Nonexistence of the Oppenheimer-Phillips process in low-energy deuteron-nucleus collisions

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It is shown that the electric polarizability of the deuteron produces negligible effect on the cross section of deuteron induced rearrangement reactions even at extremely low energies. This assessment is based on simple analytical formulas, derived on the basis of N-particle scattering theory by means of the general two-potential formalism, including Coulomb and exchange effects. It is shown on the basis of general physical arguments that the polarizability effects at very low energies are entirely contained in a multiplicative enhancement factor that differs from 1 by at most a few percent. As a result enhancement of (d,p) relative to (d,n) reactions is not possible by the Oppenheimer-Phillips mechanism.

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I. INTRODUCTION

The loosely bound deuteron is easily polarized by a strong electric field. As Oppenheimer and Phillips noted long ago [1], the effective polarization of the deuteron gives rise to a P-wave component in the deuteron internal wave function. They argued that the degree of polarization depends strongly on the binding energy via the adiabatic polarizability of the bound state and enhances the cross section of (d,p) reactions as compared to (d,n) cross sections. The existence of this so-called Oppenheimer-Phillips (OP) process was not fundamentally questioned until recently when Koonin and Mukherjee [2] and Austern [3] advanced arguments based on three-body models that polarizability forces cause small distortions in the initial wave function but otherwise have negligible effects in inelastic reactions. It is the purpose of the present paper to strengthen this conclusion by offering more general arguments.

Early estimates of the effect of deuteron polarizability predicted, on the basis of adiabatic polarizability and considerations equivalent to a first order Born approximation, a deviation of the elastic from the Rutherford cross section of only a few percent for heavy nuclei and for energies near the Coulomb barrier [4,5]. Subsequently Clement [6] improved on the adiabatic approximation and by an extensive numerical evaluation of the second order Born approximation, which in addition to the virtual breakup processes takes into account the real breakup, found a large effect on the elastic scattering cross section. In order to resolve the contradiction Dickens and Perey included the dipole part of the adiabatic polarizability potential in an optical model code and numerically solved the Schrödinger equation [7]. Their results confirmed the early estimates [4,5], and showed that polarizability effects were indeed very small in elastic scattering. Later on it was found by Clement himself that the large effects obtained previously were due to numerical errors in his code.

Subsequent work established, on the basis of a three-body model, a number of properties of the multipole components of the polarizability potential [8] and further confirmed that the long-range polarizability contribution to the deuteron optical potential produces effects of almost negligible importance in elastic scattering [9].

Deuteron polarizability came again into focus because of a controversy about p-d scattering lengths obtained by independent three-body calculations [10,11]. The difference between the results was attributed to the fact that in the presence of inverse power potentials, such as the dipole part of the polarizability potential, the usual Coulomb-modified effective-range theory breaks down and gives an infinite scattering length, as was pointed out long ago [12]. This led to the elaboration of a modified effective-range theory that yields finite scattering lengths even in the presence of long-range polarization forces [13,14]. It was also shown that experimental data can be extrapolated in a stable way to zero energy and yield precisely the new modified scattering length [15].

The latest revival of interest in deuteron polarizability was the result of "cold fusion" experiments and the OP process was invoked to explain the suppression of neutron production reactions in fusion reactions involving deuterons [16,17]. However, Koonin and Mukherjee [2], using a three-body model and a second order Born approximation, and Austern [3], using a more sophisticated approximation to the same three-body model, showed with detailed numerical calculations that the OP process cannot provide the suppression required by the experimental claims [17]. Similar conclusions were reached by Bencze [18] on the basis of the optical theorem and a general theory of long-range modified scattering lengths. Finally, it should also be mentioned that more recent measurements at very low energies do not find any appreciable enhancement of (d,p) relative to (d,n) cross sections [19].

The basic problem is that the physical picture on which the model of Oppenheimer and Phillips is based is not consistent with the modern understanding of the dynamics of the collision process. In the adiabatic picture the Coulomb field of the nucleus induces a dipole moment in the deuteron internal state and the dipole is oriented in such a way that the

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constituent proton is located at a larger distance from the nucleus. Accordingly, the capture of the neutron by the nucleus should have a higher probability and the (d,p) reaction cross sections should be enhanced in comparison with (d,n) cross sections.

A full dynamical treatment of the deuteron-nucleus collision dictates a completely different physical picture. As a result of the polarizability of the deuteron by the Coulomb field an attractive secondary interaction — the so-called polarizability potential — is generated in the elastic channel that has a long-range local tail. The dipole part of this potential decreases as the inverse fourth power of the distance, thus giving mostly a long-range contribution to the deuteron optical potential. Since no permanent electric dipole moment can be induced, considerations based on the orientation of the internal state are totally misleading. Whatever the effects are, they show up only in the breakup channel component of the three-body wave function. At very low energies, i.e., well below the breakup threshold, this component is asymptotically rapidly vanishing and its contribution to the various reaction amplitudes is negligible.

In the present work a very simple estimate of the effects of deuteron polarizability is derived by evaluating the enhancement of the penetrability due to long-range polarizability forces. In Sec. II it is shown on the basis of N-particle scattering theory and some rather general physical considerations that for every reaction amplitude all polarizability effects are contained in a multiplicative factor that depends only on the initial channel of the reation. In Sec. III an approximate analytical expression is obtained for that factor and found to be accurate to a relative error of a few percent over the entire periodic table. The numerical results show that the polarizability effect is so small, a few percent, that one can in fact claim the Oppenheimer-Phillips process to be nonexistent. A concluding discussion is found in Sec. IV, and an appendix contains some technical error estimates.

II. POLARIZABILITY ENHANCEMENT FACTOR

In the previous section it was established that in low-energy deuteron-induced nuclear reactions one can account for the polarizability of the deuteron by including a polarizability potential in the asymptotic analysis of the initial deuteron-nucleus channel. This is efficiently done within an N-body two-potential formalism [20] and yields the expression

$$\mathscr{A}_{\beta n:\alpha m}(\mathbf{q}_\beta;\mathbf{q}_\alpha) = N_\alpha^{1/2}\langle \Psi_{\beta n}^{\mathscr{S}(-)}(\mathbf{q}_\beta)|V^\alpha + V_c^\alpha - W^\alpha|\alpha m \chi_\alpha^{(+)}(\mathbf{q}_\alpha)\rangle \tag{2.1}$$

for the transition amplitude from the initial deuteron-nucleus channel to some final inelastic channel. In Eq. (2.1) the final channel is represented by a fixed, but arbitrary, representative of a set of physically indistinguishable channels that differ only by interchange of neutrons or of protons. The indices β, n, and $\mathbf{q}_\beta$ refer, respectively, to the partitioning of the nucleons into the bound clusters of the channel, the set of quantum numbers of the clusters, and the Jacobi momenta of relative motion of the clusters. Similar considerations and notation apply to the initial deuteron-nucleus channel, where N_α denotes the number of distinct members of the set of physically indistinguishable initial channels. The state $|\Psi_{\beta n}^{\mathscr{S}(-)}(\mathbf{q}_\beta)\rangle$ represents the correctly antisymmetrized and normalized solution to the N-body Schrödinger equation that has the incoming scattering boundary conditions appropriate to the final channel [21,22]. The symbols V^α and V_c^α denote, respectively, the sums of the strong nuclear and the Coulomb interactions acting between the nucleons of the different fragments. The two-body potential

$$W^\alpha(r) = \frac{\lambda}{r} + U(r) \tag{2.2}$$

is the sum of the asymptotic Coulomb potential and the polarizability potential $U(r)$ acting between the deuteron and the target nucleus with $\lambda = 2MZe^2A/(A+2)\hbar^2$. Here Z and A denote the charge and mass numbers of the target nucleus, and M denotes the nucleon mass. Finally, $|\alpha m\chi_\alpha^{(+)}(\mathbf{q}_\alpha)\rangle = |\alpha m\rangle \otimes |\chi_\alpha^{(+)}(\mathbf{q}_\alpha)\rangle$, where $|\alpha m\rangle$ denotes the product of the bound states of the deuteron and target nucleus, and $|\chi_\alpha^{(+)}(\mathbf{q}_\alpha)\rangle$ is a distorted wave state (with outgoing boundary conditions) describing the relative motion of the deuteron and the target nucleus under the influence of W^α.

It is generally assumed that the main effect of the Coulomb interaction between the fragments is adequately represented by the asymptotic two-body Coulomb potential and the polarizability potential U. The residual interaction V_c^α-W^α is negligible in the asymptotic channel region, while at short distances it can be neglected in comparison with the strong nuclear interactions V^α. Thus one can write to a good accuracy

$$\mathscr{A}_{\beta n:\alpha m}(\mathbf{q}_\beta;\mathbf{q}_\alpha) \approx \int d\mathbf{r} N_\alpha^{1/2}\langle \Psi_{\beta n}^{\mathscr{S}(-)}(\mathbf{q}_\beta)|V^\alpha|\alpha m\mathbf{r}\rangle\langle \mathbf{r}|\chi_\alpha^{(+)}(\mathbf{q}_\alpha)\rangle. \tag{2.3}$$

At very low energies the wave function in the incoming channel is dominated by the S-wave component so that the scattering amplitude can be further simplified:

$$\mathscr{A}_{\beta n:\alpha m}(\mathbf{q}_\beta;\mathbf{q}_\alpha) \approx \int d\mathbf{r} N_\alpha^{1/2}\langle \Psi_{\beta n}^{\mathscr{S}(-)}(\mathbf{q}_\beta)|V^\alpha|\alpha m r \ell \mu\rangle \bigg|_{\ell=0,\mu=0} f^{(+)}(|\mathbf{q}_\alpha|,r). \tag{2.4}$$

In Eq. (2.4) the radial wave function can be written as $f^{(+)}(k,r)=N(k)F(k,r)$, where $F(k,r)$ is determined by the S-wave Volterra-type integral equation [23,24]

$$F(k,r)=[1+A(k,r)]F^C(k,r)-B(k,r)G^C(k,r), \tag{2.5}$$

$$A(k,r)=k^{-1}\int_0^r ds G^C(k,s)U(s)F(k,s), \tag{2.6}$$

$$B(k,r)=k^{-1}\int_0^r ds F^C(k,s)U(s)F(k,s), \tag{2.7}$$

with F^C and G^C denoting the regular and irregular S-wave Coulomb functions, respectively. The normalization factor $N(k)$ is uniquely determined by the asymptotic boundary condition

$$f^{(+)}(k,r)\to_{r\to\infty}\sin[\phi^C(k,r)+\delta^{C,p}], \tag{2.8}$$

where $\delta^{C,p}$ is the Coulomb-modified phase shift of the polarizability potential U and $\phi^C(k,r)$ is the Coulomb S-wave phase function,

$$\phi^C(k,r)=kr-(\lambda/2k)\ln(2kr)+\arg\Gamma(1+i\lambda/2k). \tag{2.9}$$

Substituting Eq. (2.5) into Eq. (2.8) and using the known asymptotic properties of the Coulomb functions yields

$$N(k)=\{[1+A(k,\infty)]^2+[B(k,\infty)]^2\}^{-1/2}. \tag{2.10}$$

The polarizability potential U represents only the long-range polarizability effects [9]; i.e., it describes only the long-range tail of the deuteron optical potential. At short distances its effect cannot be uniquely isolated and should be combined with the short-range nuclear potential. Hence the local representation of U can be considered to vanish inside some radius R that is large on the nuclear scale,

$$U(r)=0 \quad \text{for} \quad r\leqslant R. \tag{2.11}$$

Consequently, $f^{(+)}(k,r)=N(k)F^C(k,r)$ for $r\leqslant R$. On the other hand, the short range of the nuclear forces restricts the integration in Eq. (2.4) to the nuclear interior. As a consequence one can write

$$\mathscr{A}_{\beta n:\alpha m}(\mathbf{q}_\beta;\mathbf{q}_\alpha)\approx N(k)\int d\mathbf{r} N_\alpha^{1/2}\langle\Psi_{\beta n}^{\mathscr{S}(-)}(\mathbf{q}_\beta)|V^\alpha|\alpha m r \ell \mu\rangle\Big|_{\ell=0,\mu=0} F^C(|\mathbf{q}_\alpha|,r). \tag{2.12}$$

The isolation of polarizability effects in a multiplicative factor $N(k)$ that depends only on the initial channel is, of course, not surprising on physical grounds. However, it has noteworthy consequences. It follows trivially that the ratio of reaction cross sections with $(d\sigma_{\beta\alpha}/d\Omega_\beta)$ and without polarizability effects $(d\tilde{\sigma}_{\beta\alpha}/d\Omega_\beta)$ is independent of the particular final asymptotic channel,

$$\frac{d\sigma_{\beta\alpha}/d\Omega_\beta}{d\tilde{\sigma}_{\beta\alpha}/d\Omega_\beta}\equiv P(k)=N^2(k)=\{[1+A(k,\infty)]^2+[B(k,\infty)]^2\}^{-1}. \tag{2.13}$$

The *polarizability enhancement factor* $P(k)$ is the multiplicative modification to the Coulomb penetrability needed to describe the low-energy effects of the deuteron polarizability.

III. EVALUATION OF THE POLARIZABILITY ENHANCEMENT

In order to determine the polarizability factor $P(k)$ a specific form has to be chosen for the potential U and the integrals $A(k,\infty)$ and $B(k,\infty)$ need to be evaluated numerically.

A relatively simple estimate can be obtained at low energies, however, for which the value of $P(k)$ at zero energy provides a reliable measure of the importance of polarizability effects. It is known [14] that $B(0,\infty)=0$ and that

$$A(0,\infty)=\int_0^\infty ds G^C(s)U(s)F(r), \tag{3.1}$$

where $F(r)$ is determined by the zero-energy Volterra equations

$$F(r)=[1+A(r)]F^C(r)-B(r)G^C(r), \tag{3.2}$$

$$A(r)=\int_0^r ds G^C(r)U(s)F(s), \tag{3.3}$$

$$B(r)=\int_0^r ds F^C(r)U(s)F(s). \tag{3.4}$$

The zero-energy S-wave Coulomb functions can be expressed in terms of modified Bessel functions by [14]

$$F^C(r)=\sqrt{r/\lambda}I_1(2\sqrt{\lambda r}) \quad \text{and} \quad G^C(r)=2\sqrt{\lambda r}K_1(2\sqrt{\lambda r}). \tag{3.5}$$

The zero-energy polarizability enhancement factor is thus given by

$$P(0)=|1+A(0,\infty)|^{-2}. \tag{3.6}$$

A good analytical estimate of the integral $A(0,\infty)$ can be obtained by the Born approximation in which $F(r)$ is replaced in the integrand by $F^C(r)$. This approximation, usually reserved for high energies or high angular momenta, is justified in the Appendix.

For the explicit form of the polarization potential only the long-range tail is important, since effects due to the short-

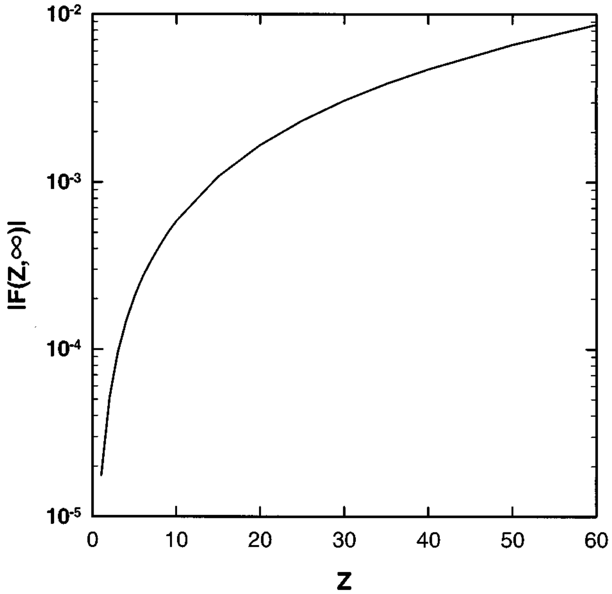

FIG. 1. The dependence of $F(Z,\infty)$ on the nuclear charge Z.

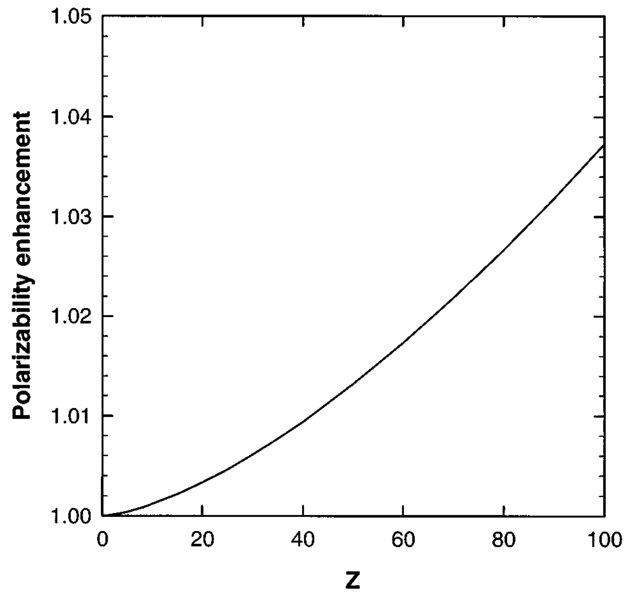

FIG. 2. The dependence of the polarizability enhancement $P=1-2F(Z,\infty)$ on the nuclear charge Z for infinitely heavy nuclei.

range part can be included into those produced by the nuclear interactions. This allows a lower cutoff form to be used [14],

$$U(r)=\begin{cases} 0 & \text{for } r<R, \\ -(\alpha Z\lambda/2)r^{-4} & \text{for } r\geqslant R. \end{cases} \tag{3.7}$$

Evaluating $A(0,\infty)$ for this potential in the first Born approximation yields

$$A(0,\infty)\approx -F(Z,A), \tag{3.8}$$

where

$$F(Z,A)=8\alpha\lambda^3 Z\int_X^\infty ds\,\frac{I_1(s)K_1(s)}{s^5},$$

$$=2.1632\times 10^{-2}\alpha Z^4 \tag{3.9}$$

$$\times\left(\frac{A}{A+2}\right)^3\int_X^\infty ds\,\frac{I_1(s)K_1(s)}{s^5}, \tag{3.10}$$

where $X=2(\lambda R)^{1/2}$.

The case of an infinitely heavy nucleus provides an upper limit to the function $F(Z,A)$. Taking the numerical values $R=20$ fm and $\alpha=0.632$ fm^3 allows the numerical evaluation of that function with the aid of the standard integration and Bessel function routines of Mathematica. As a function of the upper limit of integration the integral converges very rapidly and by 500 an accuracy of five digits is obtained. The value of the integral is insensitive to the choice of the lower cutoff radius R as long as R is larger than the nuclear radius. The results of the calculations are shown in Fig. 1.

By combining Eq. (3.6) with Eq. (3.8) one can express the polarizability enhancement factor of any deuteron induced reaction as

$$P(0)\approx|1-F(Z,A)|^{-2}. \tag{3.11}$$

Since the numerical calculations show that $F(Z,A)$ is small, Eq. (3.11) can be expanded into a power series around unity and one can write to a very good approximation

$$P(0)\approx 1+2F(Z,A). \tag{3.12}$$

The integral in $F(Z,A)$ can be approximated by an analytical expression if the asymptotic form of the modified Bessel functions [25] is used in the integrand of Eq. (3.10). The leading terms in the integral can then be expressed in terms of inverse power functions to yield the approximate analytical form

$$P(0)\approx 1+3.730\times 10^{-5}Z^{3/2}\left(\frac{A}{A+2}\right)^{1/2} - 1.793\times 10^{-6}Z^{1/2}\left(\frac{A}{A+2}\right)^{-1} - 2.346\times 10^{-7}Z^{-1/2}\left(\frac{A}{A+2}\right)^{-3/2}. \tag{3.13}$$

The approximate expression turns out to be accurate to three digits throughout the periodic table.

The polarization enhancement is shown in Fig. 2 for the case of an infinitely heavy nucleus. As can be seen, a 3% percent increase of the cross section is reached only in the transuranium region, where the exponentially decreasing penetration factor reduces the cross section to practically zero.

IV. DISCUSSION

The considerations of the present work are based on exact multiparticle scattering theory and the use of the powerful two-potential formalism [20]. As a result exchange effects can be fully taken into account without limiting the validity of the conclusions reached [21,22]. The exact two-potential formalism leads to the following physical picture of the deuteron induced reactions at very low energies.

At very low energies and at large distances the motion of the deuteron in the Coulomb field of the nucleus is described by an effective three-body wave function for the neutron, the proton, and the nucleus considered as a point charge. The projection of this wave function onto the deuteron-nucleus two-cluster channel gives a distorted wave that can be generated by the two-body channel Coulomb interaction acting between the center of mass of the deuteron and the point nucleus and by the long-range tail of the polarizability potential. This is the first stage of the two-potential picture. By including the polarizability potential into the dynamics of the asymptotic channel in fact the boundary conditions are redefined. The next stage of the formalism introduces the nuclear interactions as well as the residual electromagnetic interactions in the reaction amplitude with the distorted wave for the initial channel. The considerations up to this point are exact and lead to the amplitude in Eq. (2.1). In actual calculations certain approximations have to be invoked, as described in Sec. II. However, it is important to emphasize that, due to the methods used, polarizability effects can now be estimated without performing the lengthy and detailed numerical calculations previously required [2,3].

The effect of deuteron polarizability on the various deuteron-induced reaction cross sections has been seen to be negligible at very low energies. Since the polarizability enhancement is accounted for by a multiplicative factor, polarizability effects cannot change the ratios of certain cross sections and in particular cannot cause enhancement or suppression of certain special processes as has been predicted by Oppenheimer and Phillips [1]. This result is in conflict with the data of Ref. [17] which reported a considerable enhancement of (d,p) reactions as compared to (d,n) reactions on a lithium target at the astrophysical energy of 60 keV.

Recently deuteron polarizability was also thought to have an important effect on the astrophysical factor of certain fusion-type reactions extracted from experimental data [26,27]. However, subsequent study showed that at astrophysical energies atomic or molecular screening effects are far more important and indeed must be taken into account in order to explain the experimental data [28,29]. Thus at low energies deuteron polarizability effects are negligible, or in other words the Oppenheimer-Phillips process seems to be ineffective and not able to explain the suppression of neutron-producing reactions in "cold-fusion"-type experiments [16,17]. It is important to realize, however, that even if polarization effects were several orders of magnitude larger, the drastic decrease of the cross sections due to the Coulomb penetration factor would still prevent any appreciable enhancement to be noticed.

Another region of interest where polarizability effects were thought to have important effects on the reaction dynamics was the case of the collision of deuterons with heavy nuclei at energies below the Coulomb barrier [6]. Optical model calculations with an exact account for the polarizability potential by Dickens and Perey [7] on the other hand put an end to such suggestions.

While our present considerations show the negligible effect of deuteron polarizability at astrophysical energies, the basic dynamical picture of the deuteron-nucleus collision process [2,9] strongly suggests that in a proper dynamical treatment the Oppenheimer-Phillips process does not exist at all, a conclusion also supported by recent experimental data [19].

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APPENDIX

In this appendix we justify the approximation of the zero-energy S-wave integral

$$A \equiv \int_0^\infty ds G^C(s) U(s) F(s) \tag{A1}$$

by its Born approximation

$$A_B \equiv \int_0^\infty ds G^C(s) U(s) F^C(s). \tag{A2}$$

This will be done by adapting the methods of the appendixes of Ref. [14], which require that the potential U satisfy the condition

$$\int_0^\infty ds s |U(s)| < \infty. \tag{A3}$$

The function F is defined by the integral equation

$$F(r) = [1 + A(r)] F^C(r) - B(r) G^C(r), \tag{A4}$$

in which

$$A(r) \equiv \int_0^r ds G^C(s) U(s) F(s), \tag{A5}$$

$$B(r) \equiv \int_0^r ds F^C(s) U(s) F(s). \tag{A6}$$

The functions F^C and G^C are the zero-energy Coulomb wave functions,

$$F^C(r) = \sqrt{r/\lambda} I_1(2\sqrt{\lambda r}), \tag{A7}$$

$$G^C(r) = 2\sqrt{\lambda r} K_1(2\sqrt{\lambda r}), \tag{A8}$$

where λ is defined immediately following Eq. (2.2) and I_1 and K_1 are standard modified Bessel functions. It will turn out to be important that F^C is a positive monotonically increasing function and that G^C is a positive monotonically decreasing function.

To derive the necessary error estimates it is convenient to define the functions

$$a(r)\equiv\int_0^r ds|G^C(s)U(s)[F(s)-F^C(s)]|, \tag{A9}$$

$$b(r)\equiv\int_0^r ds|F^C(s)U(s)[F(s)-F^C(s)]|, \tag{A10}$$

$$a_B(r)\equiv\int_0^r ds|F^C(s)||G^C(s)||U(s)|, \tag{A11}$$

$$b_B(r)\equiv\int_0^r ds|F^C(s)|^2|U(s)|. \tag{A12}$$

All these are differentiable, positive, and monotonically increasing functions of r.

It is straightforward to use Eq. (A4) to show that

$$b'(r)\leqslant|F^C(r)U(r)I_B(r)|+a(r)b_B'(r)+b(r)a_B'(r), \tag{A13}$$

where the prime denotes differentiation with respect to r and

$$I_B(r)=\int_0^r dsF^C(s)U(s)[G^C(s)F^C(r)-F^C(s)G^C(r)]. \tag{A14}$$

Equation (A13) is readily integrated to yield

$$b(r)\leqslant e^{a_B(r)}\int_0^r dse^{-a_B(s)}[|F^C(s)U(s)I_B(s)|+a(s)b_B'(s)]. \tag{A15}$$

Similar manipulations lead to

$$a'(r)\leqslant|G^C(r)U(r)I_B(r)|+a(r)a_B'(r)+b(r)|G^C(r)|^2|U(u)|. \tag{A16}$$

It follows from Eq. (A14) and the triangle inequality that

$$|G^C(r)U(r)I_B(r)|\leqslant a_B(r)a_B'(r)+\int_0^r ds|F^C(s)U(s)G^C(r)||F^C(s)U(r)G^C(r)|. \tag{A17}$$

Because of the monotonicity properties of F^C and G^C, the right-hand side of Eq. (A17) is increased if $G^C(r)$ is replaced by $G^C(s)$ in the first factor of the integrand and if $F^C(s)$ is replaced by $F^C(r)$ in the second factor. This yields

$$|G^C(r)U(r)I_B(r)|\leqslant 2a_B(r)a_B'(r). \tag{A18}$$

Similar exploitation of the monotonicity properties leads to

$$|G^C(r)|^2|U(u)|b(r)\leqslant a_B'(r)e^{a_B(r)}\int_0^r dse^{-a_B(s)}[|G^C(s)U(s)I_B(s)|+a(s)a_B'(s)], \tag{A19}$$

$$\leqslant a_B'(r)e^{a_B(r)}\int_0^r dse^{-a_B(s)}a_B'(s)[2a_B(s)+a(s)]. \tag{A20}$$

Since a_B and a are both monotonically increasing functions, this last inequality leads immediately to

$$|G^C(r)|^2|U(u)|b(r)\leqslant a_B'(r)(e^{a_B(r)}-1)[2a_B(r)+a(r)]. \tag{A21}$$

Substituting Eqs. (A18) and (A21) into Eq. (A16) yields

$$a'(r)\leqslant a_B'(r)e^{a_B(r)}[2a_B(r)+a(r)]. \tag{A22}$$

The integrating factor $W(r)=\exp[-1+\exp a_B(r)]$ allows one to integrate Eq. (A22), with the result

$$a(r)\leqslant 2W(r)\int_0^r dsW^{-1}(s)e^{a_B(s)}a_B(s)a_B'(s), \tag{A23}$$

$$\leqslant a_B(r)W(r)\int_0^r dsW^{-1}(s)e^{a_B(s)}a_B'(s), \tag{A24}$$

$$\leqslant 2a_B(r)[W(r)-1]. \tag{A25}$$

Combining the definitions of A and A_B with Eq. (A25) yields the final inequality

$$|A-A_B|\leqslant a(\infty)\leqslant a_B(\infty)[W(\infty)-1]. \tag{A26}$$

It is evident from Eq. (A26) that the Born approximation is accurate if $a_B(\infty)$ is sufficiently small. This is indeed the case for the potential $U(r)$ of Eq. (3.7), for which one has

$$-A_B=a_B(\infty)=F(Z,A), \tag{A27}$$

where $F(Z,A)$ is defined in Eq. (3.10). Numerical evaluation of $F(Z,\infty)$ shows that the Born approximation is accurate to better than 4% for all $Z\leqslant 100$.

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