. The minimal and maximal elements of the set of partitions of n particles with respect to the partial ordering $\supseteq$ are the n-cluster and one-cluster partitions, respectively. These are denoted by $\underline{0}:=(1)\cdots(n)$ and $\underline{1}:=(1\cdots n)$.

The set of all partitions of n particles is denoted by $\mathcal{P}$ while $\mathcal{P}'$ denotes all partitions except the one-cluster partition. The set of $\mathcal{P}$ with the relations $\cap$ and $\cup$ forms an abstract lattice [16]. The zeta and Möbius functions for the partition lattice are integer-valued functions on $\mathcal{P}\times\mathcal{P}$ defined by

$$\Delta(a,b)=\begin{cases} 1 & \text{if } a \supseteq b, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Delta^{-1}(a,b)=\begin{cases} (-)^{n_a}\displaystyle\prod_{l=1}^{n_a}(-)^{n_{b_l}}(n_{b_l}-1)! & \text{if } a \supseteq b, \\ 0 & \text{otherwise.} \end{cases}$$

The zeta function satisfies

$$\Delta(a\cap b,c)=\Delta(a,c)\Delta(b,c),$$

$$\Delta(a,b\cup c)=\Delta(a,b)\Delta(a,c). \tag{2.11}$$

It is useful to define

$$\zeta_a := -\Delta^{-1}(\underline{1},a)(1-\delta_{\underline{1}a}). \tag{2.12}$$

It follows as a consequence of this definition that

$$\sum_{a\in\mathcal{P}} \zeta_a \Delta(a,b) = (1-\delta_{\underline{1}b}). \tag{2.13}$$

Partitions provide an economical means for classifying operators that act on the n-particle Hilbert spaces. An operator in the n-particle Hilbert space has connectivity c if (1) it has no interactions involving particles in different clusters of the partition c and (2) it vanishes in the limit that any interaction involving particles in the same cluster of c vanishes. The notation

$$A = [A]_c \tag{2.14}$$

indicates that the operator A is equal to its c-connected part or has connectivity c. In general n-particle operators of interest can be expanded as

$$A = \sum_{c\in\mathcal{P}} [A]_c . \tag{2.15}$$

It follows from the definition that the operator obtained from A by turning off all interactions involving particles in different clusters of b, which is denoted by $(A)_b$, is

$$(A)_b = \sum_{c\in\mathcal{P}} \Delta(b,c)[A]_c . \tag{2.16}$$

This expansion can be inverted using the Möbius function

$$[A]_b = \sum_{c\in\mathcal{P}} \Delta^{-1}(b,c)(A)_c . \tag{2.17}$$

The notation $(A)^b$ denotes the residual interactions

$$(A)^b = A - (A)_b . \tag{2.18}$$

With the above notation Eqs. (2.13), (2.15), and (2.17) imply

$$A - [A]_{\underline{1}} = \sum_{a\in\mathcal{P}'} \zeta_a (A)_a , \tag{2.19}$$

When this formula is applied to Poincaré generators it determines the relation between the n-body generators and the subsystem generators. For example, when A is the Hamiltonian it gives the relation

$$H = \sum_{a\in\mathcal{P}'} \zeta_a (H)_a + [H]_{\underline{1}} . \tag{2.20}$$

where $[H]_{\underline{1}}$ is an n-body interaction and $(H)_a$ is the sum of the subsystem Hamiltonians for the particles in each cluster of a.

C. Group theory

The generalized orthogonality relation [17] is used in what follows:

$$\int_{\mathrm{SU}(2)} dR(2j_1+1) D^{j_1 *}_{\mu_1\nu_1}(R) D^{j_2}_{\mu_2\nu_2}(R) = \delta_{j_1 j_2}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}. \tag{2.21}$$

The derivation uses known properties of rotation matrices and can be found in Ref. [17]. From this relation it is possible to deduce the conjugate relation

$$\delta(R-R') = \sum_{j\mu\nu} (2j+1) D^{j*}_{\mu\nu}(R') D^{j}_{\mu\nu}(R). \tag{2.22}$$

In these formulas dR is the Haar measure on SU(2).

The D functions in these expressions are given by the homogeneous polynomials of degree $2j$ with real coefficients in the SU(2) matrix elements:

$$D^j_{\mu\nu}(R) = \sum_{\rho=0}^{2j} \frac{(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!]^{1/2}}{(j+\mu-\rho)!\rho!(\rho-\mu+\nu)!(j-\nu-\rho)!} \times R_{11}^{j+\mu-\rho} R_{12}^{\rho} R_{21}^{\rho-\mu+\nu} R_{22}^{j-\nu-\rho} . \tag{2.23}$$

D. Quantum mechanics: General

The principle of special relativity assumes that all inertial coordinate systems are related by Poincaré transformations. In a quantum theory with a Hilbert space $\mathcal{H}$, the requirement that probabilities have identical values in all inertial coordinate systems [1,2] is equivalent to the existence of a unitary representation $U(\Lambda,a)$ of the Poincaré group acting on $\mathcal{H}$.

The infinitesimal generators of the space-time translations define the four-momentum operator P^μ and the infinitesimal generators of Lorentz transformations define an antisymmetric tensor $J^{\mu\nu}$. These operators are Hermitian operators on $\mathcal{H}$ satisfying the commutation relations

$$[P^\mu, P^\nu]_- = 0, \tag{2.24}$$

$$[J^{\alpha\beta}, P^\gamma]_- = i(g^{\alpha\gamma}P^\beta - g^{\beta\gamma}P^\alpha), \tag{2.25}$$

$$[J^{\alpha\beta}, J^{\gamma\delta}]_- = i(J^{\alpha\gamma}g^{\beta\delta} - J^{\alpha\delta}g^{\beta\gamma} + J^{\beta\delta}g^{\alpha\gamma} - J^{\beta\gamma}g^{\alpha\delta}). \tag{2.26}$$

The Pauli-Lubanski vector

$$W^\mu := \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}P_\alpha J_{\beta\gamma} \tag{2.27}$$

is an additional Hermitian operator that commutes with the four-momentum operator and transforms like a four-vector under Lorentz transformations. The mass and spin are invariant Hermitian operators related to the generators by

$$M^2 := -P^\mu P_\mu , \quad M^2 j^2 := W^\mu W_\mu . \tag{2.28}$$

Spin vectors $\vec{j}$ satisfying $\vec{j}\cdot\vec{j} = j^2$ and

$$[j_i, j_j]_- = i\epsilon_{ijk} j_k \tag{2.29}$$

are defined by

$$(0,\vec{j}_x)^\mu = \frac{1}{M}[\hat{B}_x^{-1}(p)]^\mu{}_\nu W^\nu, \tag{2.30}$$

where $\hat{B}_x^{-1}(p)$ is the matrix of operators obtained by replacing all occurrences of p by the corresponding component of the four-momentum operator in the expression for the boost $B_x^{-1}(p)$. Because of the operator nature of $\hat{B}_x^{-1}(p)$, the quantity $\vec{j}_x$ in (2.30) is *not* the spatial part of a four-vector. It Wigner rotates under Lorentz transformations. The canonical spin is the spin vector constructed using $B_x(p)=B_c(p)$, the canonical boost, in Eq. (2.30). The light front spin and helicity are spin vectors constructed by replacing the canonical boost $\hat{B}_c^{-1}(p)$ by the light front boost or helicity boost [9]. Different spin vectors are related by generalized Melosh rotations. The operator $\vec{j}$ is taken to be the canonical spin operator when there is no subscript x.

The state of a structureless quantum mechanical particle of mass m and spin j is represented by a vector in a mass m spin j irreducible representation of the Poincaré group.

Vectors in an irreducible representation space are simultaneous eigenstates of a complete set of commuting Hermitian operators that are functions of the Poincaré generators (for fixed mass and spin). Common choices of commuting Hermitian operators are the three-momentum $\vec{p}$ and the three-component of the canonical spin $\vec{j}\cdot\hat{e}_3$, the 4-velocity $v:=p/m$ and the three-component of the canonical spin, and the light front components $p^+:=p^0+p^3,\vec{p}_\perp(p^1,p^2)$ of the four-momentum and the three-component of the light front spin $\vec{j}_f\cdot\hat{e}_3$ [9].

For example, a basis of states for a single particle of mass m and spin j is the generalized eigenstates of the linear momentum $\vec{p}$ and three-component of the canonical spin $\vec{j}\cdot\hat{e}_3$:

$$|\vec{p},\nu[m,j]\rangle. \tag{2.31}$$

The normalization is arbitrary and taken to be

$$\langle\vec{p},\mu[m,j]|\vec{p}',\mu'[m,j]\rangle=\delta^3(\vec{p}-\vec{p}')\,\delta_{\mu\mu'}. \tag{2.32}$$

Alternative representations of the single particle states that are appropriate for point form and front form dynamics are discussed in the proof of theorem 5 in the Appendix.

The one-particle representation space $\mathcal{H}_1=\mathcal{H}_{mj}$ is the Hilbert space of square integrable functions

$$\psi(\vec{p},\mu):=\langle\vec{p},\mu[m,j]|\psi\rangle, \tag{2.33}$$

with scalar product

$$\langle\psi_1|\psi_2\rangle:=\int d^3p\sum_{\mu=-j}^{j}\psi_1^*(\vec{p},\mu)\psi_2(\vec{p},\mu). \tag{2.34}$$

The mass-m spin-j irreducible unitary representation of the Poincaré group on $\mathcal{H}_1$ is

$$U_1(\Lambda,a)|\vec{p},\mu[m,j]\rangle:=e^{-ip'\cdot a}|\vec{p}',\mu'[m,j]\rangle \times\sqrt{\frac{\omega_m(\vec{p}')}{\omega_m(\vec{p})}}D^j_{\mu'\mu}[R_w(\Lambda,p)], \tag{2.35}$$

where $p'=\Lambda p$. The transformation properties of the wave functions follow from Eq. (2.35):

$$\langle\vec{p}',\mu'[m,j]|U(\Lambda,a)|\psi\rangle=e^{-ip'\cdot a}\sqrt{\frac{\omega_m(p')}{\omega_m(p)}}D^j_{\mu'\mu} \times[R_w(\Lambda,p)]\psi(\vec{p},\mu). \tag{2.36}$$

E. Quantum mechanics: Many particle

The Hilbert space for a system of n particles is the tensor product of n one-particle spaces:

$$\mathcal{H}=\underbrace{\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_1}_{n\text{ times}}. \tag{2.37}$$

The n-particle dynamics is given by a unitary representation $U(\Lambda,a)$ on $\mathcal{H}$. The mass and spin operators associated with the representation $U(\Lambda,a)$ are functions of infinitesimal generators. For dynamical models some of the generators necessarily contain interactions.

A bound state of mass λ and spin j is a simultaneous eigenstate of M^2 and j^2 with λ^2 in the point spectrum of M^2. Any bound state vector can be decomposed into simultaneous eigenstates of the mass, spin, momentum, and three-component of a spin vector. These eigenstates span a Poincaré-invariant subspace of $\mathcal{H}$ containing vectors of the form

$$|\psi\rangle=\int d^3p\sum_{\mu=-j}^{j}|\vec{p},\mu[\lambda,j]\rangle\chi(\vec{p},\mu). \tag{2.38}$$

The generalized eigenstates $|\vec{p},\mu[\lambda,j]\rangle$ define a mapping $\Phi_{\lambda,j}$ from the mass-λ spin-j irreducible representation space $\mathcal{H}_{\lambda,j}$ spanned by the $\chi(\vec{p},\mu)$'s to the Poincaré-invariant subspace of the n-particle Hilbert space spanned by the vectors $|\vec{p},\mu[\lambda,j]\rangle$. With this interpretation Eq. (2.38) becomes

$$|\psi\rangle=\Phi_{\lambda,j}|\chi\rangle \tag{2.39}$$

and, by definition,

$$U(\Lambda,a)\Phi_{\lambda,j}=\Phi_{\lambda,j}U_{\lambda,j}(\Lambda,a), \tag{2.40}$$

where $U_{\lambda,j}(\Lambda,a)$ is the mass-λ spin-j irreducible representation of the Poincaré group on $\mathcal{H}_{\lambda,j}$. The representation $U_{\lambda,j}(\Lambda,a)$ has the same form as Eq. (2.35) with m replaced by the mass eigenvalue λ.

The generalization of the above to treat multiparticle scattering follows the discussion in Ref. [7]. The first step is to give a general formulation of the asymptotic conditions that define many-particle scattering states. This is most conveniently done by constructing an auxiliary Hilbert space of scattering asymptotes. Scattering asymptotes are labeled by channels. In order to define a channel, fix a partition b. Note that for any partition b the n-particle Hilbert space $\mathcal{H}$ can be

expressed as a tensor product of the n_{b_i}-particle Hilbert spaces associated with the particles in the ith cluster of b:

$$\mathcal{H}=\otimes_{i=1}^{n_b}\mathcal{H}_{b_i}. \tag{2.41}$$

Define $U_b(\Lambda,a)$ by

$$U_b(\Lambda,a):=\otimes_{i=1}^{n_b}U_{b_i}(\Lambda,a), \tag{2.42}$$

where $U_{b_i}(\Lambda,a)$ the representation of the Poincaré group corresponding to the particles in the ith cluster of b. Let M_{b_i} be the mass operator associated with the representation $U_{b_i}(\Lambda,a)$. There is a channel β associated with the partition b if each M_{b_i} has a bound eigenstate with mass $\lambda_{\beta i}$ and spin j_{β_i}. Any such eigenstate is a linear combination of generalized eigenstates of the subsystem three-momentum and three-component of subsystem canonical spin:

$$|\psi_{\beta_i}\rangle=\int d^3p\sum_{\mu=-j_{\beta_i}}^{j_{\beta_i}}|\vec{p},\mu[\lambda_{\beta_i},j_{\beta_i}]\rangle\chi_{\beta_i}(\vec{p},\mu). \tag{2.43}$$

Following the treatment of n-particle bound states, the eigenstates $|\vec{p},\mu[m_{\beta_i},j_{\beta_i}]\rangle$ define a mapping Φ_{β_i} from the space $\mathcal{H}_{\beta_i}$ of square integrable functions $\chi_{\beta_i}(\vec{p},\mu)$ to the $U_{b_i}(\Lambda,a)$-invariant subspace of $\mathcal{H}_{b_i}$ spanned by the eigenstates $|\vec{p},\mu[m_{\beta_i},j_{\beta_i}]\rangle$. With this interpretation Eq. (2.43) can be expressed as

$$|\psi_{\beta_i}\rangle=\Phi_{\beta_i}|\chi_{\beta_i}\rangle. \tag{2.44}$$

It follows that

$$U_{b_i}(\Lambda,a)\Phi_{\beta_i}=\Phi_{\beta_i}U_{f\beta_i}(\Lambda,a), \tag{2.45}$$

where $U_{f\beta_i}(\Lambda,a)$ is a mass-λ_{β_i} spin-j_{β_i} irreducible representation of the Poincaré group associated with a bound state of the particles in the cluster b_i. It can be obtained from Eq. (2.35) by replacing the mass and spin with the eigenvalues λ_{β_i} and j_{β_i}.

Define the channel Hilbert space $\mathcal{H}_\beta$ for the channel β as

$$\mathcal{H}_\beta=\otimes_{i=1}^{n_b}\mathcal{H}_{\beta_i} \tag{2.46}$$

and the mapping Φ_β from $\mathcal{H}_\beta$ to $\mathcal{H}$ by

$$\Phi_\beta=\otimes_{i=1}^{n_b}\Phi_{\beta_i}. \tag{2.47}$$

The channel representation of the Poincaré group on $\mathcal{H}_\beta$ is defined by

$$U_{f\beta}(\Lambda,a):=\otimes_{i=1}^{n_b}U_{f\beta_i}(\Lambda,a). \tag{2.48}$$

It follows from Eqs. (2.42), (2.45), and (2.48) that the intertwining relations

$$U_b(\Lambda,a)\Phi_\beta=\Phi_\beta U_{f\beta}(\Lambda,a) \tag{2.49}$$

are satisfied for each channel β. Channels with $n_b>1$ are called scattering channels. The set of channels also includes a channel with $n=1$ for each n-particle bound state.

The asymptotic Hilbert space $\mathcal{H}_f$ is the direct sum of all of the channel spaces, including channels for the n-particle bound states:

$$\mathcal{H}_f:=\oplus_\beta\mathcal{H}_\beta. \tag{2.50}$$

An injection operator that maps $\mathcal{H}_f$ to the n-particle Hilbert space $\mathcal{H}$ is defined by

$$\Phi=\sum_\beta\Phi_\beta, \tag{2.51}$$

where the channel sum runs over bound and scattering channels. The asymptotic representation of the Poincaré group on $\mathcal{H}_f$ is defined by

$$U_f(\Lambda,a)=\sum_\beta U_{f\beta}(\Lambda,a), \tag{2.52}$$

where $U_{f\beta}(\Lambda,a)$ is interpreted to be zero on the orthogonal complement of the β channel subspace.

With this notation it is possible to treat all of the bound and scattering channels simultaneously. Scattering states are solutions of the relativistic Schrödinger equation

$$|\Psi^\pm(t)\rangle=U(I,t)|\Psi^\pm(0)\rangle \tag{2.53}$$

satisfying an incoming $(-)$ or outgoing $(+)$ wave asymptotic condition:

$$\lim_{t\to\pm\infty}\||\Psi^\pm(t)\rangle-\Phi U_f(I,t)|\chi\rangle\|=0. \tag{2.54}$$

States $|\chi\rangle$ in a subspace of $\mathcal{H}_f$ corresponding to n-particle bound states particles automatically satisfy Eq. (2.54) for both time limits. For states $|\chi\rangle$ in channel subspaces involving more than one cluster Eq. (2.54) is equivalent to the existence of the strong limits

$$\lim_{t\to\pm\infty}\||\Psi^\pm(0)\rangle-U(I,-t)\Phi U_f(I,t)|\chi\rangle\|=0 \tag{2.55}$$

for some state $|\Psi^\pm(0)\rangle$. The model is asymptotically complete if both limits exist and are separately complete. In all that follows the dynamics is assumed to be asymptotically complete. It follows that

$$|\Psi^\pm(0)\rangle=\Omega_\pm(H,\Phi,H_f)|\chi\rangle, \tag{2.56}$$

where the wave operator $\Omega_\pm(H,\Phi,H_f)$ is a mapping from $\mathcal{H}_f$ to $\mathcal{H}$ given by the strong limit

$$\Omega_\pm(H,\Phi,H_f):=\lim_{t\to\pm\infty}U(I,-t)\Phi U_f(I,t). \tag{2.57}$$

The assumed asymptotic completeness of the incoming or outgoing wave scattering states plus the bound states implies that wave operators $\Omega_\pm(H,\Phi,H_f)$ are unitary mappings from $\mathcal{H}_f$ to $\mathcal{H}$ satisfying

$$I_{\mathcal{H}}=\Omega_+(H,\Phi,H_f)\Omega_+^\dagger(H,\Phi,H_f)$$
$$=\Omega_-(H,\Phi,H_f)\Omega_-^\dagger(H,\Phi,H_f). \quad (2.58)$$

This implies unitarity of the scattering operator

$$S=\Omega_+^\dagger(H,\Phi,H_f)\Omega_-(H,\Phi,H_f). \quad (2.59)$$

The scattering operator, considered as a mapping on $\mathcal{H}_f$, is Poincaré invariant if

$$[S,U_f(\Lambda,a)]=0. \quad (2.60)$$

The following theorems are useful in what follows

Theorem 1. Consider a relativistic quantum mechanical model with asymptotically complete wave operators. A sufficient condition for the scattering operator to be Poincaré invariant is for the asymptotically complete wave operators to satisfy the intertwining relations

$$U(\Lambda,a)\Omega_\pm(H,\Phi,H_f)=\Omega_\pm(H,\Phi,H_f)U_f(\Lambda,a). \quad (2.61)$$

Relativistic models formulated in one of Dirac's forms of the dynamics satisfy

$$U(\Lambda,a)\Phi=\Phi U_f(\Lambda,a) \quad (2.62)$$

for (Λ,a) in one of the kinematic subgroups E, L, or F, corresponding to the instant, point, or front form, respectively. Dirac's forms of the dynamics are distinguished by the property that the kinematic subgroup (E, L, or F) of the dynamical representation of the Poincaré group is identical to that of the noninteracting representation.

Equation (2.62) implies invariance of the wave operators with respect the appropriate kinematic subgroup.

Sufficient conditions for Poincaré invariance of the scattering operator in each of Dirac's forms of the dynamics are given by the following theorems.

Theorem 2a. A sufficient condition for Poincaré invariance in an instant form dynamics with asymptotically complete wave operators is

$$\lim_{t\to\pm\infty}\|[\Phi-U^\dagger(B_c(p),0)\Phi U_f(B_c(p),0)]U_f(I,t)|\chi\rangle]\|=0 \quad (2.63)$$

for all canonical boosts $B_c(p)$.

Theorem 2b. A sufficient condition for Poincaré invariance in a point form dynamics with asymptotically complete wave operators is

$$\lim_{t\to\pm\infty}\|[\Phi-U^\dagger(I,\vec{a})\Phi U_f(I,\vec{a})]U_f(I,t)|\chi\rangle]\|=0 \quad (2.64)$$

for all spatial translations.

Theorem 2c. A sufficient condition for Poincaré invariance in a light front dynamics with asymptotically complete wave operators is

$$\lim_{t\to\pm\infty}\|[\Phi-U^\dagger(R,0)\Phi U_f(R,0)]U_f(I,t)|\chi\rangle]\|=0 \quad (2.65)$$

for all rotations R.

Theorem 1 is proved in Ref. [7] while theorems 2a–2c are proved in the Appendix. They imply restrictions on the interactions in the nontrivial dynamical generators specific to each form of the dynamics.

Spacelike cluster properties provide the necessary connection between the relativistic dynamics of few- and many-body problems. Spacelike cluster properties require that when the clusters of particles in the partition b are asymptotically separated into subsystems, the dynamical representation of $U(\Lambda,a)$ looks like the tensor product of the subsystem representations.

In order to formulate this condition precisely let β be a scattering channel corresponding to the partition b. Let c be any partition satisfying $c\supseteq b$. This means that the asymptotically bound clusters in the channel β are grouped in clusters of the partition c.

Define the operator on $\mathcal{H}_\beta$ that translates the individual clusters c of ''particles'' in the channel space β:

$$T_{c;f\beta}(x_1,\ldots,x_{n_c})=\otimes_{i=1}^{n_c}[\otimes_{b_j\in c_i}U_{fb_j}(I,x_i)], \quad (2.66)$$

where i labels the clusters of c and j labels cluster of b that are contained in the ith cluster of c. A channel β is compatible with a partition c if $c\subseteq b$. Spacelike cluster properties can be formulated as

$$\lim_{\min|\vec{x}_i-\vec{x}_j|\to\infty}\|[U(\Lambda,a)-U_c(\Lambda,a)]$$
$$\times\Phi_\beta T_{c;f\beta}(x_1,\ldots,x_{n_c})|\chi_\beta\rangle\|=0 \quad (2.67)$$

in the limit that the minimum spacelike separation between the clusters of c becomes infinite. This must hold for all channels compatible with the partition c. The representation $U(\Lambda,a)$ satisfies algebraic cluster properties if it becomes the tensor product $U_c(\Lambda,a)$ when the interactions involving particles in different clusters of the partition c are turned off. Formally, operators that satisfy $(A)_b=A_b$, where $(A)_b$ is obtained by turning off the interactions between particles in different clusters of b and A_b is built up from tensor products of subsystem generators, are said to satisfy algebraic cluster properties. Since the operation in Eq. (2.67) separates particles in different clusters of c, it follows that spacelike cluster properties follow from algebraic cluster properties provided the interactions are of sufficiently short range.

The difficulty is that for Bakamjian-Thomas models $U(\Lambda,a)$ does not become $U_c(\Lambda,a)$ when the interactions between particles in different clusters of c are turned off. Specifically,

$$(U(\Lambda,a))_c\neq U_c(\Lambda,a). \quad (2.68)$$

Scattering equivalences are used in what follows to transform Bakamjian-Thomas representations to representations that satisfy spacelike cluster properties. A number of theorems on scattering equivalence are needed in what follows. A unitary operator A on $\mathcal{H}$ is called a scattering equivalence if it leaves the scattering operator $S=\Omega_+^\dagger\Omega_-$ unchanged. Invariance of the scattering operator $S_1=S_2$ is equivalent to

$$\Omega_{1+}\Omega_{2+}^\dagger=\Omega_{1-}\Omega_{2-}^\dagger=:A_{12}. \quad (2.69)$$

The intertwining properties of the wave operators give $A_{12}H_2=H_1A_{12}$.

Wave operators related by

$$\Omega_{\pm}(AHA^{\dagger},\Phi',H_f)=A\Omega_{\pm}(H,\Phi,H_f) \tag{2.70}$$

are scattering equivalent.

Theorem 3. A necessary and sufficient condition for the unitary operator A to satisfy Eq. (2.70) is for

$$\lim_{t\to\pm\infty}\|(A\Phi-\Phi')U_f(t,I)|\chi\rangle\|=0 \tag{2.71}$$

to vanish for both time limits.

Theorem 4. Any instant form representation is scattering equivalent to an instant form Bakamjian-Thomas representation.

Theorem 5. Any instant form Bakamjian-Thomas representation is scattering equivalent to both a point form and a front form Bakamjian-Thomas representation.

The proof of theorems 3 and 4 are given in [7] and the proof of theorem 5 is given in the Appendix.

III. BASIS CONSTRUCTION

In this section the basis vectors used in the formulation of the dynamical models are constructed.

The kinematic representation of the Poincaré group on $\mathcal{H}$ is the tensor product of n one-particle representations:

$$U_0(\Lambda,a):=\underbrace{U_1(\Lambda,a)\otimes\cdots\otimes U_1(\Lambda,a)}_{n\text{-times}}. \tag{3.1}$$

The representation $U_0(\Lambda,a)$ is reducible. There are many ways to reduce this n-fold tensor product to a linear superposition of mutually orthogonal irreducible subspaces. For any collection of n four-momenta a Lorentz boost can be used to transform to a coordinate system where the total momentum vanishes. The resulting coordinate system can be rotated so that the momentum of particle 1 points in the $+\hat{z}$ direction, while an additional rotation about the $\hat{z}$ axis can be used to bring the momentum of particle 2 to the positive x part of the x-z plane. This coordinate system can be considered as a ''body-fixed'' momentum coordinate system for the n-particle system.

With this motivation the desired irreducible representations are defined below.

Definition 1. Let $\Sigma_{l=1}^{n}\vec{k}_l=\vec{0}$ with $\vec{k}_1=(0,0,|\vec{k}_1|)$ and $\vec{k}_2=(k_{2x},0,k_{2z})$ with $k_{2x}\geqslant 0$ and define the state

$$|[\vec{p},\mu;j]\nu;\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle:=U_0(B_c(p),0)\int_{\mathrm{SU}(2)}dRD^{j*}_{\mu\nu}(R)U_0(R,0)|\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle\sqrt{\frac{m_0}{\omega_{m_0}(\vec{p})}}, \tag{3.2}$$

where

$$p:=B_c(p)(m_0,\vec{0}),\quad m_0=\sum\omega_i(\vec{k}_i), \tag{3.3}$$

μ_i are canonical spin magnetic quantum numbers, and

$$|\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle=|\vec{k}_1,\mu_1\rangle\otimes,\ldots,\otimes|\vec{k}_n,\mu_n\rangle. \tag{3.4}$$

The state vectors $|[\vec{p},\mu;j]\nu;\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle$ are the desired basis states. These states have the following properties.

Theorem 6. The state vector $|[\vec{p},\mu;j]\nu;\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle$ *transforms as a mass-*m_0 *spin-*j *irreducible representation of the Poincaré group:*

$$U(\Lambda,a)|[\vec{p},\mu;j]\nu;\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n\rangle=e^{-ip'\cdot a}|[\vec{p}',\mu';j]\nu;k_1,\mu_1,\ldots,k_n,\mu_n\rangle\sqrt{\frac{\omega_{m_0}(p')}{\omega_{m_0}(p)}}D^{j}_{\mu'\mu}[R_w(\Lambda,p)] \tag{3.5}$$

where $p':=\Lambda p$.

The proof, which uses the invariance of the SU(2) Haar measure, is by direct calculation.

The relation to the single-particle states is given by the Clebsch-Gordan coefficients

$$\begin{aligned}\langle\vec{p}_1,\mu_1,\ldots,\vec{p}_n,\mu_n|[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n\rangle=\delta\Big(\sum_i\vec{p}_i-\vec{p}\Big)\int_{\mathrm{SU}(2)}dR\,\delta^3(\vec{p}_1-[B(\vec{p}/m_0)R]k_1)\cdots\delta^3\\ \times(\vec{p}_{n-1}-[B(\vec{p}/m_0)R]k_{n-1})\sqrt{\frac{m_0}{\omega_{m_0}(\vec{p})}}D^{j*}_{\mu\nu}(R)\prod_{i=1}^{n}\sqrt{\frac{\omega(p_i)}{\omega(k_i)}}D^{j_i}_{\mu_i\nu_i}\\ \times\{(R_w[B_c(p)R],k_i)\}.\end{aligned} \tag{3.6}$$

The integral over SU(2) can be done by observing that if we define

$$\vec{k}_1''=R\vec{k}_1, \quad \vec{k}_2''=R\vec{k}_2, \tag{3.7}$$

then

$$dR=\frac{1}{16\pi^2}d\hat{k}_1''d\hat{k}_2''\delta(\hat{k}_1''\cdot\hat{k}_2''-\hat{k}_1\cdot\hat{k}_2), \tag{3.8}$$

where the integral around one of the azimuthal angles must be over 4π rather than 2π since SU(2) is a double cover of SO(3). After performing the integrations the resulting fixed value of R is the rotation

$$R=\begin{pmatrix} 1 & \vec{0} \\ 0 & [\hat{k}_2'-\hat{k}_1'(\hat{k}_1'\cdot\hat{k}_2')]/[1-(\hat{k}_1'\cdot\hat{k}_2')^2]^{1/2} \\ 0 & \hat{k}_1'\times\hat{k}_2'/[1-(\hat{k}_1'\cdot\hat{k}_2')^2]^{1/2} \\ 0 & \hat{k}_1' \end{pmatrix}, \tag{3.9}$$

which transforms $k_1':=B^{-1}(p)p_1$ to k_1 with $\vec{k}_1=(0,0,|\vec{k}_1|)$ and $k_2':=B^{-1}(p)p_2$ to k_2 with $\vec{k}_2=(k_{2x},0,k_{3y})$ and $k_{2x}\geqslant 0$.

Evaluation of the integral gives

$$\begin{aligned}
&\langle\vec{p}_1,\mu_1,\ldots,\vec{p}_n,\mu_n|[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n\rangle \\
&\quad=\frac{1}{16\pi}\sum_{\pm}\ \delta\Big(\sum_i\ \vec{p}_i-\vec{p}\Big)\frac{\delta(|\vec{k}_1'|-|\vec{k}_1|)}{|\vec{k}_1|^2}\ \frac{\delta(|\vec{k}_2'|-|\vec{k}_2|)}{|\vec{k}_2|^2}\,\delta(\hat{k}_1'\cdot\hat{k}_2'-\hat{k}_1\cdot\hat{k}_2) \\
&\quad\times\prod_{i=3}^{n-1}\ \delta(\vec{k}_i'-\vec{k}_i)\ \sqrt{\frac{\omega_{m_0}(\vec{p})}{m_0}}D^{j*}_{\mu\nu}(\pm R)\prod_{i=1}^{n}\ \sqrt{\frac{\omega(k_i)}{\omega(p_i)}}D^{j_i}_{\mu_i\nu_i}\{(R_w[B_c(p)(\pm R)],k_i)\},
\end{aligned} \tag{3.10}$$

where $k_i'=R^{-1}B^{-1}(p)p_i$ and R is the rotation (3.9). The sum is over the two SU(2) matrices that correspond to the rotation (3.9). It arises because the integral over the Haar measure picks up both elements of SU(2). For integer spins both terms in the sum are identical and it is sufficient to replace $\Sigma_\pm$ by multiplication by 2.

The action of the permutation operator can be computed in this basis. Let σ be the permutation that takes $i\to i'=\sigma(i)$. Let $R(\sigma)$ be the rotation that takes $k_{1'}$ to the z axis and $k_{2'}$ to the positive half x-z plane:

$$R(\sigma)=\begin{pmatrix} 1 & \vec{0} \\ 0 & [\hat{k}_2'-\hat{k}_1'(\hat{k}_1'\cdot\hat{k}_2')]/[1-(\hat{k}_1'\cdot\hat{k}_2')^2]^{1/2} \\ 0 & \hat{k}_1'\times\hat{k}_2'/[1-(\hat{k}_1'\cdot\hat{k}_2')^2]^{1/2} \\ 0 & \hat{k}_1' \end{pmatrix}. \tag{3.11}$$

It follows from the definitions that, for a permutation σ,

$$U(\sigma)|[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n\rangle=|[\vec{p},\mu;j]\nu';\vec{k}_1',\nu_1',\ldots,\vec{k}_n',\nu_n'\rangle D^{j*}_{\nu'\nu}[R(\sigma)]D^{j_1}_{\nu_1'\nu_1}[R(\sigma)]\cdots D^{j_n}_{\nu_n'\nu_n}[R(\sigma)]. \tag{3.12}$$

The space reflection operation has a similar form

$$U(P)|[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n\rangle=|[-\vec{p},\mu;m_0,s]\nu';k_1',\nu_1',\ldots,k_{n'},\nu_{n'}\rangle D^{j*}_{\nu'\nu}[R(P)]D^{j_1}_{\nu_1'\nu_1}[R(P)]\cdots D^{j_n}_{\nu_n'\nu_n}[R(P)], \tag{3.13}$$

where

$$k_i' = -R(P)k_i$$

and $R(P)$ is the rotation about the y axis through π:

$$R(P)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.14}$$

IV. TWO-BODY INTERACTION AND SCATTERING EQUIVALENCE

The saturation of nuclear binding energy with particle number suggests that it should be sufficient to model nuclei using dynamical operators with only two- and three-body interactions. In nonrelativistic quantum mechanics two-body interactions are determined from the two-body problem. Cluster properties imply that the many-body Hamiltonian includes sums of the two-body interactions involving each pair of interacting particles. Additional three-body interactions are also possible and can be deduced from three-body models. Cluster properties determine their contribution to the many-body Hamiltonian. This can be repeated for more complicated many-body interactions.

Although generalizations of this procedure to relativistic systems are possible, the decomposition of the generators of the Poincaré group into linear combinations of subsystem interactions that is dictated by cluster properties fails to satisfy the Poincaré commutation relations without additional many-body interactions.

Fortunately this construction works in the special case that only a single pair of particles interact. In this case the representation of the Poincaré group is a tensor product

$$U_b(\Lambda,a) := U_{(12)}(\Lambda,a)\otimes U_{(3)}(\Lambda,a)\otimes\cdots\otimes U_{(n)}(\Lambda,a), \tag{4.1}$$

where b is the partition $(12)(3)(4)\cdots(n)$, with corresponding formulas when pairs other than particles 1 and 2 interact. The representation $U_{(12)}(\Lambda,a)$ is assumed to be an instant form Bakamjian-Thomas representation. The tensor product of subsystem Bakamjian-Thomas representations is not a Bakamjian-Thomas representation. Specifically, the canonical spin operator for $U_b(\Lambda,a)$ differs from the canonical spin operator for a system of n noninteracting particles.

In this section we apply theorem 4 to the mass operator M_b associated with the tensor product representation (4.1) to obtain a scattering-equivalent Bakamjian-Thomas mass operator $\overline{M_b}$. These Bakamjian-Thomas mass operators can be combined to make an n-body Bakamjian-Thomas mass operator including all of the two-body interactions.

Theorem 4 is special to the instant form of the dynamics. The other forms can be treated by first transforming them to an instant form representation, applying theorem 4, and then transforming back to the initial form of the dynamics. The necessary transformations are discussed in the next section and are given in the Appendix.

Consider a partition b of the form $b=[(12)(3)\cdots(n)]$. The mass operator M_b is related to the two-body mass operator M_{12} by

$$M_b=[(\sqrt{M_{12}^2+\vec{p}_{12}^{\,2}}+\sqrt{M_{0,n-2}^2+\vec{p}_{n-2}^{\,2}})^2-(\vec{p}_{12}+\vec{p}_{n-2})^2]^{1/2}, \tag{4.2}$$

where M_{12} is the interacting two-body Bakamjian-Thomas mass operator tensor product with the identity on the spectator Hilbert space and $M_{0,n-2}$ is the kinematically invariant mass operator for a system of $n-2$ free particles tensor product with the identity on $\mathcal{H}_{12}$. The quantities $\vec{p}_{12}$ and $\vec{p}_{n-2}$ denote the momentum of the interacting pair and the $n-2$ spectator particles. The operator M_b is not a Bakamjian-Thomas type of operator in the n-particle Hilbert space. This can be seen by evaluating the matrix element in the basis of Sec. III. These matrix elements have a nontrivial dependence on the total momentum and kinematic spin.

The procedure for constructing the interaction between particles 1 and 2 is to construct M_b and use scattering-equivalence theorems to find a scattering equivalent Bakamjian-Thomas interaction $\overline{M_b}$. The scattering equivalence is not explicitly needed to construct the interaction; all that is needed is the result of applying this transformation to M_b.

The desired two-body interaction is then

$$\overline{v_b} := \overline{M_b} - M_0. \tag{4.3}$$

In order to compute this interaction note that explicit calculation shows that

$$\langle \vec{p},\mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]|M_b|\vec{p}',\mu';j'[\nu',\vec{k}_1',\nu_1,\ldots,\vec{k}_n',\nu_n]\rangle = \delta(\vec{p}-\vec{p}')$$
$$\times\langle \mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]\|M_b(\vec{p})\|\mu';j'[\nu',\vec{k}_1',\nu_1,\ldots,\vec{k}_n',\nu_n]\rangle \tag{4.4}$$

has nonzero matrix elements for $j\neq j'$. Theorem 4 states that this operator is scattering equivalent to a mass operator $\overline{M_b}$ with matrix elements of the form (4.4) with the kernel in Eq. (4.4) that multiplies the delta function replaced by the same expression with $\vec{p}=0$. This operator is independent of $\vec{p}$ and diagonal in j. The kernel of $\overline{M_b}$ can be computed using the spectral representation of the two-body solutions. To compute M_b first note that by definition

$$\langle \vec{p},\mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]|\overline{M_b}|\vec{p}',\mu';j'[\nu',\vec{k}_1',\mu_1',\ldots,\vec{k}_n',\mu_{n'}]\rangle := \delta(\vec{p}-\vec{p}')$$
$$\times\langle \mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]\|M_b(0)\|\mu';j'[\nu',\vec{k}_1',\mu_1',\ldots,\vec{k}_n',\mu_n']\rangle. \tag{4.5}$$

The kinematical rotational invariance and the orthogonality relation (2.21) give

$$\langle \mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]\| M_b(0)\|\mu';j'[\nu',\vec{k}'_1,\mu'_1\ldots,\vec{k}'_n,\mu'_n]\rangle$$

$$=\frac{\delta_{jj'}\delta_{\mu\mu'}}{2j+1}\sum_{\mu''=-j}^{j}\int d^3p\langle\vec{0}\mu'';j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]|M_b|\vec{p}\mu'';j[\nu',\vec{k}'_1,\mu'_1,\ldots,\vec{k}'_n,\mu'_n]\rangle. \tag{4.6}$$

Inserting the definition of the basis vectors, Eq. (3.2), and the spectral expansion of the two-body mass operator, Eq. (4.2), gives

$$\langle \mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]\|M_b(0)\|\mu';j'[\nu',\vec{k}'_1,\mu'_1,\ldots,\vec{k}'_n,\mu'_n]\rangle$$

$$=\frac{\delta_{jj'}\delta_{\mu\mu'}}{2j+1}\int_{\mathrm{SU(2)}}dRD^{j*}_{\nu\nu'}(R)\langle\vec{k}_1,\mu_1,\vec{k}_2,\mu_2\|j_{12},\lambda_{12},\mu_{12}\rangle\left[\sqrt{\lambda_{12}^2+(\vec{k}_1+\vec{k}_2)^2}+\sum_{l=3}^{n}\sqrt{\vec{k}'^2_l+\mu^2}\right]$$

$$\times D^{j_{12}}_{\mu_{12},\mu'_{12}}(R)\langle j_{12},\lambda_{12},\mu'_{12}\|\vec{k}'_1,\mu'_1,\vec{k}'_2,\mu'_2\rangle\prod_{m=3}^{n}\left[\frac{\delta(k_m-k'_m)}{k_m^2}Y^{l_m}_{r_m}(\hat{k}_m)D^{l_m}_{r_mr'_m}(R)Y^{*l_m}_{r'_m}(\hat{k}'_m)D^{1/2}_{\mu_m\mu'_m}(R)\right], \tag{4.7}$$

where $\langle j_{12},\lambda_{12},\mu'_{12}\|\vec{k}'_{12},\mu'_1,\vec{k}'_2,\mu'_2\rangle$ are the reduced two-body wave functions of the interacting 12 pair:

$$\langle\vec{p},j_{12},\lambda_{12},\mu'_{12}|\vec{k}'_1,\mu'_1,\vec{k}'_2,\mu'_2\rangle=\delta(\vec{p}-\vec{k}_1-\vec{k}_2)\langle j_{12},\lambda_{12},\mu'_{12}\|\vec{k}'_1,\mu'_1,\vec{k}'_2,\mu'_2\rangle \tag{4.8}$$

and

$$\langle j_{12},\lambda_{12},\mu'_{12}\|\vec{k}_1,\mu_1,\vec{k}_2,\mu_2\rangle=\langle j_{12},\lambda_{12}|q_{12},l,s\rangle\left[\frac{\omega_1(q_{12})\omega_2(q_{12})\omega_{12}(p_{12})}{\omega_1(k_1)\omega_2(k_2)m_{012}}\right]^{1/2}\langle j_{12},\mu_{12}|l,\mu_l,s,\mu_s\rangle$$

$$\times\langle s\mu_s|\tfrac{1}{2},\mu'_1,\tfrac{1}{2},\mu'_2\rangle Y^{*l}_{\mu_l}(\hat{q}_{12})D^{1/2}_{\mu'_1\mu_1}[B^{-1}(q_{12})B^{-1}(p_{12})B^{-1}(k_1)]$$

$$\times D^{1/2}_{\mu'_2\mu_2}[B^{-1}(q_{12})B^{-1}(p_{12})B^{-1}(k_2)], \tag{4.9}$$

with $p_{12}=k_1+k_2$, $q_{12}=B^{-1}(p_{12})k_1$, and $m_{012}^2=-(k_1+k_2)^2$. The amplitudes $\langle j_{12}\lambda_{12}|q,l,s\rangle$ are solutions to the two-body mass eigenvalue equation

$$(\sqrt{\vec{q}^2+m_1^2}+\sqrt{\vec{q}^2+m_2^2}-\lambda_{12})\langle j_{12},\lambda_{12}|q,l,s\rangle=-\int\langle q',l',s'|v^{j_{12}}_{12}|q,l,s\rangle q'^2dq'\langle j_{12},\lambda_{12},|q',l',s'\rangle, \tag{4.10}$$

corresponding to both bound and scattering states.

The matrix elements in Eq. (4.7) have a simple expression in terms of coefficients defined by the following integral over the rotations,

$$I^{j:j_{12},l_3,\cdots l_n,1/2\cdots1/2}_{\nu\nu':\nu_{12},\nu'_{12},r_3,r'_3,\ldots,r_n,r'_n,\mu_3,\mu'_3\cdots\mu_n,\mu'_n}=\frac{1}{2j+1}\int_{\mathrm{SU(2)}}dRD^{j*}_{\nu\nu'}(R)D^{j_{12}}_{\mu_{12}\mu'_{12}}(R)\prod_{m=3}^{n}[D^{l_m}_{r_mr'_m}(R)D^{1/2}_{\mu_m\mu'_m}(R)], \tag{4.11}$$

given by

$$\langle \mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]\|M_b(0)\|\mu';j'[\nu',\vec{k}'_1,\mu'_1,\ldots,\vec{k}'_n,\mu'_n]\rangle$$

$$=\delta_{jj'}\delta_{\mu\mu'}I^{j:j_{12},l_3,\ldots,l_n,1/2\ldots,1/2}_{\nu\nu':\nu_{12},\nu'_{12},r_3,r'_3\ldots,r_n,r'_n,\mu_3,\mu'_3\ldots,\mu_n,\mu'_n}\langle\vec{k}_1,\mu_1,\vec{k}_2,\mu_2\|j_{12},\lambda_{12},\mu_{12}\rangle\left[\sqrt{\lambda_{12}^2+(\vec{k}_1+\vec{k}_2)^2}+\sum_{l=3}^{n}\sqrt{\vec{k}_l^2+m^2}\right]$$

$$\times\langle j_{12},\lambda_{12},\mu'_{12}\|\vec{k}'_1,\mu'_1,\vec{k}'_2,\mu'_2\rangle\prod_{m=3}^{n}\left[\frac{\delta(k_m-k'_m)}{k_m^2}Y^{l_m}_{r_m}(\hat{k}_m)Y^{*l_m}_{r'_m}(\hat{k}'_m)\right]. \tag{4.12}$$

These coeffieicents can be computed exactly using group theoretical techniques or by direct integration using Eqs. (2.21) and (2.23).

The above expression corresponds to the case that particles 1 and 2 interact. For any other choice of an interacting pair of particle ij let σ be the permutation that interchanges particles 1 with i and 2 with j. Since $U^\dagger(\sigma)M_bU(\sigma)=M_{(ij)(1)\cdots(n)}$, it follows using Eq. (3.12) that

$$\begin{aligned}
&\langle[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|\overline{M}_{(ij)(1)\cdots(n)}|[\vec{p}',\mu';j']\nu';\vec{k}_1',\nu_1',\ldots,\vec{k}_n',\nu_n'\rangle\\
&\quad=\langle[\vec{p},\mu;j]\nu;\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|U^{\dagger}(\sigma)\overline{M}_{(12)(1)\cdots(n)}U(\sigma)|[\vec{p}',\mu';j']\nu';\vec{k}_1',\nu_1',\ldots,\vec{k}_n',\nu_n'\rangle\\
&\quad=D^{j*}_{\nu\nu''}[R^{-1}(\sigma)]D^{j_1}_{\nu_1\nu_1''}[R^{-1}(\sigma)]\cdots D^{j_n}_{\nu_n\nu_n''}[R^{-1}(\sigma)]\langle[\vec{p},\mu;j]\nu'';\vec{k}_1'',\nu_1',\ldots,\vec{k}_n''\nu_n''|\overline{M}_{(12)(3)\cdots(n)}|\\
&\qquad\times[\vec{p}',\mu';j]\nu''';\vec{k}_1''',\nu_1''',\cdots,\vec{k}_n'''\rangle D^{j*}_{\nu'''\nu'}[R(\sigma)]D^{j_1}_{\nu_1'''\nu_1'}[R(\sigma)]\cdots D^{j_n}_{\nu_n'''\nu_n'}[R(\sigma)],
\end{aligned}\tag{4.13}$$

where $k_i''=R(\sigma)k_i$ and $k_i'''=R(\sigma)k_i'$.

V. MODEL DYNAMICS

The two-body interactions defined by (4.3), (4.6), (4.7), and (4.13) are all Bakamjian-Thomas interactions. Each one commutes with both the kinematic generators and the kinematic spin. Although they fail to cluster properly, each interaction has the property that when it is added to the kinematic mass operator the result is scattering equivalent to a system of two interacting particles and $n-2$ noninteracting spectators that clusters properly.

These Bakamjian-Thomas interactions can be added to construct a Bakamjian-Thomas mass operator that includes interactions involving all pairs of particles. This follows because each term in this mass operator commutes with all kinematic generators and the kinematic spin. Simultaneous eigenstates of the mass, spin, and momentum can be chosen to transform irreducibly with respect to a Bakamjian-Thomas representation of the Poincaré group.

The n-particle mass eigenstates are solutions to the eigenvalue problem

$$\overline{M}|\Psi\rangle=\lambda|\Psi\rangle,\tag{5.1}$$

where

$$\overline{M}=M_0+\sum_{\{b|n_b=n-1\}}(\overline{M}_b-M_0)=M_0+\sum_{\{b|n_b=n-1\}}\overline{v}_b.\tag{5.2}$$

In this representation the wave functions have the form

$$\begin{aligned}
&\langle\vec{p},\mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n]|\vec{p}',\mu';j',\psi\rangle\\
&\quad=\delta^3(\vec{p}-\vec{p}')\delta_{\mu\mu'}\delta_{jj'}\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle.
\end{aligned}\tag{5.3}$$

The reduced wave functions $\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle$ satisfy

$$\begin{aligned}
&\Big(\lambda-\sum\ \omega_m(\vec{k}_i^{\,2})\Big)\langle\nu,\vec{k}_1,\nu_1,\ldots\vec{k}_n,\nu_n|j;\phi\rangle\\
&\quad=\int\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n\|\overline{v_b^j}\|\nu',\vec{k}_1',\nu_1',\ldots,\vec{k}_n',\nu_n'\rangle\\
&\qquad\times d|\vec{k}_1|dk_{2x}dk_{2z}\theta(k_{2x})d^3k_3\cdots d^3k_n\delta\Big(\sum\ \vec{k}_i\Big)\\
&\qquad\langle\nu',\vec{k}_1',\nu_1',\ldots,\vec{k}_n',\nu_n'|j;\phi\rangle,
\end{aligned}\tag{5.4}$$

where $\langle\cdots\|v_b\|\cdots\rangle$ indicates that the $\delta^3(\vec{p}-\vec{p}')\delta_{jj'}\delta_{\mu\mu'}$ have been factored out of the matrix element defined by (4.3), (4.7), and (4.13).

The dynamics $U(\Lambda,a)$ is defined by the requirement that the complete set of eigenstates (5.3) of the mass operator (5.2) transform like mass-λ spin-j irreducible representations of the Poincaré group.

For a system of more than a few particles, this equation is a complicated many-body problem. However, it has the useful feature that it can be used to compute variational solutions of the form

$$\begin{aligned}
&\langle\vec{p}\mu;j[\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\nu_n]|\vec{p}',\mu',j'\phi\rangle\\
&\quad=\delta(\vec{p}-\vec{p}')\delta_{\mu\mu'}\delta_{jj'}\langle\nu,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle,
\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}
&\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle\\
&\quad=\sum_n\ c_n\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi_n\rangle
\end{aligned}\tag{5.6}$$

and $\langle\nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi_n\rangle$ are suitably chosen expansion functions. This approximation corresponds to projecting the mass operator M on a subspace of the Hilbert space. The equations for the coefficients have the form of a generalized eigenvalue problem:

$$\begin{aligned}
&\sum_n\ \langle j\phi_l|\sum\ w_m(\vec{k}_i^{\,2})|j\phi_n\rangle c_n+\sum_b\ \sum_n\ \langle j\phi_l\|\overline{v_b}\|j\phi_n\rangle c_n\\
&\quad=\lambda\sum_n\ \langle j\phi_l|j\phi_n\rangle c_n.
\end{aligned}\tag{5.7}$$

The resulting equations give coefficients c_n that define the variational eigenstates. The eigenvalues are variational (upper) bounds on the exact mass eigenstates λ_n.

The variational approximation has the important property that it exactly preserves the Poincaré invariance. Specifically, the approximate eigenstates $|\vec{p},\mu,j,\phi\rangle=|\vec{p},\mu,j,\lambda_n\rangle$ defined by (5.5), (5.6), and (5.7) transform irreducibly with respect to the unitary representation of the Poincaré group:

$$\overline{U_v}(\Lambda,a)|\vec{p},\mu,j,\lambda_n\rangle = e^{-ip'_{\lambda_n}\cdot a}|\vec{p}',\mu',j,\lambda_n\rangle \times \sqrt{\frac{\omega_{\lambda_n}(\vec{p}')}{\omega_{\lambda_n}(\vec{p})}} D^j_{\mu'\mu} \times [B^{-1}(\vec{p}'/\lambda_n)\Lambda B(\vec{p}/\lambda_n)], \tag{5.8}$$

where λ_n is the variational mass eigenvalue and $p' = \Lambda(\omega_{\lambda_n}(\vec{p}),\vec{p})$.

The special advantage of the simultaneously coupled basis is that variational wave functions of a given spin, parity, and exchange symmetry can be constructed using the operators $U(\sigma)$ and $U(P)$ given in Eq. (3.12) and (3.13). Expansion functions of a given exchange symmetry can be generated from a given expansion function by symmetrizing:

$$\langle \nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle = c\sum_\sigma (-)^{|\sigma|}\langle \nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|U(\sigma)|j;\phi\rangle, \tag{5.9}$$

where the sum is over all permutations of identical particles and the phase factor is $-$ for odd transposition of fermions. The coefficient c is a normalization constant. Variational wave functions of a given parity are constructed using expansion functions of the form

$$\langle \nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\phi\rangle = \frac{1}{\sqrt{2}}[\langle \nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|j;\chi\rangle \pm \langle \nu,\vec{k}_1,\nu_1,\ldots,\vec{k}_n,\nu_n|U(P)|j;\chi\rangle]. \tag{5.10}$$

The result is a set of approximate eigenstates $|\vec{p},\mu,j,\lambda_n\rangle$ that are linear combinations of the symmetrized expansion functions. These states define reduced wave functions for states that transform as a mass-λ_n spin-j irreducible representation of the Poincaré group. Thus exact Poincaré invariance, permutation symmetry, and reflection symmetry are preserved at each level of approximation. Successive approximations improve the mass eigenvalue.

Equation (5.4), with the interactions defined in Sec. III and the approximations defined in Eq. (5.7), is the main result of this paper. It shows how to embed two-body interactions in models with symmetric coupling schemes and how to formulate systematic Poincaré-invariant approximations of a given spin, parity, and exchange symmetry.

A shortcoming of this model and the associated approximations is that they explicitly violate spacelike cluster properties. In the next section this model is interpreted as a first approximation to a model that clusters properly. The goal of the next section is to obtain an understanding of the corrections needed to restore cluster properties.

The irreducible basis states defined in definition 1 use the linear momentum and canonical spin to label vectors in an irreducible subspace. The degeneracy quantum numbers include ''body-fixed'' components of the linear momentum and the three-component of the canonical spin. It is easy to construct similar irreducible basis vectors that replace all of the linear momentum variables by either four-velocities or light front momentum components. In addition the spins can all be Melosh rotated to light front spins or helicities. Because theorem 4 is special to instant form dynamics, the construction of two-body interactions that are scattering equivalent to Bakamjian-Thomas interactions is most naturally performed using instant form dynamics. The results can be transformed to front- or point form representation with the transformations used in the proof of theorem 5 in the Appendix.

VI. SPACELIKE CLUSTER PROPERTIES

The purpose of this section is to deduce the relationship between the models defined by solving Eq. (5.4) in the previous section to models that satisfy spacelike cluster properties. This relationship can be used to determine the specific corrections needed to restore spacelike cluster properties.

The spacelike cluster property is the requirement that the dynamical representation of the Poincaré group approximate a tensor product of subsystem representations when the particles in the different clusters are separated beyond the range of their mutual interactions. The mathematical formulation of this property is given in Eq. (2.67). Spacelike cluster properties provide the connection between the relativistic few- and many-body problem and are needed to interpret experiments that test special relativity on isolated subsystems.

In what follows the notion of scattering equivalence is used extensively. Scattering equivalences are a subgroup of the group of unitary operators. The type of scattering equivalence discussed in theorem 4 that transforms any instant form representation of the Poincaré group to an instant form Bakamjian-Thomas representation is denoted by B_x. The type of scattering equivalence discussed in theorem 5 that transforms an instant form Bakamjian-Thomas representation of the Poincaré group to a front- (or point) form Bakamjian-Thomas representation is denoted by C_x. These two types of scattering equivalences are used to inductively construct representations of the Poincaré group satisfying cluster properties. A overbar is used to indicate a Bakamjian-Thomas representation $\overline{U}(\Lambda,a)$ of the Poincaré group.

The construction is discussed in detail for the three- and four-body systems. Explicit inductive algorithms are given in the n-particle case. The general construction follows closely the construction given in [7]. Theorem 4 of this paper is theorem 3.4 of Ref. [7]. This theorem does not have a general extension to the front or point-form of the dynamics. There is an extension in the case of the front form [18], but it does not hold with the generality of theorem 4. In this paper, the scattering equivalence C_x is used to transform to representations where theorem 4 can be used.

The construction of a unitary representation of the Poincaré group satisfying cluster properties is by upward induction on the number of particles and downward induction on the number of clusters for a fixed number of particles.

The starting point is a two-body Bakamjian-Thomas model. Let b be a partition of a n-particle system with n-1 clusters. This corresponds to n-2 free particles and a single interacting pair. The Bakamjian-Thomas method [3,7,10,9]

can be used to construct a two-body unitary representation of the Poincaré group in any of Dirac's forms of the dynamics. Let

$$U_b(\Lambda,a):=\otimes_{i=1}^{n_b}\overline{U_{b_i}}(\Lambda,a) \tag{6.1}$$

be the tensor product of the subsystem representations corresponding to the partition b. This tensor product includes 1 two-body Bakamjian-Thomas representation and n-2 one-body representations.

Since each of the subsystem representations associated with the clusters of the partition b is a Bakmjian-Thomas representation (trivially in the one-body case), they are each scattering equivalent to an instant form Bakamjian-Thomas representation by the scattering equivalences $C_{b_i}^\dagger$ of theorem 5. In the case that the starting representation is already an instant form representation $C_{b_i}^\dagger=I$, it follows that

$$U_b^I(\Lambda,a):=[\otimes C_{b_i}^\dagger]U_b(\Lambda,a)[\otimes C_{b_i}]=\otimes[C_{b_i}^\dagger\overline{U_{b_i}}(\Lambda,a)C_{b_i}] \tag{6.2}$$

is a tensor product of instant form Bakamjian-Thomas representations that are scattering equivalent to $U_b(\Lambda,a)$. Thus for each $n-1$ cluster partition b there is a scattering equivalence that transforms the representation $U_b(\Lambda,a)$ to a tensor product of instant form Bakamjian-Thomas representations.

The tensor product of instant form Bakamjian-Thomas representations does not commute with the kinematic canonical spin operator and is consequently not a Bakamjian-Thomas representation. It is, however, an instant form representation and is scattering equivalent to a Bakamjian-Thomas representation by the scattering equivalence B_b of theorem 4:

$$\overline{U_b^I}(\Lambda,a):=B_bU_b^I(\Lambda,a)B_b^\dagger. \tag{6.3}$$

The representation $\overline{U_b^I}(\Lambda,a)$ is exactly the representation constructed in Sec. III using the mass operator $\overline{M}_b$, the kinematic instant form Poincaré generators, and the kinematic canonical spin.

By theorem 5 the representation (6.3) is scattering equivalent to a Bakamjian-Thomas representation in the chosen form of the dynamics by a scattering equivalence C_b:

$$\overline{U_b}(\Lambda,a):=C_b\overline{U_b^I}(\Lambda,a)C_b^\dagger. \tag{6.4}$$

Since the scattering equivalences form a group, the product

$$A_b:=C_bB_b(\otimes C_{b_i}^\dagger) \tag{6.5}$$

is a scattering equivalence that maps a tensor product of subsystem representations to a Bakamjian-Thomas representation in any form of the dynamics:

$$\overline{U_b}(\Lambda,a)=A_bU_b(\Lambda,a)A_b^\dagger. \tag{6.6}$$

The construction up to this point is limited to n-particle systems where a single pair of particles interact.

The next step is to consider the case of three interacting particles. In order to formulate the limit of an n-particle system clustering into a tensor product of subsystems that include three-particle subsystems we need a three-particle representation that clusters to a tensor product of a two-particle representation and a single-particle representation.

In general the spin operators $\vec{j}_b$ associated with the tensor product representation $U_b(\Lambda,a)$ for different $n-1$ cluster partitions b are distinct and nonkinematic. Because of this, it is difficult to combine the mass operators M_b corresponding to different interacting pairs in a manner that preserves the group representation properities. The problem can be solved by replacing the M_b's by the scattering equivalent $\overline{M}_b$'s defined by

$$\overline{M}_b=A_bM_bA_b^\dagger, \tag{6.7}$$

where A_b is given by Eq. (6.5). The spin operator for each $\overline{M}_b$ is the kinematic n-particle spin operator. It follows that the three-body mass operator

$$\overline{M}(3):=\overline{M}_{(12)(3)}+\overline{M}_{(23)(1)}+\overline{M}_{(31)(2)}-2M_{(1)(2)(3)} \tag{6.8}$$

commutes with the kinematic generators and the kinematic spin in a given form of the dynamics. Thus it is the mass operator for a three-body Bakamjian-Thomas representation of the Poincaré group.

Note that for instant form Bakamjian-Thomas representations, $\overline{U}(\Lambda,a)$ can be constructed by first finding simultaneous eigenstates of $\overline{M}(3)$, $\vec{P}$, j^2, and $\vec{j}_c\cdot\hat{z}$. This is done by diagonalizing $\overline{M}(3)$ in a basis of simultaneous eigenstates of the kinematic operators $\vec{P}$, j^2, and $\vec{j}\cdot\hat{z}$ and any additionally kinematically invariant degeneracy parameters. The representation $\overline{U}(\Lambda,a)$ is fixed by the requirement that these eigenstates transform as a unitary irreducible representation of the Poincaré group with the mass and spin labeled by the eigenvalues of j^2 and $\overline{M}(3)$. The analogous construction works in the point and front forms except that the momentum is replaced by either the four-velocity or the light front component of the four-momentum and in the case of the front form the $\hat{z}$ component of the canonical spin is replaced by the $\hat{z}$ component of the light front spin.

To investigate cluster properties consider the behavior of the scattering equivalences A_b in the limit that the interactions between particles in the same cluster of b are turned off. For a two-cluster partition b of a three-particle system $M_b\to M_{(1)(2)(3)}$ when the interactions between the interacting pair of particles are turned off. The corresponding representation $U_b(\Lambda,a)$ becomes the kinematic representation, which is a Bakamjian-Thomas representation, in the same limit. With the definitions (6.5) scattering equivalence A_b becomes the identity.

Thus, in the limit that all interactions involving particle (3) are turned off, $\overline{M}(3)$ becomes

$$\begin{aligned}\overline{M}(3)&=\sum_{n_b=2}A_bM_bA_b^\dagger-2M_{(1)(2)(3)}\to IM_{(1)(2)(3)}I\\&\quad+IM_{(1)(2)(3)}I+A_{(12)(3)}M_{(12)(3)}A_{(12)(3)}^\dagger-2M_{(1)(2)(3)}\\&=A_aM_aA_a^\dagger=\overline{M}_a,\end{aligned} \tag{6.9}$$

where $a=(12)(3)$. It follows that

$$\overline{U}(\Lambda,a)\to A_a U_a(\Lambda,a)A_a^\dagger=\overline{U_a}(\Lambda,a)\neq U_a(\Lambda,a) \tag{6.10}$$

in the same limit, which shows the manifest violations of cluster properties due to the presence of the A_b's.

A new scattering-equivalent representation with the correct limit can be constructed by introducing a new scattering equivalence. The new scattering equivalence is designed to cancel off each of the A_b's in each asymptotic region. Define the Sokolov product of the operators $A_{(12)(3)}$, $A_{(23)(1)}$, and $A_{(31)(2)}$ by adding the Cayley transform of each operator and constructing the inverse Cayley transform of the result:

$$\mathsf{S}[A_{(12)(3)}A_{(23)(1)}A_{(31)(2)}] := \frac{1+i(\alpha_{(12)(3)}+\alpha_{(23)(1)}+\alpha_{(31)(2)})}{1-i(\alpha_{(12)(3)}+\alpha_{(23)(1)}+\alpha_{(31)(2)})}, \tag{6.11}$$

$$\alpha_{(ij)(k)} := i\frac{I-A_{(ij)(k)}}{I+A_{(ij)(k)}}. \tag{6.12}$$

The Sokolov product of unitary operators is unitary and symmetric under interchange of operator ordering within the product. In what follows the Sokolov product of scattering equivalences is assumed to be a scattering equivalence. The scattering equivalence $A(3)$ is defined as the Sokolov product of the scattering equivalences A_b for all two cluster partitions:

$$A(3) := \mathsf{S}[A_{(12)(3)}A_{(23)(1)}A_{(31)(2)}]. \tag{6.13}$$

The transformed mass operator and unitary representation of the Poincaré group are defined by

$$\begin{aligned} M(3) &:= A^\dagger(3)\overline{M}(3)A(3), \\ U(\Lambda,a) &= A^\dagger(3)\overline{U}(\Lambda,a)A(3). \end{aligned} \tag{6.14}$$

In the limit that the interactions involving particle i are turned off these operators become

$$A(3)\to \mathsf{S}[A_{(jk)(i)}\cdot I\cdot I]=A_{(jk)(i)} \tag{6.15}$$

and

$$U(\Lambda,a)\to A^\dagger_{(jk)(i)}\overline{U}_{(jk)(i)}(\Lambda,a)A_{(jk)(i)}=U_{(jk)(i)}(\Lambda,a), \tag{6.16}$$

which shows that $U(\Lambda,a)$ defined by Eq. (6.14) satisfies algebraic cluster properties. One expects that for intercluster interactions of sufficiently short range the representation $U(\Lambda,a)$ defined above will satisfy Eq. (2.67).

The scattering-equivalence $A(3)$ transforms a Bakamjian-Thomas three-body system to a scattering-equivalent three-body system that algebraically clusters. Since cluster properties fix the three-body generators of the Poincaré group in terms of the one- and two-body generators, up to overall three-body interactions, the net result of this construction is to generate additional new three-body interactions needed to maintain Poincaré invariance. The three-body interactions generated have a momentum dependence dictated by cluster properties. This distinguishes them from three-body Bakamjian-Thomas interactions that can always be added to $\overline{M}(3)$

The representation $\overline{U}(\Lambda,a)$ in Eq. (6.14) is exactly the representation of the Poincaré group constructed in the previous section for the instant form three-body case. Since this representation is scattering equivalent to a dynamical model that clusters, it follows that the model of Sec. III gives the same mass eigenvalues and scattering cross sections as the model that clusters. This means that in the three-body case no additional corrections are needed. This result was established in 1965 by Coester in [4]. Unfortunately the conclusion is not valid for systems of more than three particles.

The four-body system is the simplest system that exhibits the most general difficulties. Proceeding to the four-body system, consider both two- and three-cluster partitions of four particles. For each of these partitions define the representations

$$U_b(\Lambda,a) := \otimes_{i=1}^{n_b} U_{b_i}(\Lambda,a), \tag{6.17}$$

where each $U_{b_i}(\Lambda,a)$ is the representation of the Poincaré group for the physical one-, two-, and three-body subsystems that satisfy algebraic cluster properties. For the two-body subsystems these are two-body Bakamjian-Thomas representations while for the three-body subsystems it is the representation (6.14) defined above.

In the two-body case $U_{b_i}(\Lambda,a)=\overline{U}_{b_i}(\Lambda,a)$ are Bakamjian-Thomas representations while in the three-body case $U_{b_i}(\Lambda,a)$ is related to a Bakamjian-Thomas representation by the scattering equivalence $A(3)$ in Eq. (6.13) which is denoted by A_{b_i}. The general result can be expressed in the form

$$U_{b_i}(\Lambda,a)=A^\dagger_{b_i}\overline{U}_{b_i}(\Lambda,a)A_{b_i}, \tag{6.18}$$

where A_{b_i} is the identity for two-particle clusters and the scattering equivalence $A(3)$ for each of the three-particle clusters.

The next step is to generalize Eq. (6.5) to the four-body case by defining the scattering equivalences

$$D_b := C_b B_b(\otimes C^\dagger_{b_i}A_{b_i}). \tag{6.19}$$

The scattering equivalences C_b, B_b, and C_{b_i} have the same interpretation as the corresponding quantities appearing in Eq. (6.5). The scattering equivalences D_b transform the tensor product of the physical subsystem representations associated with the particles in each cluster of the partition b to Bakamjian-Thomas representations. The tensor product representations are the required limiting representations when interactions between particles in different clusters of b are turned off while the Bakamjian-Thomas representations are convenient for adding interactions. The new factors A_{b_i}, which do not appear in Eq. (6.5), are needed to convert the physical representations of the three-body subsystems discussed above to Bakamjian-Thomas representations.

The four-body problem has an additional complication that does not occur in the three-body case. To understand the

problem consider the three partitions $a=(123)(4)$, $b=(12)(34)$, and $c=(12)(3)(4)$ and turn off all interactions involving particle 3. In this limit D_a and D_b defined by Eq. (6.19) both become scattering equivalences $(D_a)_c$ and $(D_b)_c$ with the property that $(D_a)_c$, $(D_b)_c$, and D_c each map $U_c(\Lambda,a)$ to (possibly) different scattering-equivalent Bakamjian-Thomas representations. This is more complicated than in the three-body case where all of the limiting forms were the identity, which facilitated the cancellations in Eq. (6.9). In order to get similar cancellations in the four-body system it is desirable to redefine D_a and D_b so they both become D_c in the limit that interactions involving particle 3 are turned off. Since interactions involving particle 1 or 2 could also be turned off, the general condition

$$(D_a)_d=D_{a\cap d}, \quad (D_b)_d=D_{b\cap d} \tag{6.20}$$

is required, where d is any partition of the four-body system. This ensures that all of these operators D_a are built out of the same b-connected $[D]_b$'s.

This can be done by replacing the D_b's defined by Eq. (6.19) by new scattering equivalences A_b with this property defined as follows:

$$A_d:=D_d, \quad n_d\geqslant 3 \tag{6.21}$$

for all three- and four-cluster partitions. For two-cluster partitions, such as a and b above, define A_a using the A_d's for $n_d\geqslant 3$ in the Sokolov products

$$A_a:=D_a\mathsf{S}_{a\supset d\supset 0}[(D_a)_d^{\dagger}A_{a\cap d}], \tag{6.22}$$

where the factors in the Sokolov product correspond to the three-cluster partitions d contained in a (or b, respectively). Specifically for the case $a=(123)(4)$ the Sokolov product is the inverse Cayley transform of the sum of the Cayley transforms of $(D_a)_d^{\dagger}A_{a\cap d}$ for the partitions $d=(12)(3)(4)$, $(13)(2)(4)$, and $(23)(1)(4)$. For the case $b=(12)(34)$ the product is over the two partitions $d=(1)(2)(34)$ and $(12)(3)(4)$. The remaining two-cluster partitions are treated similarly. The definition of the A_b for two-cluster b's in Eq. (6.22) only involves three-cluster A_c's. The operators A_b so defined have the property that if the interactions involving particles in different clusters of a partition c are turned off, then

$$(A_b)_c\rightarrow(D_b)_c(D_b^{\dagger})_cA_{b\cap c}=A_{b\cap c}. \tag{6.23}$$

This follows because all but one term become the identity in the Sokolov product when one of the two clusters of b is broken up.

This construction extends to all partitions of two or more clusters, generating scattering equivalences A_b that map each tensor product representation to a Bakamjian-Thomas representation in the desired form of the dynamics:

$$A_bU_b(\Lambda,a)A_b^{\dagger}=\overline{U_b}(\Lambda,a), \tag{6.24}$$

$$\overline{M_b}:=A_b\overline{M_b}A_b^{\dagger}. \tag{6.25}$$

By construction the operators A_b satisfy the following algebraic cluster property:

$$A_b\rightarrow(A_b)_c=(A)_{b\cap c}. \tag{6.26}$$

It follows that in the same limit (6.24) becomes

$$A_{b\cap c}U_{b\cap c}(\Lambda,a)A_{b\cap c}^{\dagger}=\overline{U_{b\cap c}}(\Lambda,a). \tag{6.27}$$

As in the three-body case let $\overline{M_{b_i}}$ denote the mass operators associated with the barred representations and define a four-body Bakamjian-Thomas mass operator:

$$\overline{M}=\sum_{a\neq\underline{1}}\zeta_aA_aM_aA_a^{\dagger}. \tag{6.28}$$

This is the generalization of the expansion (6.8) for the four (n)-body system. This can be used to construct a Bakamjian-Thomas representation $U(\Lambda,a)$ exactly as was done in the three-body case.

As in the three-body case we seek a scattering equivalence A that clusters as $A\rightarrow A_a$ when the interactions involving particles in different cluster of a are turned off. The desired operator can be constructed using Sokolov products

$$A=A(4):=\frac{1+i\alpha}{1-i\alpha}, \quad \alpha:=\sum_{a\neq 1}\zeta_a\alpha_a, \tag{6.29}$$

$$\alpha_a:=i\frac{1-A_a}{1+A_a}. \tag{6.30}$$

Note that Eqs. (6.29) and (6.30) define the Sokolov product

$$\mathcal{S}_{a\neq\underline{1}}[A_a^{\zeta_a}]. \tag{6.31}$$

Because the individual A_a's are designed to satisfy algebraic cluster properties $A_a\rightarrow A_{a\cap b}$ when the interactions between particles in the cluster of b are turned off, it follows that $\alpha_a\rightarrow\alpha_{a\cap b}$ and

$$\alpha:=\sum_{a\neq\underline{1}}\zeta_a\alpha_a,$$

$$\sum_{a\neq\underline{1}}\zeta_a\alpha_{a\cap b}=-\sum_{a\neq\underline{1}}\Delta^{-1}(\underline{1},a)\Delta(a,c)\Delta(b,c)[\alpha]_c=\alpha_b, \tag{6.32}$$

which means that $A\rightarrow A_b$ in this limit. Thus we define

$$U(\Lambda,a)=A^{\dagger}\overline{U(\Lambda,a)}A, \tag{6.33}$$

which by design has the algebraic cluster properties

$$U(\Lambda,a)\rightarrow A_a^{\dagger}\overline{U_a}(\Lambda,a)A_a=U_a(\Lambda,a). \tag{6.34}$$

This construction can be continued by induction for any number of particles. No new complications arise. The steps in the general construction are outlined below. For the case of n particles assume that all k-body representations of the Poincaré group with $k<n$ have been constructed. By induction these all are assumed to satisfy algebraic cluster properties and are each scattering equivalent to a Bakamjian-Thomas representation. Specifically they satisfy

$$[U(\Lambda,a)]_b=U_b(\Lambda,a), \tag{6.35}$$

$$U(\Lambda,a)=A^{\dagger}(k)\overline{U}(\Lambda,a)A(k). \tag{6.36}$$

For each partition b of n particles with at least two clusters define the tensor products of subsystem dynamics,

$$U_b(\Lambda,a):=\otimes_{i=1}^{n}U_{b_i}(\Lambda,a), \tag{6.37}$$

and the scattering equivalences

$$D_b:=C_bB_b(\otimes C_{b_i}^{\dagger}A_{b_i}). \tag{6.38}$$

The A_{b_i}'s are the $A(k)$'s discussed above and the remaining operators are as defined in the three- and four-body cases. For each $c\subset b$ define $(D_b)_c$ by formally turning off interactions in the formal expression for D_b between particles in different clusters of the partition c. For each b define recursively A_b beginning with the n-1 cluster partitions and proceeding downward in number of clusters:

$$A_b:=D_b,\quad n_b=n-1. \tag{6.39}$$

Assuming A_c have been defined for all partitions with $k>n-m$ clusters define A_b for $n_b=n-k$ by

$$A_b:=D_b\mathsf{S}_{c\subset b}[[(D_b)_c^{\dagger}A_c]^{-\Delta^{-1}(b,c)}]. \tag{6.40}$$

Continue until A_b have been defined for all partitions with at least two clusters.

With this definition, in the limit that the interactions between particles in different clusters of the partition d are turned off, A_b becomes

$$(A_b)_d=(D_b)_d\mathsf{S}_{c\subset b}[[(D_b)_{c\cap d}^{\dagger}A_{c\cap d}]^{-\Delta^{-1}(b,c)}]. \tag{6.41}$$

Let $(\alpha_b)_{c\cap d}$ denote the Cayley transform $(D_b)_{c\cap d}^{\dagger}A_{c\cap d}$. The Cayley transform of $\mathsf{S}_{c\subset b}[[(D_b)_{c\cap d}^{\dagger}A_{c\cap d}]^{-\Delta^{-1}(b,c)}]$ is

$$\sum_c \Delta^{-1}(b,c)(\alpha_b)_{c\cap d}=(\alpha_b)_{b\cap d}. \tag{6.42}$$

Inverting the Cayley transform gives

$$(D_b)_d^{\dagger}A_{b\cap d}, \tag{6.43}$$

which implies

$$(A_b)_d=(D_b)_d[(D_b)_d^{\dagger}A_{b\cap d}]=A_{b\cap d} \tag{6.44}$$

and the relations

$$A_bU_b(\Lambda,a)A_b^{\dagger}=\overline{U_b}(\Lambda,a) \tag{6.45}$$

and

$$[A_bU_b(\Lambda,a)A_b^{\dagger}]_c=A_{b\cap c}U_{b\cap c}(\Lambda,a)A_{b\cap c}^{\dagger}=\overline{U_{b\cap c}}(\Lambda,a). \tag{6.46}$$

Next define the scattering equivalence

$$A:=\mathsf{S}_{a\neq\underline{1}}[A_a^{\zeta_a}] \tag{6.47}$$

and the Bakamjian-Thomas mass operator

$$\overline{M}=\sum_{a\neq\underline{1}}\zeta_aA_a^{\dagger}M_aA_a, \tag{6.48}$$

where M_a is the mass operator associated with $U_a(\Lambda,a)$. This defines a Bakamjian-Thomas representation of the Poincaré group $\overline{U}(\Lambda,a)$. Given this representation define the n-body representation by

$$U(\Lambda,a):=A\overline{U}(\Lambda,a)A^{\dagger}. \tag{6.49}$$

It can be verified, following the methods of [7], that this construction defines a $U(\Lambda,a)$ satisfying algebraic clustering. It is also unitarily equivalent to a pure Bakamjian-Thomas representation, thus completing the induction.

In order to understand the relation of this general construction to the model of Sec. V we consider the structure of the operator A.

The complete restoration of cluster properties, using the scattering equivalences as outlined above, is complicated to implement in applications. To understand the relation of the model discussed in Sec. V to a model that satisfies cluster properties consider the cluster decomposition of the Cayley transform of A:

$$\alpha:=\sum_{a\neq\underline{1}}\zeta_a\alpha_a=\sum_{a\neq\underline{1}}[\alpha]_a, \tag{6.50}$$

where $[\alpha]_a$ is the a-connected part of α defined by

$$[\alpha]_a:=\sum_b\Delta_{ab}^{-1}\alpha_b. \tag{6.51}$$

Define the Hermitian operators

$$\alpha(k)=\sum_{\{a|n_a\geqslant n+1-k\}}[\alpha]_a \tag{6.52}$$

and the unitary operators

$$A(k):=A=\frac{1+i\alpha(k)}{1-i\alpha(k)}. \tag{6.53}$$

The operator $\alpha(k)$ has terms with at most k-particle correlations.

Also define the unitary operators

$$A(k{:}k-1):=A(k)A^{\dagger}(k-1). \tag{6.54}$$

These definitions lead to the following decomposition of the scattering equivalence A:

$$A=A(n{:}n-1)\cdots A(3{:}2)A(2). \tag{6.55}$$

The mass operator M for the model satisfying spacelike cluster properties can be expanded as

$$M=A\overline{M}A^{\dagger}=A(n{:}n-1)\cdots A(3{:}2)A(2)$$
$$\times\left[\sum_{a\neq 1}[\overline{M}]_a\right]A^{\dagger}(2)A^{\dagger}(3{:}2)\cdots A^{\dagger}(n{:}n-1)$$
$$=A(n{:}n-1)\cdots A(3{:}2)A(2)\left[\sum_{\{a|n_a\geqslant n-1\}}[\overline{M}]_a\right]A^{\dagger}(2)$$
$$\times A^{\dagger}(3{:}2)\cdots A^{\dagger}(n{:}n-1)+A(n{:}n-1)\cdots A(4{:}3)$$
$$\times A(3)\left[\sum_{\{a|n_a=n-2\}}[\overline{M}]_a\right]A^{\dagger}(3)A^{\dagger}(4{:}3)\cdots$$
$$\times A^{\dagger}(n{:}n-1)+\cdots A(n{:}n-1)A(n-1)$$
$$\times\left[\sum_{\{a|n_a=2\}}[\overline{M}]_a\right]A^{\dagger}(n-1)A^{\dagger}(n{:}n-1)$$
$$+A_n[\overline{M}]_{\underline{1}}A^{\dagger}(n). \tag{6.56}$$

In order to understand this decomposition note that the representation $U(\Lambda,a)$ clusters properly only in the limit that all interactions vanish. The operator $A(2)$ includes those terms necessary to restore cluster properties in the limit that all particles are separated except for a single pair of particles. The operator $A(3{:}2)$ adds the additional operators needed to ensure algebraic cluster properties when the system is broken up into $n-2$ or $n-1$ clusters. The corrections generated by $A(k,k-1)$ involve correlations associated with k cluster partitions. To the extent that two-body correlations are the dominant corrections needed to restore cluster properties $A\rightarrow A(2)$ should be a good approximation. Note that $A(2)$ still has many-body correlations. The point is that the role of these terms is to restore cluster properties only in the limit that all but two particles are asymptotically separated. Each successive correction restores cluster properties into a smaller number of partitions.

This suggests the following sequence of approximations. The first one,

$$M(2)=A(2)\left[\sum_{\{a|n_a\geqslant n-1\}}[\overline{M}]_a\right]A^{\dagger}(2), \tag{6.57}$$

clusters properly only when all but two particles are separated. It includes correlations needed for clustering into $n-1$ clusters. The operators $A(2)$ also generate many-body interactions that ensure that $M(2)$ and $\overline{M}(2):=A^{\dagger}(2)M(2)A(2)$ have the same spectrum and cross section. It does not cluster properly when separated into fewer than $n-1$ clusters. In the instant form case $M(2)$ is exactly the mass operator of the previous section. This exhibits precisely the relation of the model of the previous section and a model satisfying cluster properties. In the point- and instant form cases $M(2)=C(2)\hat{M}(2)C^{\dagger}$, where in this expression $\hat{M}$ is the mass operator of the previous section and C is the transformation of theorem 5.

The next approximation

$$M(3)=A(3{:}2)M(2)A(2)A^{\dagger}(3{:}2)+A(3)$$
$$\times\left[\sum_{n_a=3}[\overline{M}]_a\right]A^{\dagger}(3) \tag{6.58}$$

clusters properly when the system is separated into at least n-2 clusters. This differs from the previous approximation by the addition of many-body interactions involving at least three particles. These many-body interactions modify the spectrum and cross section of the previous approximation.

Similarly, successive approximations of the form

$$M(k)=A(k{:}k-1)M(k-1)A(k-1)A^{\dagger}(k{:}k-1)+A(k)$$
$$\times\left[\sum_{n_a=k}[\overline{k}]_a\right]A^{\dagger}(k) \tag{6.59}$$

modify the previous approximation. The new interactions ensure that the model clusters properly when the system is separated into at least n-k+1 subsystems.

This procedure can be continued. The last step includes the additional n-body interactions needed for the resulting model to satisfy full algebraic cluster properties.

These arguments show that the model of the previous section is the first step in a series of approximations that satisfy cluster properties into subsystems of at least n-k clusters by including additional $(k+1)$-body interactions. These approximations should be good when the many-body correlations needed to restore clustering become weaker as the number of correlated particles increases.

The approximation scheme provides a formal framework for including systematic corrections that restore cluster properties. Successive approximations add new many-body correlations involving more and more particles. The leading approximation is the pure Bakamjian-Thomas model discussed in Sec. IV. It is scattering equivalent to a model that only clusters properly when the system is broken up into at least $n-1$ clusters. On the other hand, even in the crudest approximations, both Poincaré invariance and relevant discrete symmetries remain exact.

VII. SUMMARY

In this paper induced representations of the Poincaré group are used to construct a basis of n-particle states that transform irreducibly with respect to the noninteracting representation of the Poincaré group. Interactions that are scattering equivalent to input two-body interactions and commute with the free particle spin operator are constructed. The interactions are combined to make a mass operator that commutes with the noninteracting spin. The problem of constructing an interacting representation of the Poincaré group is reduced to one of diagonalizing this mass operator on a subspace of fixed kinematic spin. The choice of basis leads to equations that have simple properties under exchange of identical properties and space reflection. Both the Poincaré invariance and discrete symmetries are easily preserved under the approximation. This formalism can be used as the basis for fully relativistic variational calculations of nuclear wave functions. The interactions are most naturally formulated in an instant-form representation but the resulting in-

stant form interactions can be transformed to any of Dirac's forms of the dynamics.

The model discussed is known to violate spacelike cluster propertes. The relationship of the model formulated to a model satisfying cluster properties is discussed. Cluster properties require that the two-body interactions in the N-body problem be identical to the two-body interactions in the two-body problem. Dynamically generated (3,4, $\ldots,N$)-body interactions are needed to restore the Poincaré commutation relations. In our model the interactions do not cluster properly, but each of the two-body interactions gives the same two-body phase shifts and binding energies as the original two-body interactions, with the additional property that they can be combined without violating the commutation relations. The analysis in Sec. VI shows that a model satisfying cluster properties can be obtained from the model of this paper by an alternating sequence of unitary transformations and the addition of (3,4, $\ldots,N$)-body interactons. This relationship implies that the model will be a good approximation to a model satisfying cluster properties when the Cayley transforms of the operators $A_{(1)\cdots(ij)\cdots(n)}$ defined by Eq. (6.5) are small.

APPENDIX A

Proof of theorem 2. For the instant form case let $\lambda:=B^0{}_0(p)$ and $\eta^i:=B^i{}_0(p)$. We have

$$0=\lim_{t\to\pm\infty}\|[\Phi-U^\dagger[B(p)]\Phi U[B(p)]]U_f(t)|\chi\rangle\|=\lim_{t\to\pm\infty}\|[U(-t)\Phi-U(-t)U^\dagger[B(p)]\Phi U[B(p)]]U_f(t)|\chi\rangle\|$$

$$=\lim_{t\to\pm\infty}\|[U(-t)\Phi U_f(t)-U^\dagger[B(p)]U(-\lambda t,-\vec{\eta}t)\Phi U_f(\lambda t,\vec{\eta}t)U_f[B(p)]]|\chi\rangle\|$$

$$=\lim_{t\to\pm\infty}\|[U(-t)\Phi U_f(t)-U^\dagger[B(p)]U(-\lambda t)\Phi U_f(\lambda t)U_f[B(p)]]|\chi\rangle\|,$$

where we have used $\Phi U_f(\vec{x})=U(\vec{x})\Phi$. Since λ is a positive constant, it can be absorbed into the definition of t as $t\to\pm\infty$. Taking the limits gives

$$\Omega_\pm(H,\Phi,H_f)|\chi\rangle=U^\dagger[B(p)]\Omega_\pm(H,\Phi,H_f)U_f[B(p)]|\chi\rangle. \tag{A1}$$

These relations coupled with the kinematic identities (2.62) imply Poincaré invariance of the wave operators.

For the point form we have

$$\lim_{t\to\pm\infty}\|[\Phi-U^\dagger(I,\vec{a})\Phi U_f(I,\vec{a})]U_f(t)|\chi\rangle\|=\lim_{t\to\pm\infty}\|[U(-t)\Phi-U(-t)U^\dagger(I,\vec{a})\Phi U_f(I,\vec{a})]U_f(t)|\chi\rangle\|$$

$$=\lim_{t\to\pm\infty}\|[U(-t)\Phi U_f(t)-U^\dagger(I,\vec{a})U(-t)\Phi U_f(t)U_f(I,\vec{a})]|\chi\rangle\|,$$

which immediately gives

$$\Omega_\pm(H,\Phi,h_f)|\chi\rangle=U^\dagger(I,\vec{a})\Omega_\pm U_f(I,\vec{a})|\chi\rangle, \tag{A2}$$

which can be combined with the kinematic relations (2.62) for Lorentz invariance to give Poincaré invariance of the wave operators.

The last case is the front form where

$$\lim_{t\to\pm\infty}\|[\Phi-U^\dagger(R,0)\Phi U_f(R,0)]U_f(I,t)|\chi\rangle\|=\lim_{t\to\pm\infty}\|[U(I,-t)\Phi U_f(I,t)-U(I,-t)U^\dagger(R,0)\Phi U_f(R,0)U_f(I,t)]|\chi\rangle\|$$

$$=\lim_{t\to\pm\infty}\|[U(I,-t)\Phi U_f(I,t)-U^\dagger(R,0)U(I,-t)\Phi U_f(I,t)U_f(R,0)]|\chi\rangle\|,$$

which gives

$$\Omega_\pm(H,\Phi,H_f)|\chi\rangle=U^\dagger(R,0)\Omega_\pm(H,\Phi,H_f)U_f(R,0)|\chi\rangle. \tag{A3}$$

Proof of theorem 5. Evaluate the instant-form operators in a basis of the form

$$|\vec{p},\mu,j,d\rangle, \tag{A4}$$

where $\vec{p}$ is the kinematic linear momentum, μ is the magnetic quantum number associated with the canonical spin, j is the canonical spin, and d denotes a collection of kinematically invariant degeneracy parameters.

An instant form Bakamjian-Thomas mass operator in this representation necessarily has the form

$$\langle\vec{p},\mu,j,d|M|\vec{p}',\mu',j',d'\rangle=\delta^3(\vec{p}-\vec{p}')\,\delta_{\mu\mu'}\delta_{jj'}\langle d\|M(j)\|d'\rangle. \tag{A5}$$

The spectrum and cross sections are determined by the reduced matrix element $\langle d\|M(j)\|d'\rangle$. The basis $|\vec{p},\mu,j,d\rangle$ is related to the bases $|p^+,\vec{p}_\perp,\mu,j,d\rangle$ and $|\vec{v},\mu,j,d\rangle$ by the unitary transformations defined by the kernels

$$\langle \vec{p},\mu,j,d|p'^+,\vec{p}'_\perp,\mu',j',d'\rangle = \delta(p^+ - \sqrt{m(d')^2+\vec{p}'^2} - p'^3)\,\delta^2(\vec{p}_\perp - \vec{p}'_\perp)\,\sqrt{\frac{p^+}{\omega_{m(d)}(\vec{p})}}\,\delta_{jj'}\delta_{dd'}D^j_{\mu\mu'}[R_{cf}(p',d')] \quad \text{(A6)}$$

and

$$\langle \vec{p},\mu,j,d|\vec{v}',\mu',j',d'\rangle = \delta(\vec{p} - m(d')\vec{v})|m(d)|^{3/2}\delta_{jj'}\delta_{dd'}\delta_{\mu\mu'}. \quad \text{(A7)}$$

The rotation $[R_{cf}(p',d')]$ in Eq. (A6) is a Melosh rotation that transforms the front form spin into the canonical spin.

It follows that the interactions M_{F} and M_{L} defined by the kernels

$$\langle p^+,\vec{p}_\perp,\mu,j,d|M_{\mathsf{F}}|p'^+,\vec{p}'_\perp,\mu',j',d'\rangle = \delta(p^+ - p'^+)\,\delta^2(\vec{p}_\perp - \vec{p}'_\perp)\,\delta_{\mu\mu'}\delta_{jj'}\langle d\|M(j)\|d'\rangle \quad \text{(A8)}$$

and

$$\langle \vec{v},\mu,j,d|M_{\mathsf{L}}|\vec{v}',\mu',j',d'\rangle = \delta^3(\vec{v} - \vec{v}')\,\delta_{\mu\mu'}\delta_{jj'}\langle d\|M(j)\|d'\rangle \quad \text{(A9)}$$

have the same reduced matrix elements and thus the same spectrum and scattering matrix elements. The desired scattering equivalencies have the general form

$$A := \Omega_\pm(H_{\mathsf{F}},\Phi_{\mathsf{F}},H_f)B_{\mathsf{FE}}\Omega^\dagger_\pm(H,\Phi,H_f), \quad \text{(A10)}$$

$$A := \Omega_\pm(H_{\mathsf{L}},\Phi_{\mathsf{P}},H_f)B_{\mathsf{LE}}\Omega^\dagger_\pm(H,\Phi,H_f), \quad \text{(A11)}$$

where B_{FE} and B_{LE} are the kinematic unitary transformations that change the $\vec{p},\mu$ to either the light front components of the four-momentum and magnetic quantum number associated with the light front spin or the vector components of the four-velocity and the magnetic quantum number associated with the canonical spin.

These transformations can be explicitly given in terms of eigenfunction expansions

$$A = \sum \int |p^+,\vec{p}_\perp,\mu,d^\pm;\mathsf{F}\rangle D^j_{\mu\mu'}[R_{fc}(p,m(d))]d^3p\,\sqrt{\frac{p^+}{\omega_{m(d)}(\vec{p})}}\langle \vec{p}mu',d^\pm;\mathsf{E}| \quad \text{(A12)}$$

or

$$A = \sum \int |p^+,\vec{p}_\perp,\mu,d^\pm;\mathsf{L}\rangle d^3p\,m(d)^{-3/2}\langle \vec{v}mu,d^\pm\mathsf{E}|, \quad \text{(A13)}$$

where $d^\pm$ is a label for all bound state and scattering degeneracy labels and $m(d)$ is the mass eigenvalue when the degeneracy quantum numbers have the value d. The invariance of the scattering operator ensures that this expression has a value independent of the choice of asymptotic condition. The notation F, L, and E indicates the kinematic subgroup of the relevant mass operator.

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