Nucleon–α-particle interactions from inversion of scattering phase shifts

N. Alexander, K. Amos, B. Apagyi,* and D. R. Lun
School of Physics, University of Melbourne, Parkville 3052, Victoria, Australia
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Scattering amplitudes have been extracted from (elastic scattering) neutron-alpha (n-α) differential cross sections below threshold using the constraint that the scattering function is unitary. Real phase shifts have been obtained therefrom. A modification to the Newton iteration method has been used to solve the nonlinear equation that specifies the phase of the scattering amplitude in terms of the complete (0 to 180°) cross section since the condition for a unique and convergent solution by an exact iterated fixed point method, the ''Martin'' condition, is not satisfied. The results compare well with those found using standard optical model search procedures. Those optical model phase shifts, from both n-α and p-α (proton-alpha) calculations in which spin-orbit effects were included, were used in the second phase of this study, namely to determine the scattering potentials by inversion of that phase shift data. A modified Newton-Sabatier scheme to solve the inverse scattering problem has been used to obtain inversion potentials (both central and spin orbit) for nucleon energies in the range 1 to 24 MeV. The inversion interactions differ noticeably from the Woods-Saxon forms used to give the input phase shifts. Not only do those inversion potentials when used in Schrödinger equations reproduce the starting phase shifts but they are also very smooth, decay rapidly, and are as feasible as the optical model potentials of others to be the local form for interactions deduced by folding realistic two-nucleon g matrices with the density matrix elements of the α particle.

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I. INTRODUCTION

The nucleon-alpha (N-α) scattering system below threshold is of interest given the simplicity one assumes with the target structure. The closed s-shell configuration one favors to describe the alpha particle encourages belief that the inherent five-body problem can be assessed in diverse ways, ranging from its representation by effective real local and phenomenological optical model potentials, through treatment as a microscopic optical model interaction defined by folding appropriate effective two-nucleon (NN) g matrices with the density matrix elements of the alpha particle, to its analysis with Yakubovskii-Faddeev equations. One seeks the best representation of the N-α interaction also because there is evidence that the $A=6$ nuclei are well described by the three-body ($\alpha+2n$) system [1].

Simple functional forms for the N-α interaction have been chosen in most cases and Gaussian form factors have been used commonly to facilitate application. While one might expect attractive interactions to be chosen as Dubovichenko *et al.* [2] have done, Danilin *et al.* [1] used a repulsive potential parametrized to fit the low energy (to 25 MeV) N-α scattering phase shifts. But a better effective local N-α interaction is needed given the results found to date with mass-6 calculations. A candidate is the conventional optical model potential form and a comprehensive study with that of fits to the n-α and p-α cross sections and analyzing powers at low energies (1–20 MeV) and below threshold was made by Satchler *et al.* [3]. They noted that the dominant s- and p-wave phase shifts need be supplemented by small d-wave values to get good fits to measured data, but that the exact values were not well determined with their calculations. Unlike a previous attempt at a global analysis [4], that study presumed an interaction essentially independent of the energy. One concern with these approaches, however, is that true nonlocality has been ignored. It is well known that folding a two-nucleon interaction, even one chosen to be local in form, with the density matrix elements of the alpha particle leads to a nonlocal N-α interaction [5], and one for which the range of the nonlocality is close to being the same as the effective radius of the complete interaction. As a consequence, derivation of equivalent local potentials was found to be delicate [5] but the result was always an energy-dependent interaction. Lassaut and Vinh Mau also noted that medium effects upon the NN g matrices (which are actually required in the folding model) were influential upon the result, as was the choice of NN interaction. Best results were found with NN interactions that gave good saturation properties. It has been noted [6] also that the the effects of nonlocality upon the wave functions may be like those of a repulsive local potential. Thus we anticipate that the most appropriate form of an equivalent local N-α interaction is one that is energy dependent and perhaps with some repulsive character.

All ''direct'' studies of scattering, however, begin with assumptions about the form of the phenomenological optical potentials or of the NN interactions that are to be folded to define the N-α (nonlocal) interaction. Often those chosen model forms involve many variable parameters, and in the case of the phenomenological optical model potentials, ambiguities exist with regards to the parameter sets of both discrete (different parameter sets giving equivalent fits to measured data) and continuous type. The well-known Vr^2 link is an example of the latter. Inverse scattering theory [7] offers an alternative approach. Therewith one begins with scattering functions that fit the data of interest and, if neces-

* Permanent address: Department of Physics, Technical University of Budapest, Budafoki út 8, Budapest, Hungary.

sary, map them to a utilitarian functional form. Then the equations that define the (local) potential via inversion are to be solved. The potentials that result are linked to the inverse scattering method used and questions of uniqueness arise, but there is no essential *a priori* assumption made as to the shape of the interaction that results. Inverse scattering studies made so far belong in one of two groupings, identified according that the energy or angular momentum is considered as the spectral parameter. The Marchenko inversion method [7] (fixed angular momentum) has been used recently [8] in an analysis of the s-wave scattering with the n-α system to define an energy-independent local interaction. But that study dealt with all energies and particularly concerned the phase shift values above the first threshold. Those authors did note though that Pauli-forbidden bound states may exist, and certainly do with the phenomenological potentials [3], but that those forbidden states could be removed under a supersymmetric transformation giving a phase-shift-equivalent potential. That ultimate potential was repulsive and similar in shape to that chosen by Danilin *et al.* [1]. The other class of inversion methods has fixed energy. Such are appropriate when one wishes to extract a local interaction from fixed energy data such as the differential cross sections, polarizations, etc. Of the methods constituting this class, the Lipperheide-Fiedeldey [9] ones have been used very successfully to analyze nuclear heavy ion collision data, proton elastic scattering from nuclei, the scattering of electrons from atoms and molecules, and atom-atom scattering data [10]. But those methods are best suited to high energy conditions and for scattering in which many partial waves contribute. At low energies and with cases of relatively few significant partial wave components, the Newton-Sabatier method [11], or a modified version of it [12,13], may be more appropriate. Cross-section data from nuclear and atomic scattering have been analyzed in that way [14].

In part, our interest with the low energy n-α and p-α data stems from the fact that much data lie below the first nonelastic scattering threshold and for which, therefore, the physical S function is unitary. Then it becomes possible to use the generalized unitarity theorem to extract the actual scattering amplitudes from the measured data and therefrom, by Legendre integration, to specify the scattering phase shifts. For n-α scattering we have determined a method to carry out such a global phase shift analysis [15] and herein we make use of the resultant phase shifts as input to a fixed energy inverse scattering theory. In particular, we have used the modified Newton-Sabatier method [12] to specify the local (real) potentials for each set of those phase shifts at each energy. In this analysis, the spin attributes of the scattering process have been ignored. They have not in the second set of inversion analyses that we report.

With those second set of analyses, we seek to apply the modified Newton-Sabatier fixed energy inverse scattering theory to define thereby local potentials that are phase equivalent to the (local) phenomenological interactions that others [3,4] have specified by fits to the available low energy data from both neutron and proton scattering off of alpha particles. Those phenomenological optical model potentials include spin-orbit components which we may also seek by inversion since, in a recent paper [16], a distorted wave approximation procedure has been given to extract the central and spin-orbit interactions by inversion of the sets of phase shifts $\delta_{\ell}^{(\pm)}$. The superscripts denote $j=\ell\pm\frac{1}{2}$, respectively. The results are energy-dependent smooth interactions and the spin-orbit potentials are much weaker than the central ones except at incident energies about 1 MeV at which there is a shape resonance. The most noticeable thing about these inversion potentials though is that they are quite different from the Woods-Saxon forms that were used to define the scattering phase shifts input to the inversion process.

The inversion potentials have been checked for accuracy and stability. The accuracy has been measured by solving directly the Schrödinger equations including those inversion potentials and comparing the phase shifts so given with the original input sets. At worst, differences of 2–5 % were found. The stability of the results was assessed by repeated calculations with slight variations to the technical parameter values involved in the chosen inversion method. Essentially the same potentials result in all cases.

II. THEORETICAL BACKGROUND

In this section we review, briefly, the three theoretical aspects involved in the inversion studies we have made. First we review the use of the generalized unitarity theorem to extract the scattering amplitude, and thus phase shifts, from a known (complete) differential cross section when the scattering is below the first nonelastic threshold. A fuller account is given in Ref. [15]. Then the method of inversion that we have used, the modified Newton-Sabatier method, is presented. That scheme is appropriate for use with small sets of known phase shifts as input. A detailed account is to be found in Ref. [12]. Finally, the work of Leeb, Huber, and Fiedeldey (LHF) [16] is recounted as that enables us to define central and spin-orbit potentials from those determined by the inversion procedure with two quasi-independent sets of phase shifts built from the sets $\delta_{\ell}^{(\pm)}$.

A. Unitarity and the scattering phase shifts

If, for convenience, we ignore the spin of the neutron, then the differential cross sections for n-α scattering can be expressed in terms of scattering amplitudes,

$$f(x)=\frac{1}{k}A(x)e^{i\varphi(x)}, \tag{1}$$

where $x=\cos(\theta)$, by

$$\frac{d\sigma}{d\Omega}=|f(x)|^2=\frac{1}{k^2}A^2(x). \tag{2}$$

The magnitude and phase of those scattering amplitudes may be extracted from the measured differential cross sections, under the constraint that the scattering function is unitary [17,18], as the generalized unitarity theorem leads to an equation that specifies the phase in terms of the complete (0–180°) cross section, viz.,

$$\sin\varphi(x)=\int\int\frac{A(y)A(z)\cos[\varphi(y)-\varphi(z)]dydz}{2\pi A(x)(1-x^2-y^2-z^2+2xyz)^{1/2}}. \tag{3}$$

Therein the region of integration is the interior of an ellipse. From the scattering amplitude, it is straightforward then to find the the scattering function, and concomitantly the phase shifts, by Legendre integration, viz.,

$$S_\ell - 1 = e^{2i\delta_\ell} - 1 = ik \int_0^\pi f(\theta) P_\ell(\theta) \sin(\theta) d\theta. \qquad (4)$$

Usually, solutions of Eq. (3), or its equivalent, have been sought with iteration schemes based on the contraction mapping principle [17–19]. That approach also defines an existence condition for a solution and for its global uniqueness as well. In application though we have found difficulties with it. The physical circumstances considered [15] did not meet the domain criteria and the solutions found were not stable. Thus we considered a modification of the Newton iteration method, and by which a stable convergent result of iteration could be found. The price of finding stable convergent solutions by this means, however, is the loss of global uniqueness. In particular, the n-α scattering of interest did not meet the domain criteria to guarantee unique stable solutions for the phase functions, and so the modified Newton method had to be used.

B. Modified Newton-Sabatier inverse scattering theory

The fixed energy inverse scattering method of Newton and Sabatier [11] leads to a fundamental equation for solution functions of the form

$$\psi_\ell^U(\rho) = \psi_\ell^{U_0}(\rho) - \sum_{\ell'=0}^{\infty} c_{\ell'} L_{\ell\ell'}(\rho) \psi_{\ell'}^U(\rho), \qquad (5)$$

which is a set of simultaneous equations with coefficients all defined in terms of the partial wave eigenfunctions $\psi_\ell^{U_0}(\rho)$ of a known reference potential U_0, as $L_{\ell\ell'}$ is the calculable matrix,

$$L_{\ell\ell'}(\rho) = \int_0^\rho \psi_\ell^{U_0}(\rho') \psi_{\ell'}^{U_0}(\rho') d\rho' / \rho'^2. \qquad (6)$$

Both the reference functions $\psi_\ell^{U_0}(\rho)$ and the solution functions $\psi_\ell^U(\rho)$ satisfy radial Schrödinger equations with the given reference potential U_0 and the wanted inversion potential U, respectively. The inversion potential may then be constructed, according to the Newton-Sabatier inverse scattering theory, from

$$U(\rho) = U_0(\rho) - \frac{2}{\rho} \frac{d}{d\rho} \sum_{l=0}^{\infty} c_\ell \psi_l^{U_0}(\rho) \psi_l^U(\rho) / \rho, \qquad (7)$$

wherein the standard notations $\rho = k_{\text{c.m.}} r$, $U_0 = V_0(r)/E_{\text{c.m.}}$, $U = V(r)/E_{\text{c.m.}}$, and $E_{\text{c.m.}} = \hbar^2 k_{\text{c.m.}}^2 / 2\mu$ have been used. But in practical applications, this scheme needs to be modified. A most useful modification [12] is predicated upon knowledge of the scattering potential U beyond a given finite distance r_0. For nuclear heavy ion collisions this long range form is well established as the Coulomb potential. For electron-atom scattering, it is the induced dipole polarization (r^{-4}) potential. For the elastic scattering problems we consider, it is appropriate to assume that $U_0(\rho)$ is either zero (neutrons) or the Coulomb potential (protons) for $\rho \geq \rho_0$.

According to the modified Newton-Sabatier theory [12,13], the solution of Eq. (5) involves a two-step procedure after one truncates the summations of ℓ to the first $\ell_{\max}+1$ terms ($\ell_{\max}$ is the largest angular momentum of the input phase shifts δ_ℓ effectively contributing to the scattering). In the first step, one uses Eq. (5) at $N \geq 2$ points $\rho = \rho_1, \rho_2, \ldots, \rho_N$ $\rho_i \geq \rho_0$ (at which the wave functions have their asymptotic forms involving the scattering phase shifts) to determine the $2(\ell_{\max}+1)$ unknowns c_ℓ. A least squares method has been used for that. Then, with the repeated use of Eq. (5) for $\rho < \rho_0$, the potential $U(\rho)$ can be specified by use of Eq. (7).

The inversion calculations reported in this article were made using the computer program BIC [20] in which equidistant least squares points $\rho_1, \rho_2 = \rho_1 + \Delta\rho, \ldots, \rho_N = \rho_1 + (N-1)\Delta\rho$ are taken, and with the code PHATRA [21] to effect a convenient transformation on the phase shifts.

C. Spin-orbit interaction by LHF inversion

In their recent paper [16], Leeb, Huber, and Fiedeldey (LHF) have set out an approximate global inversion procedure by which both the central and spin-orbit potentials for the scattering of spin-1/2 particles (nucleons in our case) from spin-0 targets can be obtained from fixed energy data, e.g., differential cross sections and polarizations. Specifically their procedure begins with values for the phase shifts $\delta_\ell^{(\pm)}$ for all angular momenta ℓ and $j = \ell \pm \frac{1}{2}$.

The LHF method is based upon an expansion of the phase shifts where the effect of the spin-orbit potential is taken into account via a distorted wave approximation. Using

$$\delta_\ell^{(\pm)} = \delta_\ell^{(0)} + a_\ell^{(\pm)} C_\ell^{(1)} + (a_\ell^{(\pm)})^2 C_\ell^{(2)} + \cdots, \qquad (8)$$

where

$$a_\ell^{(\pm)} = \frac{2}{\hbar^2} \langle \mathbf{s} \cdot \mathbf{l} \rangle = \begin{cases} \ell \\ -(\ell+1) \end{cases} \quad \text{for } j = \begin{cases} \ell + \frac{1}{2} \\ \ell - \frac{1}{2} \end{cases}, \qquad (9)$$

the quasi-independent phase shifts

$$\tilde{\delta}_\ell = \frac{1}{(2\ell+1)} \{(\ell+1)\delta_\ell^{(+)} + \ell \delta_\ell^{(-)}\}$$
$$= \delta_\ell^{(0)} + \ell(\ell+1) C_\ell^{(2)} + \cdots,$$

$$\hat{\delta}_\ell = \frac{1}{(2\ell+1)} \{\ell \delta_\ell^{(+)} + (\ell+1)\delta_\ell^{(-)}\}$$
$$= \delta_\ell^{(0)} - C_\ell^{(1)} + (\ell^2 + \ell + 1) C_\ell^{(2)} + \cdots, \qquad (10)$$

when restricted to first order, map by separate inversions to

$$\tilde{\delta}_\ell \sim \delta_\ell^{(0)} \Leftrightarrow \tilde{V} \sim V_0 ,$$

$$\hat{\delta}_\ell \sim \delta_\ell^{(0)} - C_\ell^{(1)} \Leftrightarrow \hat{V} \sim V_0 - \frac{1}{2} V_{\text{s.o.}} . \tag{11}$$

Therein V_0 and $V_{\text{s.o.}}$ are the approximations for the central and spin-orbit potentials. Leeb, Huber, and Fiedeldey [16] note that this expansion remains exact through second order. Thus by forming the quasi-independent phase shifts, one may then apply any appropriate fixed energy inversion scheme [7] that defines inversion potentials from a single set of phase shifts, to determine $\tilde{V}$ and $\hat{V}$ separately, and then from Eq. (11) obtain the central and spin-orbit interactions.

III. DISCUSSION OF RESULTS

The N-α potentials that we have determined by inversion are shown in two subsections. The first presents results found by using the phase shifts we have determined from the scattering amplitudes specified by the unitarity condition and the generalized flux theorem. In the second, we discuss the inversion potentials that were obtained using the phase shifts specified by the best phenomenological optical model potentials [3] and the LHF prescription to specify the spin-orbit field.

A. N-α potentials using the unitary condition phase shifts

We have solved the nonlinear phase equations for the scattering of 14.9, 16.4, 20.0, and 23.7 MeV neutrons scattering off of alpha particles. The specifics have been published [15] but are given again in brief for completeness. The data [22] are incomplete and we have had to interpolate and extrapolate to specify full (0–180°) cross sections for use in solution of the nonlinear equations for the phases of the scattering amplitudes. It was also necessary to minimize effects of nonstatistical errors and that was achieved by applying generalized cross validation [23] to each data set. As noted previously [15], the ''Martin'' condition [19] is not met by these data sets and so we used a modified Newton iteration method to solve the nonlinear equations for the phase of the scattering amplitudes. Legendre integration of the resultant scattering amplitudes then gave reliably the (purely real) scattering phase shifts to $\ell = 4$ (see Table I in Ref. [15]).

Using those phase shifts in the modified Newton-Sabatier inversion scheme gave potentials that are real, smooth, and energy dependent. We classify them as the $G1$ set of potentials and they are given in the top segment of Fig. 1. They vary with energy most markedly at small radii and below 0.5 fm in particular, but the actual data are not sensitive to details of these potentials near to the origin.

The unitarity condition approach has been used to specify only the lowest set of phase shifts. The incomplete nature of the actual data, the approximations needed to find stable solutions for the phases of the scattering amplitudes, and all numerics involved preclude specification of reliable phase shifts for $\ell > 4$ or 5. The values of the phase shifts at low values of ℓ also should be taken as reliable to perhaps a degree at best too. Thus, in the first studies we made, we set the higher ℓ phase shifts exactly to zero. To assess how those small higher ℓ phase shifts might influence results, we next made a set of calculations using higher partial waves (than those tabled in Ref. [15]) matched to each tabled unitarity set and varying according to ℓ^{-2}. The inversion potentials that result, and which we designate as the $G2$ potentials, are also real and smooth, but now less energy dependent than the $G1$ set. They are displayed in the bottom segment of Fig. 1.

TABLE I. The original s- and p-wave phase shifts that were obtained from the unitarity condition analyses of n-α cross sections compared with those found by using the two sets of inversion potentials, $G1$ and $G2$.

E (MeV)	$\delta_0^{(\text{orig})}$	$\delta_0^{(\text{inv})}(G1)$	$\delta_0^{(\text{inv})}(G2)$
14.9	1.824	1.823	1.827
16.4	1.846	1.845	1.844
20.0	1.742	1.739	1.741
23.7	1.733	1.727	1.729
	$\delta_1^{(\text{orig})}$	$\delta_1^{(\text{inv})}(G1)$	$\delta_1^{(\text{inv})}(G2)$
14.9	1.180	1.164	1.158
16.4	1.423	1.404	1.405
20.0	1.360	1.345	1.343
23.7	1.287	1.286	1.274

Both potential sets were used in Schrödinger equations whose solutions lead to predicted phase shifts and cross sections. In comparison with the data, the $G1$ results are better than those obtained using the $G2$ potentials. That was expected given that high partial wave phase shifts were added arbitrarily to the unitarity based set to define the input for the inversions that gave the $G2$ interactions. But neither give fits, and to the 14.9 MeV data in particular, which are of the same quality as the original ones from which the input phase shifts

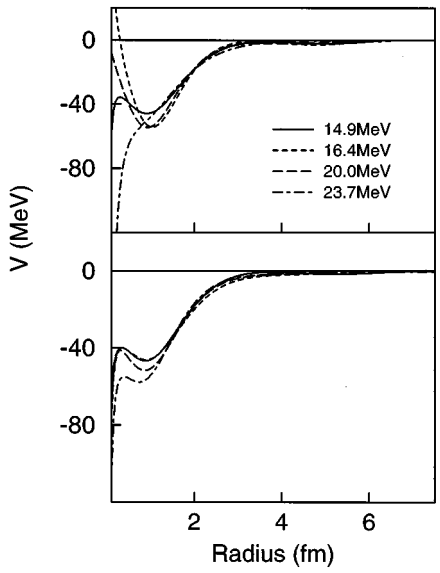

FIG. 1. The potentials obtained by inversion starting with solely the phase shifts given by Legendre integration of the unitarity formed scattering amplitudes (top) and those found when higher partial wave phase shifts vary as ℓ^{-2} (bottom). The solid, small dashed, large dashed, and dot-dashed curves give the results for 14.9, 16.4, 20.0, and 23.7 MeV incident energies.

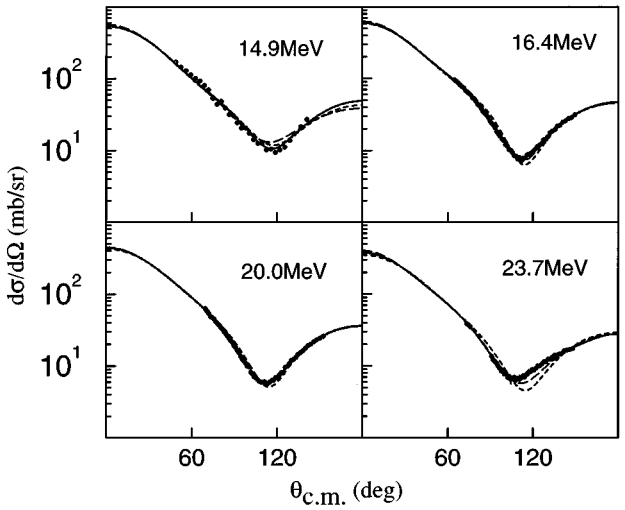

FIG. 2. The cross sections from 14.9, 16.4, 20.0, and 23.7 MeV neutrons elastically scattered off of an alpha particle compared with those evaluated using the unitarity phase shifts and shown by the solid curves. The large and small dashed curves depict the cross sections calculated using all of the phase shifts found with the $G1$ potentials and those found using the set to $\ell = 4$.

were defined. The dominant elements are the s- and p-wave phase shifts and their values are given in Table I. There is little variation between the input and various calculation results. Thus we presume that the specific values of the higher ℓ partial wave phase shifts, small though they may be, have an important effect. To test that presumption we have made calculations of cross sections based upon both the $G1$ and $G2$ potentials but with only the first few partial wave phase shifts (for $\ell < 4$ specifically) taken into account. The results are shown in Fig. 2. Those found using the restricted set of phase shifts are displayed therein by the small dashed curves. The cross sections found by using all sensible $G1$ phase shifts are displayed by the large dashed curves. They are compared with the n-α elastic scattering data and with the cross sections given by using the input (unitarity) phase shifts that are tabled in Ref. [15]. The latter are displayed by the solid curves. Clearly the set of higher ℓ phase shifts given from the potentials has a noticeable effect upon the calculations but the results are not always worse (in comparison with the actual data and unitarity phase shifts fits) than what results by using only the first few potential defined partial wave phase shifts alone. The small differences in the low ℓ phase shifts values also effect the cross sections. The approximations in the inversion process we have set down and the numerics used seem to give results accurate to about 2–5 %.

B. Inversion of the optical model phase shifts

The optical model potentials of Satchler *et al.* [3] have been used to define the sets of phase shifts $\delta_\ell^{(\pm)}$ that we have used as base data in our fixed energy inversion studies to extract central and spin-orbit potentials via the LHF [16] prescription. We have used a set of energies from 1 to 18 MeV for neutrons and protons and for which the phase shifts also have been tabulated [3].

TABLE II. The n-α phase shifts $\tilde{\delta}_\ell$ and $\hat{\delta}_\ell$ (in radians) given by the optical model calculation used in the inversion process and those that result (in parentheses) upon using the inversion potentials.

E_n	$\hat{\delta}_0 = \tilde{\delta}_0$	$\tilde{\delta}_1$	$\hat{\delta}_1$	$\tilde{\delta}_2$	$\hat{\delta}_2$
1	2.707	0.702	0.386	0.0001	0.001
	(2.696 / 2.663)	(0.69)	(0.395)	(0.002)	(0.002)
2	2.534	1.475	0.876	0.0011	0.001
	(2.497 / 2.497)	(1.451)	(0.871)	(0.0043)	(0.007)
5	2.20	1.699	1.28	0.0089	0.0082
	(2.230 / 2.234)	(1.638)	(1.265)	(0.0067)	(0.018)
10	1.87	1.62	1.34	0.038	0.034
	(1.870 / 1.874)	(1.6)	(1.33)	(0.034)	(0.032)
15	1.655	1.47	1.22	0.077	0.068
	(1.650 / 1.653)	(1.45)	(1.2)	(0.082)	(0.065)
18	1.55	1.38	1.14	0.102	0.087
	(1.550 / 1.549)	(1.36)	(1.13)	(0.103)	(0.086)

The first step in these studies was to form the quasi-independent sets of phase shifts $\hat{\delta}_\ell$ and $\tilde{\delta}_\ell$ at each energy. Those phase shifts for the s, p, and d waves that were input to the inversion programs are listed in Table II for the n-α cases considered. They are shown without parentheses. When the separate inversion potentials $\hat{V}$ and $\tilde{V}$ were used in Schrödinger equations, the sets shown in parentheses in that table resulted. Clearly the inversion interactions reproduce the starting set of quasi-independent phase shifts quite well. We have shown two values for the s-wave recalculations. The first of each pair of values is the result for the s-wave scattering from $\hat{V}$ while the second is that from the use of $\tilde{V}$. The two sets of recalculated s-wave phases are very similar, as they should be, and are very good reproductions of the input values. Likewise the p- and d-wave phase shifts match as well with at worst a 5% variation in a few of the (small) d-wave results. With the dominant s- and p-wave phase shifts the error is less than 2%. Similar results were obtained with our inversions of the proton phase shifts.

The n-α potentials found by inversion are shown in Fig. 3 for energies ranging from 1 to 18 MeV. The $\tilde{V}$ and $\hat{V}$ potentials are shown in the top and middle segments of that figure and the $V_{\text{s.o.}}$ extracted from them are given in the bottom segment. In the LHF scheme, $\tilde{V}$ maps to the central interaction. These interactions are energy dependent, with most effects being noted with the low energy cases. The 10, 15, and 18 MeV interactions are quite similar. There is a marked structural change in the shapes of these inversion potentials at low energies too and a result of that is the quite strong low energy spin-orbit interaction. Cross sections and polarizations extracted using these LHF-scheme-specified central and spin-orbit interactions are shown in Fig. 4. Therein we display the original ''data,'' i.e., the cross sections and polarizations evaluated using the input sets of phase shifts $\delta_\ell^{(\pm)}$ by the circles. The solid curves represent the results we have found using the inversion potentials with no cutoff ($R_{\text{cut}} = 0$) in the Schrödinger equations, while those given by the small and large dashed curves are the results obtained with cutoff radii 0.85 and 1.25 fm, respectively. Clearly the ''data'' are sensitive to the potentials above 0.5 fm or so and

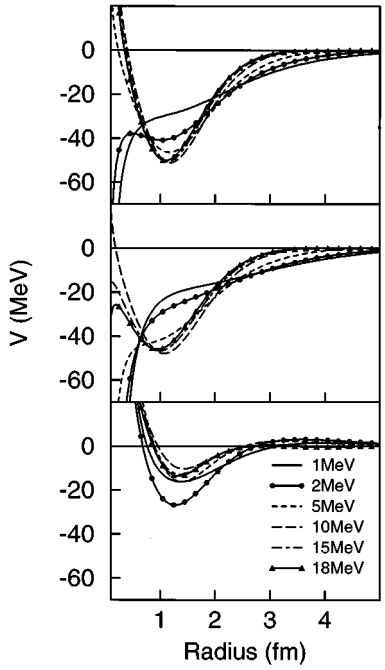

FIG. 3. The inversion potentials $\tilde{V}$ (top) and $\hat{V}$ (middle) and the spin-orbit potentials deduced from them (bottom) for n-α scattering at various energies as indicated.

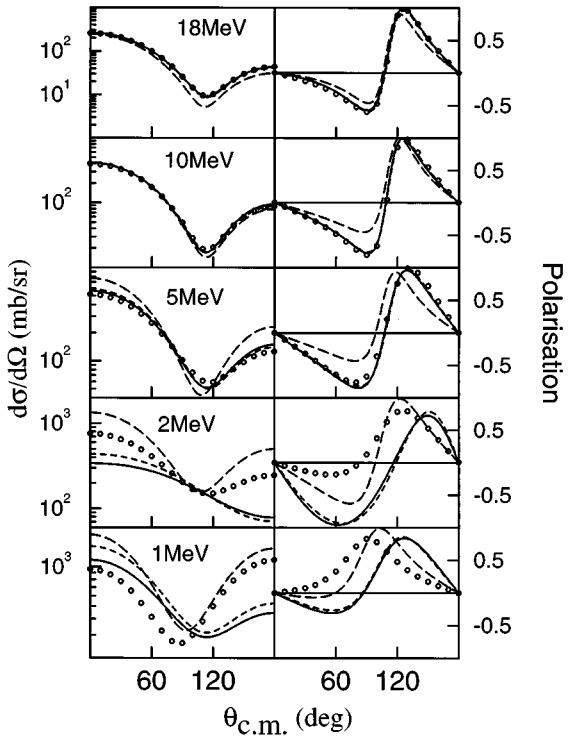

FIG. 4. The cross sections and polarizations for n-α scattering at diverse energies. The circles are the results from using the Woods-Saxon optical model potentials (from which the input phase shifts for inversion were derived). The solid curves are the results obtained by using the central and spin-orbit potentials derived by the inversion process. The small and large dashed curves display similar calculations but made with $R_{\rm cut}$ values of 0.85 and 1.25 fm, respectively.

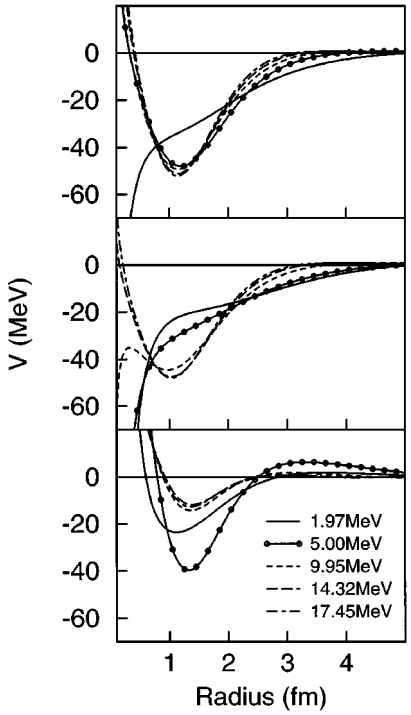

FIG. 5. As for Fig. 3 but for proton scattering.

the results are very good for the two higher energies but are quite poor at the energies coinciding with the known large resonance in the scattering.

The results for the p-α scattering are shown in Figs. 5 and 6. The potentials are given in Fig. 5 and the reproduction and sensitivity of cross sections and polarizations are shown in Fig. 6. Again there is a marked energy variation in the potentials, especially at the lowest energies with the resultant spin-orbit interactions to 5 MeV being very strong. The 9–17 MeV potentials show little energy dependence though. Again

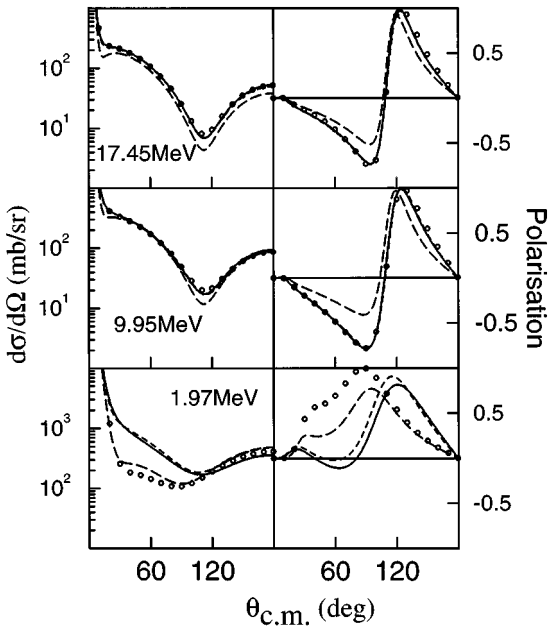

FIG. 6. As for Fig. 4 but for proton scattering.

TABLE III. The n-α phase shifts $\delta_{\ell}^{(\pm)}$ (in radians) given by the optical model calculations used in the inversion process compared with those that result upon using the inversion potentials, $\hat{V}$ and $\tilde{V}$ (in single parentheses) and with those that were obtained using the derived central and spin-orbit interactions (in double parentheses).

E_n	$\delta_0^{(+)}$	$\delta_1^{(-)}$	$\delta_1^{(+)}$	$\delta_2^{(-)}$	$\delta_2^{(+)}$
1	2.707	0.091	1.007	0.0001	0.001
	(2.696)	(0.099)	(0.986)	(0.0016)	(0.0026)
	((2.676))	((0.349))	((2.245))	((0.003))	((0.001))
2	2.534	0.275	2.075	0.0009	0.0012
	(2.497)	(0.290)	(2.032)	(0.0125)	(-0.0012)
	((2.578))	((0.865))	((2.686))	((0.013))	((0.001))
5	2.20	0.862	2.12	0.007	0.0101
	(2.23)	(0.894)	(2.01)	(0.0404)	(-0.016)
	((2.211))	((1.031))	((2.140))	((0.025))	((0.008))
10	1.87	1.06	1.9	0.0026	0.045
	(1.87)	(1.06)	(1.87)	(0.029)	(0.037)
	((1.871))	((1.072))	((1.850))	((0.029))	((0.039))
15	1.655	0.976	1.71	0.049	0.096
	(1.65)	(0.957)	(1.69)	(0.031)	(0.116)
	((1.666))	((0.970))	((1.681))	((0.042))	((0.115))
18	1.55	0.90	1.61	0.058	0.13
	(1.55)	(0.89)	(1.60)	(0.051)	(0.14)
	((1.557))	((0.899))	((1.589))	((0.057))	((0.145))

the reproduction of cross section and polarization ''data'' is very good at the higher energies and the potentials are sensitive from 0.5 fm again. The fit for ''data'' in the resonance region is again very poor.

We complete our analysis by presenting, in Table III, the $\delta_{\ell}^{(\pm)}$ phase shifts for the s, p, and d channels and from n-α scattering. The input set from Satchler *et al.* [3] is shown in the first row of each set of 3 (for each energy). Those that we obtained by using the extracted $\hat{V}$ and $\tilde{V}$ in Schrödinger equation calculations and then by inverting Eq. (10) are given next in the single parentheses while those we have obtained by using the LHF prescription for the central and spin-orbit potentials are given in the third (double parentheses) rows. The agreement with all three forms is quite good for energies 10 MeV and higher. Only a few of the small d-wave results are in error by 5–8 %. At the lower energies though, and especially in the 1–2 MeV resonance region, the LHF method results are noticeably different from the other two. The spin-orbit interactions are too strong in those cases, it seems, for the LHF first order approximation to be useful.

IV. CONCLUSIONS

As the N-α scattering up to 24 MeV incident energy lies below the first inelastic threshold, the inherent S functions are unitary and the generalized unitarity theorem leads to a specification of the phase of the scattering amplitude in terms of its magnitude (the square root of the differential cross section). To extract that scattering amplitude phase, and hence specify the complete scattering amplitude with but trivial ambiguities at most, requires that we know the differential cross section at any selected energy at all physical scattering angles. Then if that data satisfies the ''Martin'' condition, the scattering amplitude can be specified by an iterative (Newton) process and is unique. So also then are the scattering phase shifts (found by simple Legendre integration). But the measured cross sections are incomplete, necessitating extrapolation and interpolation, and the ''Martin'' condition is not satisfied by the N-α data sets considered. The latter forces use of an approximation scheme to solve the nonlinear phase equations while the former makes the input kernel ambiguous. Nevertheless, we were able to find stable and credible solutions for the scattering of 14.9–23.7 MeV neutrons from the alpha particle and so specify phase shifts (to ℓ of 3 or 4) that were consistent with ones assessed by others from their (different) analyses of the data. Those phase shifts were used as the input to an inversion calculation based upon a modified Newton-Sabatier scheme and smooth energy-dependent potentials were obtained. They were markedly different in form to the conventionally chosen Woods-Saxon potentials. The results were insensitive to details of the interactions below 0.5 fm but did vary according to what choice one makes for the small valued phase shifts.

Given the very reasonable fits to both n-α and p-α data, differential cross sections, and polarizations, for energies in the range 1–20 MeV that were obtained with a global phenomenological optical model potential study [3], we have also applied our inversion method to select sets of those phase shifts. We adapted the approximation scheme of Leeb, Huber, and Fiedeldey [16] so that we could estimate the central and spin-orbit potentials from the inversion potentials of two sets of quasi-independent phase shifts. Smooth forms resulted with the 10–20 MeV potentials being quite similar and of the form of a finite range well with a short range repulsion. At lower energies the potentials found were markedly varied and quite long ranged. None were at all like the phenomenological optical model potentials from which the input phase shifts were specified. But at the low energies, a strong resonance is known to exist in the input scattering phase shift structure. With inversion, its presence will be reflected by a strong variation in the form of the calculated potentials. Also the spin-orbit strength appears to be quite strong at low energies and so the LHF prescription may not be appropriate then. However, it is of note that our potentials for 1–2 MeV neutrons scattering from the α-particle are very similar to those found from a recent fixed angular momentum inversion study [24].

We surmise that our present calculations, which we have found to be accurate to a few percent and for energies above the low-lying resonance in particular, are a class of potentials phase equivalent to the usually chosen (phenomenological) ones. The inference of a repulsive short range character with some of these results and the energy variations found are characteristics others anticipate for local interactions equivalent to nonlocal ones. As this class of potentials is based in part upon the use of the generalized unitarity theorem to ascertain phase shift data and also upon an inversion process with which there is essentially no *a priori* bias as to the form of the local interaction derived, we believe that the set is particularly appropriate for use in other (few body) studies predicated upon a local (effective) n-α interaction in the low energy continuum which fits the observed scattering data. Of course there are ambiguities as we have shown herein be-

tween the phenomenological forms usually taken and the inversion set we have found. Furthermore, it is well known that there are supersymmetric partners to any phase shift specific potential according to the number of allowed (spurious) bound states taken. Other information needs to be taken into account to improve upon that situation and such might be a regularization of the inversion studies based upon the interactions determined by folding the underlying NN g matrices with the density matrix elements of the α particle. Prior to that, however, we will need to understand how our results might be interpreted as a nonlocal equivalent and how they can compare with ones obtained by such a fully microscopic and antisymmetric calculation.

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