Model gluon propagator and pion and ρ-meson observables

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(Received 3 August 1995)

A one parameter, model confined-gluon propagator is employed in a phenomenological application of the Dyson-Schwinger and Bethe-Salpeter equations to the calculation of a range of π- and ρ-meson observables. Good agreement is obtained with the data. The calculated quark propagator does not have a singularity on the real-p^2 axis. A mass formula for the pion, involving only the vacuum, dressed quark propagator, is presented and shown to provide an accurate estimate of the mass obtained via a direct solution of the Bethe-Salpeter equation.

PACS number(s): 14.40.Aq, 12.38.Lg, 12.39.Pn, 12.40.Yx

I. INTRODUCTION

The Dyson-Schwinger equations (DSE's) provide a useful, semiphenomenological tool for the study of QCD. These coupled integral equations relate the n-point (Schwinger) functions of QCD to each other. They provide a nonperturbative, Poincaré invariant framework that enables one to correlate hadronic observables through the properties of the Schwinger functions of the elementary excitations in QCD, i.e., the Schwinger functions of quarks and gluons. [Quark and gluon propagators (two-point functions) are examples of such Schwinger functions.] This makes it particularly suitable for addressing questions such as confinement and dynamical chiral symmetry breaking and also hadronic spectroscopy and interactions. This approach is reviewed in Ref. [1] and has recently been applied to the study of π-π scattering [2], the electromagnetic pion form factor [3], ρ-ω mixing [4], and the anomalous $\gamma^*\pi^0\to\gamma$-transition form factor [5].

It is possible to obtain information about such Schwinger functions via a numerical simulation of a lattice-QCD action [6–8]. However, in addition to the usual problems associated with identifying and establishing the existence of the continuum limit, and recovering the global symmetries of QCD, this also requires gauge fixing on the spacetime lattice. Gauge fixing eliminates a number of gauge-equivalent gauge-field configurations, thereby leading to poorer statistics. It does not eliminate all such configurations, however. One is left with Gribov copies, i.e., gauge configurations in the gauge-fixed ensemble that are not distinct but are related by topologically nontrivial gauge transformations [9–11]. This entails an overcounting problem in the evaluation of gauge-fixed correlation functions. Present studies are encouraging, having established that this approach to the calculation of gauge-fixed QCD Schwinger functions is feasible [7]. However, the problems identified above entail that they are currently qualitatively and quantitatively unreliable.

Presently the most reliable estimates of the behavior of quark and gluon Schwinger functions are obtained in DSE studies. The DSE's are a tower of coupled equations and a solution is only tractable if this tower is truncated. Truncation procedures that preserve the global symmetries of QCD are easy to construct and implement. This has not yet been accomplished for the local symmetry in QCD; however, progress is being made following the realization of the importance of the nonperturbative structure of the fermion–gauge-boson vertex [1,12–16]. This introduces an uncertainty in the infrared, i.e., for $k^2<1-2$ GeV^2. However, this uncertainty is merely quantitative. There is general agreement on the qualitative features of the quark and gluon two-point Schwinger functions, i.e., (1) that the gluon two-point function is significantly enhanced at small spacelike k^2 [1,17–19] and that this entails an enhancement of the momentum-dependent quark mass function [1,20–24], and (2) that for $k^2>1-2$ GeV^2 the two-loop, renormalization group improved, perturbative results are quantitatively reliable.

Some phenomenological DSE studies have employed a parametrization of the two-point quark Schwinger function based on these results, for example, Refs. [2–5]. Such studies are phenomenologically efficacious. However, they involve the addition of new parameters when applied to systems involving other than u and d quarks.

The introduction of new parameters is unnecessary when the propagator for a quark of a given flavor is obtained directly from a quark DSE whose kernel is determined by the two-point gluon Schwinger function and the quark-gluon vertex. This procedure correlates the propagators for quarks of different flavors via the parameters in the gluon two-point function. There have been studies that employ this approach, for example, Refs. [21,25,26]. However, it is computationally more intensive and the studies therefore addressed the calculation of a smaller class of observables. The present study is a first step in extending this latter approach.

Herein we employ a one parameter model gluon propagator (gluon two-point Schwinger function), motivated by the results of Refs. [17–19], in a calculation of a range of π- and ρ-meson observables. The one parameter is a mass scale that can be interpreted as marking the transition between the perturbative and nonperturbative domains. This model gluon propagator provides the kernel for a quark DSE, which is solved to obtain the quark propagator (quark two-point

0556-2813/96/53(1)/390(9)/$06.00

Schwinger function) for real $p^2 \in (-\infty,\infty)$. These two Schwinger functions provide the kernel of the meson Bethe-Salpeter equation (BSE), whose solution yields the meson mass and Bethe-Salpeter amplitude, which is a necessary element in the calculation of decay constants and scattering lengths, for example. The single mass parameter determines all of these Schwinger functions and is varied to obtain a good fit to a range of calculated π observables. This illustrates the utility and economy of the approach.

In studying the pion BSE we derive a mass formula for the pion, which involves only the vacuum, dressed quark propagator, valid to all orders in m_R, the renormalized current quark mass. Our numerical studies show that this formula provides an excellent estimate of the mass that is obtained by actually solving the BSE.

The model gluon propagator is discussed in Sec. II and the quark DSE in Sec. III. The pion mass formula is presented in Sec. IV. Our numerical results are discussed in Sec. V and we summarize and conclude in Sec. VI.

II. MODEL GLUON PROPAGATOR

In Euclidean metric [27] the Landau gauge gluon propagator is

$$g^2 D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{g^2}{k^2[1+\Pi(k^2)]}, \tag{1}$$

where $\Pi(k^2)$ is the gluon vacuum polarization. Setting $\mathscr{Z}_1^{\rm gh} = \mathscr{Z}_3^{\rm gh}$, where $\mathscr{Z}_1^{\rm gh}$ is the renormalization constant for the ghost-gluon vertex and $\mathscr{Z}_3^{\rm gh}$ that for the ghost wave function,

$$\Delta(k^2) \equiv \frac{g^2}{k^2[1+\Pi(k^2)]} \tag{2}$$

satisfies the same renormalization group equation as the QCD running coupling constant $\alpha(k^2)$ [28] and hence

$$[g^2 D_{\mu\nu}(k)]_R = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{4\pi\alpha(k^2)}{k^2}. \tag{3}$$

This is sometimes described as the ‘‘Abelian approximation’’ because it entails the QED-like Ward identity $Z_1 = Z_2$, where Z_1 is the quark-gluon vertex renormalization constant and Z_2 is the quark wave function renormalization constant [1].

The two-loop renormalization group expression for the running coupling constant only receives small ($\sim$10%) corrections from higher orders for spacelike $k^2 > 1$ GeV^2 and hence can be said to provide an accurate representation on this domain. For $k^2 < 1$ GeV^2, however, $\alpha(k^2)$ is not known and can only be calculated nonperturbatively. The current status of such studies is summarized in Ref. [1] and, as remarked in Sec. I, gluon-DSE studies agree on the qualitative behavior of $\alpha(k^2)$ at small k^2. Present phenomenological quark-DSE studies rely on an *Ansatz* for $\alpha(k^2 < 1\,{\rm GeV}^2)$ motivated by these gluon-DSE studies.

Herein we consider a parametrization suggested by the Landau gauge studies of Ref. [19], which revealed a strong enhancement in the gluon propagator at small spacelike k^2 (<1 GeV^2) that could be described by an integrable singularity. We employ the one parameter form

$$\Delta(k^2) = 4\pi^2 d \left[4\,\pi^2 m_t^2 \delta^4(k) + \frac{1 - e^{(-k^2/[4m_t^2])}}{k^2} \right], \tag{4}$$

where $d = 12/(33 - 2N_f)$, with $N_f = 3$ the number of light flavors. The first term in Eq. (4) provides an integrable, infrared singularity [20], which generates long-range effects associated with confinement, and the second ensures that the propagator has the correct large spacelike-k^2 behavior, up to $\ln[k^2]$ corrections. A form similar to this has been used by other authors [21–24] with one-loop logarithmic corrections included in the second term. We neglect these terms as a simple expedient to ensure that our gluon propagator does not have a Lehmann representation and may therefore be interpreted as describing a confined particle, i.e., an elementary field with which there is no associated asymptotic state [1,29].

Since ours is a model gluon propagator, there is no reason why the coefficients of the two terms in Eq. (4) should be related in the particular fashion we have chosen. However, consider

$$\Delta(x^2) \equiv \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \Delta(k^2) = d\left[m_t^2 + \frac{1}{x^2} e^{-x^2 m_t^2} \right]. \tag{5}$$

It is clear from this that with our choice of the ratio of these coefficients the effects of $\delta^4(k)$ in Eq. (4) are completely canceled at small x^2, i.e.,

$$\Delta(x^2) \overset{m_t^2 x^2 < 1}{\simeq} \frac{d}{x^2} + O(x^2), \tag{6}$$

which is the form expected from QCD (again neglecting logarithmic corrections).

One can therefore interpret m_t as the mass scale in our model that marks the transition from the perturbative to the nonperturbative regime. Herein m_t is varied to provide a best fit to a range of calculated pion observables. [See Eq. (52) and the associated discussion.]

III. QUARK SELF-ENERGY

In Euclidean metric [27] the DSE for the quark propagator is

$$S^{-1}(p) = Z_2(i\gamma\cdot p + m_0) + \Sigma'(p), \tag{7}$$

where

$$\Sigma'(p) \equiv Z_1 \int^\Lambda \frac{d^4k}{(2\pi)^4}\, \frac{4}{3} g^2 D_{\mu\nu}(p-k) \gamma_\mu S(k) \Gamma_\nu(p,k), \tag{8}$$

with $\Gamma_\mu(p,k)$ the quark-gluon vertex, is the regularized self energy, which can be decomposed as

$$\Sigma'(p) = i\gamma\cdot p (A'(p^2) - 1) + B'(p^2). \tag{9}$$

The inverse of the renormalized quark propagator is

$$S^{-1}(p) = i\gamma\cdot p + \Sigma(p) = i\gamma\cdot p A(p^2) + B(p^2). \tag{10}$$

Herein the prime denotes regularized quantities and unprimed quantities are fully renormalized.

We employ a subtractive renormalization scheme, requiring that, at a spacelike renormalization point μ^2,

$$S^{-1}(p)|_{p^2=\mu^2}=i\gamma\cdot p+m_R, \tag{11}$$

with m_R the renormalized *current* quark mass.

In this scheme, the wave function and mass renormalization constants are given by

$$Z_2\equiv 2-A'(\mu^2,\Lambda^2) \text{ and } m_R\equiv Z_2 m_0(\Lambda^2)+B'(\mu^2,\Lambda^2), \tag{12}$$

respectively, and the renormalized self-energies are therefore obtained from

$$A(p^2,\mu^2)=1+A'(p^2,\Lambda^2)-A'(\mu^2,\Lambda^2), \tag{13}$$

$$B(p^2,\mu^2)=m_R(\mu^2)+B'(p^2,\Lambda^2)-B'(\mu^2,\Lambda^2). \tag{14}$$

In this scheme, $A(\mu^2)=1$ and $B(\mu^2)=m_R(\mu^2)$. [In the following we often write $m_R(\mu^2)$ as simply m_R, in which case the μ^2 dependence is implicit.]

The renormalized axial-vector Ward identity is

$$(p-q)_\mu i\Gamma^5_\mu(p,q)=S^{-1}(p)\gamma_5+\gamma_5 S^{-1}(q)-2m_R\Gamma^5(p,q). \tag{15}$$

The composite operators Γ^5_μ and Γ^5 are renormalized such that, at $p^2=\mu^2=q^2$, $\Gamma^5_\mu(p,q)=\gamma_\mu\gamma_5$ and $\Gamma^5(p,q)=\gamma_5$.

The chiral limit is identified as the limit in which the renormalized axial-vector current is conserved, i.e, with the limit $m_R(\mu^2)\to 0$.

A. Analysis of the large-p^2 behavior of the quark propagator

At large spacelike k^2 and p^2 one may replace the gluon propagator and the quark-gluon vertex by their asymptotic forms

$$\Delta(k^2)\to\frac{1}{k^2} \text{ and } \Gamma_\mu(p,k)\to\gamma_\mu. \tag{16}$$

In this limit $A(p^2)\equiv 1$ and $B(p^2)$ is the solution of

$$B(x)=Z_2m_0+\frac{\lambda}{4}\int_0^{\Lambda^2}dy\; y\left(\frac{1}{x}\theta(x-y)+\frac{1}{y}\theta(y-x)\right)\times\frac{B(y)}{y+B^2(y)}, \tag{17}$$

where $x=p^2$, $y=k^2$, and $\lambda=4Z_1 d$.

For x such that $B(x)^2\ll x$, i.e., for $x\gg\mu^2$, this integral equation is equivalent to the differential equation

$$\frac{d}{dx}\left(x^2\frac{d}{dx}B(x)\right)+\frac{\lambda}{4}B(x)=0, \tag{18}$$

subject to the boundary condition

$$B(\mu^2)=m_R \tag{19}$$

or

$$\left.\left(\frac{d}{dx}[xB(x)]\right)\right|_{x=\Lambda^2}=Z_2m_0. \tag{20}$$

Under the change of variables $x=\mu^2\exp(2z)$, Eq. (17) becomes

$$\ddot{B}(z)+2\dot{B}(z)+\lambda B(z)=0, \tag{21}$$

which is the equation of motion for a damped harmonic oscillator. One has critical damping for $\lambda=\lambda_C=1$ and this yields the critical coupling for dynamical chiral symmetry breaking; i.e., in the absence of the first term in Eq. (4), the model would still exhibit dynamical chiral symmetry breaking for $\lambda>1$. This behavior has been observed in QED [30] and phenomenological models of QCD without an infrared-singular model gluon propagator [31–33].

The solution of Eq. (18) consistent with Eq. (19) is

$$B(z)=\kappa e^{-z}\cos(z\sqrt{\lambda-1}+\phi), \tag{22}$$

with

$$\kappa\,\cos\phi=m_R. \tag{23}$$

In the chiral limit $m_R=0$ and hence $\phi=\pi/2$. In general κ is only determined in a complete solution of the integral equation.

The boundary conditions in Eqs. (19) and (20) are equivalent: A given value of m_R entails a given value of Z_2m_0 and vice versa. In fact, for finite Λ, $m_R=0$ generally entails $Z_2m_0\neq 0$. It follows from Eq. (22), however, that for any finite value of m_R

$$\lim_{\Lambda^2\to\infty} Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2)=0. \tag{24}$$

Equation (22) indicates that the renormalized mass function will exhibit damped oscillations about zero for $p^2>\mu^2$, a feature we observed in our numerical solutions, which were well described by Eq. (22) on $p^2\in[\mu^2,\Lambda^2]$. With the exception of Ref. [34], other DSE studies implicitly use $\mu=\Lambda$ and hence the oscillations are not observed. The oscillations were observed in Ref. [34], which addresses in detail the nonperturbative renormalization of the fermion DSE in QED.

B. Additional remarks

The "Abelian approximation" entails that $Z_1=Z_2$ in Eqs. (7) and (8). We make this identification hereafter.

In the numerical studies described below we employed the rainbow approximation

$$\Gamma_\mu(p,k)=\gamma_\mu. \tag{25}$$

This is a quantitatively reliable approximation in Landau gauge. (This is not the case in other gauges.) For example, in studies of the critical coupling for dynamical chiral symmetry breaking, a comparison of the results obtained using this approximation [30] with those obtained using more realistic vertex *Ansätze* [1,12,15,16,35] shows this approximation to be accurate to 5%. The improvements to this approximation are qualitatively important [1,12,15,16,35], being crucial to the restoration of multiplicative renormalizability and gauge covariance. However, herein a quantitatively reliable calculation scheme is sufficient and this is provided by Eq. (25) in the Landau gauge.

IV. PION MASS FORMULA

The unrenormalized BSE for the pion in the generalized-ladder approximation is, with unrenormalized n-point functions below denoted by a tilde,

$$\tilde{\Gamma}_\pi(p;P)+\int\frac{d^4q}{(2\pi)^4}\,\frac{4}{3}g^2\tilde{D}_{\mu\nu}(p-q)\,\gamma_\mu\tilde{S}(q+\tfrac{1}{2}P)\tilde{\Gamma}_\pi(q;P)\tilde{S}(q-\tfrac{1}{2}P)\,\gamma_\nu=0, \tag{26}$$

where $P=p_1+p_2$ is the total momentum and $p=(p_1-p_2)/2$ the relative momentum of the $\bar{q}$-q pair.

For the pion it is a good approximation [26,36] to write

$$\tilde{\Gamma}_\pi(p;P)=\gamma_5\tilde{F}(p^2,P^2), \tag{27}$$

in the sense that $\tilde{\Gamma}_\pi(p;P)$ is a general pseudoscalar 4×4 matrix and the right-hand side is, pointwise, a good approximation to it and the inclusion of the other allowed Dirac amplitudes alters the mass eigenvalue by $<1\%$. With this approximation Eq. (26) becomes $[C_2(R)=(N_c^2-1)/(2N_c)=4/3$ for $N_c=3]$

$$8\,N_c\tilde{F}(p^2,P^2)=3\,C_2(R)\int\frac{d^4q}{(2\pi)^4}\tilde{\Delta}(p-q)\tilde{H}(q;P), \tag{28}$$

with

$$\tilde{H}(p;P)=8\,N_c(p_+\cdot p_-\tilde{\sigma}_V^+\tilde{\sigma}_V^-+\tilde{\sigma}_S^+\tilde{\sigma}_S^-)\tilde{F}(p^2,P^2), \tag{29}$$

where we have defined $p_\pm=p\pm P/2$,

$$\tilde{\sigma}_V^\pm=\frac{\tilde{A}(p_\pm^2)}{p_\pm^2\tilde{A}(p_\pm^2)^2+\tilde{B}(p_\pm^2)^2}\quad\text{and}\quad\tilde{\sigma}_S^\pm=\frac{\tilde{B}(p_\pm^2)}{p_\pm^2\tilde{A}(p_\pm^2)^2+\tilde{B}(p_\pm^2)^2}. \tag{30}$$

Equation (28) is a convolution in four dimensions and can be rewritten as

$$0=8\,N_c\tilde{F}_P(x)-C_2(R)\;3\tilde{\Delta}(x)\tilde{H}_P(x), \tag{31}$$

with $\tilde{H}_P(x)$ the Fourier transform, with respect to p, of $\tilde{H}(p;P)$.

Multiplying the right-hand side of Eq. (31) by $(\tilde{F}_P(-x)/[3C_2(R)\tilde{\Delta}(x)])$ one can construct

$$\Pi_\pi(P)\equiv\int d^4x\left(\frac{8N_c}{3C_2(R)}\,\frac{\tilde{F}_P(-x)\tilde{F}_P(x)}{\tilde{\Delta}(x)}-\tilde{F}_P(-x)\tilde{H}_P(x)\right). \tag{32}$$

In the auxiliary-field bosonization of the global color-symmetry model [1,37] the effective action contains the term

$$\int d^4x\;d^4y\;\;\pi^i(x)\Pi_\pi(x-y)\pi^i(y), \tag{33}$$

with $\pi^i(x)$ a local field variable identified with the pion field. One sees from this that $\Pi_\pi(P)$ plays the role of the inverse propagator for the composite pion field. Further, at the solution of the BSE, $P^2=-m_\pi^2$, Eq. (31) is satisfied and hence

$$\Pi_\pi(P^2=-m_\pi^2)=0. \tag{34}$$

It has been shown [38] that for $m_0=0$ the unrenormalized BSE has a massless, $P^2=0$, solution with

$$\tilde{F}_P(x)=\tilde{F}_{P=0}(x)=\tilde{B}_{m_0=0}(x), \tag{35}$$

which is the manifestation of Goldstone's theorem in the DSE approach. Using this as an approximation for $P^2=-m_\pi^2\neq 0$, via the unrenormalized DSE,

$$\tilde{B}_{m_0=0}(x)=3\,C_2(R)\tilde{\Delta}(x)\tilde{\sigma}_S^{m_0=0}(x), \tag{36}$$

one obtains

$$\Pi_\pi(P)\approx\int d^4x\;\tilde{B}_{m_0=0}(x)[8N_c\tilde{\sigma}_S^{m_0=0}(x)-\tilde{H}_P(x)]\equiv\overline{\Pi}_\pi(P). \tag{37}$$

This is manifestly invariant under renormalization and hence one may write

$$\overline{\Pi}_\pi(P)=\int d^4x\;B_{m_R=0}(x)[8N_c\sigma_S^{m_R=0}(x)-H_P(x;m_R)], \tag{38}$$

with every quantity on the right-hand side renormalized (σ_S and H have the same form but with unrenormalized quantities replaced by renormalized ones) and evaluated with $m_R\neq 0$ unless otherwise specified.

As remarked above, $\Pi(P^2=-m_\pi^2)=0$ at the solution of the BSE. Equation (38) therefore allows one to obtain a simple pion mass formula derived from the generalized-ladder approximation to the BSE and expressed solely in terms of the massless and massive renormalized, vacuum, dressed quark propagators.

For the pion (because $m_\pi^2\simeq 0$) it is a good approximation to write

$$\overline{\Pi}_\pi(P)\approx\overline{\Pi}_\pi(0)+P^2N_\pi^2, \tag{39}$$

where

$$N_\pi^2=\left(\frac{d}{dP^2}\overline{\Pi}_\pi(P^2)\right)_{P^2=0}=\frac{N_c}{8\pi^2}\int_0^{\Lambda^2}ds\;sB_{m_R=0}(s)^2(\sigma_V^2-2[\sigma_S\sigma_S'+s\sigma_V\sigma_V']-s[\sigma_S\sigma_S''-(\sigma_S')^2]-s^2[\sigma_V\sigma_V''-(\sigma_V')^2]), \tag{40}$$

with the primes denoting differentiation with respect to $s=p^2$ and σ_V and σ_S evaluated at m_R. This is just the conventional, generalized-ladder approximation Bethe-Salpeter amplitude normalization constant, calculated neglecting small ($\sim$ 2%) $O(m_\pi^2)$ corrections.

We note that if $A(p^2)\equiv 1$, $N_\pi=f_\pi$. In general, the approximation $N_\pi\approx f_\pi$ is accurate to within 10% and the difference is a measure of the error introduced by the approximation of Eq. (27) [1]. (Also see Table I.)

Equation (39) yields the explicit pion mass formula [39]

$$m_\pi^2N_\pi^2=\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}ds\;s\frac{B_{m_R=0}(s)}{B_{m_R\neq 0}(s)}[B_{m_R\neq 0}(s)\sigma_S^{m_R=0}(s)-B_{m_R=0}(s)\sigma_S^{m_R\neq 0}(s)]. \tag{41}$$

One notes immediately that, for a given value of m_R, $m_\pi^2\to\text{const}<\infty$ as $N_c\to\infty$ and that, for arbitrary N_c, $m_\pi^2\to 0$ as $m_R\to 0$. Further, if the DSE is solved with a quark-gluon vertex that ensures multiplicative renormalizability, then m_π^2 is a renormalization point invariant and the result is independent of the cutoff Λ^2. The integral on the right-hand side of Eq. (41) is convergent in the limit $\Lambda^2\to\infty$.

From Eq. (41) one can recover what is sometimes called the Gell-Mann-Oakes-Renner relation in the form

$$m_\pi^2f_\pi^2=-m_R^{\mu^2}\langle\bar{q}q\rangle_{\rm vac}^{\mu^2}, \tag{42}$$

where

$$-\langle\bar{q}q\rangle_{\rm vac}^{\mu^2}=\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}ds\,s\,\sigma_S^{m_R=0}(s), \tag{43}$$

which is the customary definition of the vacuum condensate. However, in terms of the nonperturbatively dressed quark propagator, equality between the integrands requires the following *ad hoc* and mutually incompatible "approximations": $\forall s$,

$$B_{m_R=0}(s)\approx B_{m_R\neq 0}(s), \tag{44a}$$

$$\sigma_S^{m_R=0}(s)\approx\sigma_S^{m_R\neq 0}(s), \tag{44b}$$

$$B_{m_R\neq 0}(s)\approx m_R+B_{m_R=0}(s), \tag{44c}$$

which yields Eq. (42) when one makes the additional approximation $N_\pi\approx f_\pi$, discussed above. That these are bad "approximations" is clear; for example, Eq. (44a) has the effect of replacing a convergence factor in the integrand by unity and it is incompatible with Eq. (44c). As elucidated in Ref. [40], Eq. (42) can only be obtained if the (renormalized) current quark mass is treated strictly as a perturbation. The inadequacy of Eqs. (42) and (43) is only exposed by a careful treatment of the Dyson-Schwinger and Bethe-Salpeter equations.

We emphasize that Eq. (41) is completely consistent with the general arguments of Ref. [41]. It is derived from the generalized-ladder BSE and measures the expectation value of the explicit chiral symmetry breaking term in the pion state under the approximation that Eq. (35) is valid for $P^2 \neq 0$, which is why the right-hand side involves only vacuum quantities: massless and massive, renormalized, vacuum, and dressed quark propagators.

We demonstrate below that Eq. (41) provides an extremely accurate estimate of the pion mass obtained by solving the pion BSE in generalized-ladder approximation. [See Eq. (53) and Table I.]

In our numerical studies we are interested in the subtractively renormalized Bethe-Salpeter amplitude $F(p;P)$. This is defined in terms of the regularized amplitude $F'(p;P)$ via

$$F(p;P) \equiv F'(p;P) - F'(\mu,P), \tag{45}$$

which, in generalized-ladder approximation, is obtained as the solution of

$$F'(p;P) = Z_2\, 3C_2(R) \int^{\Lambda} \frac{d^4q}{(2\pi)^4} \Delta(p-q)(q_+ \cdot q_- \sigma_V^+ \sigma_V^- + \sigma_S^+ \sigma_S^-) F(q;P). \tag{46}$$

It is clear that all corrections to free-field behavior vanish at the renormalization point, i.e., $F(p;P)|_{p^2=\mu^2} = 0$.

Upon comparison with the DSE for $B(p^2)$ in Sec. III, it is clear that in the chiral limit ($m_R = 0$) one has

$$F(p;P) = B_{m_R=0}(p), \tag{47}$$

i.e., that Goldstone's theorem is manifest [38].

One may solve Eq. (46) numerically by introducing an eigenvalue $\lambda(P^2)$ on the right-hand side. This yields an equation that has a solution at every value of P^2. The equation can then be solved repeatedly until that P^2 is found for which $\lambda(P^2) = 1$.

The eigenvalue and eigenvector are determined by employing the Tschebyshev decomposition

$$F(p;P) = \sum_{i=1}^{\infty} F_i(p^2, P^2) U_i(\cos\beta) \tag{48}$$

and solving for the Tschebyshev moments of $F(p;P)$, which are obtained via

$$F_i(p^2, P^2) = \frac{2}{\pi} \int_0^{\pi} d\beta\, \sin^2\beta\, U_i(\cos\beta) F(p,P). \tag{49}$$

In practice we only keep the lowest moment $F_0(p^2, P^2)$, neglecting the coupling to the higher moments. This is a very good approximation for the pion [26].

For an on-shell pion $P^2 = -m_\pi^2$ and hence the right-hand side of Eq. (46) samples the quark propagator at complex values of its argument. To avoid solving the quark DSE off the real-p^2 axis we expanded $(q_+ \cdot q_- \sigma_V^+ \sigma_V^- + \sigma_S^+ \sigma_S^-)$ to $O(P^2)$ and solved the resulting equation, which involves derivatives of the propagator at real $p^2 \geqslant 0$.

V. NUMERICAL RESULTS AND PHENOMENOLOGY

We have two parameters: the mass scale m_t in the gluon propagator, which marks the transition point between the perturbative and nonperturbative domains, Eq. (6), and m_R, the renormalized current quark mass. We varied these parameters in order to obtain the best χ^2 fit to the pion observables: m_π [calculated using Eq. (41)], the weak pion decay constant [1]

$$f_\pi = \frac{N_c}{4\pi^2} \int_0^{\Lambda^2} ds\, s \frac{1}{N_\pi} F_0(s, P^2) \times [\sigma_V \sigma_S + \tfrac{1}{2} s(\sigma_V' \sigma_S - \sigma_V \sigma_S')], \tag{50}$$

r_π, and the π-π scattering lengths a_0^0, a_0^2, a_1^1, a_2^0, expressions for which are given in Ref. [2].

At each pair of parameter values the quark DSE was solved numerically with $\mu = 48\ \mathrm{fm}^{-1} = 9.47$ GeV, which is large enough to be in the purely perturbative domain, and $\Lambda = 2^{18}\ \mathrm{fm}^{-1} \sim 5461\mu$. The results were almost independent of the cutoff, doubling it leading only to a 3% change in

TABLE I. Observables calculated using the parameter values in Eq. (52). The experimental values of the π-π scattering lengths are discussed in Refs. [2,42]. The other experimental values are taken from Ref. [43]. $[B_0]$ indicates that the quantity was calculated using the approximation of Eq. (47) while $[F_0]$ indicates it was calculated using the zeroth order Tschebyshev moment obtained in a direct solution of the BSE, Sec. IV A. The anomalous coupling $g_{\pi^0\gamma\gamma}$ is discussed in Ref. [3]. See Sec. V A, for a discussion of the ρ-meson observables. The difference between N_π and f_π is a measure of the accuracy of the approximation of Eq. (27). That between the calculated and experimental values of $r_\pi N_\pi$ is a measure of the importance of final-state π-π interactions and photon-ρ-meson mixing [44]. Final-state π-π interactions are also neglected in the calculation of the scattering lengths [2] and $g_{\rho\pi\pi}$. Pion-loop corrections to m_ρ are of the order of 5% [45,46].

	Calculated	Experiment
$m_\pi^{\text{mass formula:}}[B_0]$	138.7 MeV	138.3± 0.5
$m_\pi^{\text{mass formula:}}[F_0]$	137.2	
$m_\pi^{\text{BS equation}}$	139.5	
$f_\pi[F_0]$	92.4 MeV	92.4± 0.3
$f_\pi[B_0]$	92.3	
$N_\pi[F_0]$	102	
$r_\pi[F_0]N_\pi[F_0]$	0.24	0.31± 0.004
$a_0^0[F_0]$	0.16	0.21± 0.02
$a_0^2[F_0]$	-0.041	-0.040 ± 0.003
$a_1^1[F_0]$	0.028	0.038 ± 0.003
$a_2^0[F_0]$	0.0022	0.0017 ± 0.0003
$a_2^2[F_0]$	0.0013	
$g_{\pi^0\gamma\gamma}[F_0]$	0.45	0.50 ± 0.02
$m_\rho[F_0^\rho]$	0.971 GeV	0.770 ± 0.001
$g_{\rho\pi\pi}[F_0^\rho]$	4.07	6.07 ± 0.02

f_π, for example. Our results would have been completely independent of Λ if we had employed a vertex that preserves multiplicative renormalizability. This observation provides a quantitative measure of the violation of multiplicative renormalizability when the the rainbow approximation is used in Landau gauge. It is significantly worse in other gauges. As remarked above, rainbow approximation entails a loss of gauge covariance. Our experience suggests that our results would change by no more than 10% if we had used a dressed fermion–gauge-boson vertex that ensured gauge covariance of the fermion propagator [1,15,16,35].

The formulas for the observables were then evaluated using the solution obtained and the approximation that Eq. (47) is valid for $m_R \neq 0$. After obtaining the optimal values of the parameters we recalculated the observables using the pion Bethe-Salpeter amplitude calculated as described in Sec. IV A. We found numerically that

$$F_0(p^2;P^2) \approx B_{m_R=0}(p^2). \tag{51}$$

The best χ^2 fit was obtained with

$$m_t = 0.69 \text{ GeV} \quad \text{and} \quad m_R = 1.1 \text{ MeV}. \tag{52}$$

We also carried out an extended χ^2 fit where the ratio of the coefficients of the two terms in Eq. (4) was allowed to vary. In this case the best χ^2 was obtained with the value of m_t in Eq. (52) and a ratio that agreed with that in Eq. (4) to within 2%. The data therefore requires both terms in the propagator and the cancellation of long-range effects described in Eq. (6).

The observables calculated with these parameter values are presented in Table I. One observes immediately that our one parameter model for the gluon propagator provides a good description of low energy pion observables. This improves upon the results of Refs. [2–5], in which the quark propagator was parametrized, and illustrates the connection, suggested in these articles, that may be made between hadronic observables and the quark-quark interaction.

We have made a direct comparison on the spacelike-p^2 axis of the numerical solutions for σ_V and σ_S obtained herein with the parametrized forms used in Ref. [3]. The agreement in form and magnitude is very good, which suggests that the one parameter model gluon propagator will also provide a good description of hadronic form factors.

One observes that the mass formula in Eq. (41) yields an accurate estimate of the mass obtained by solving the pion BSE. We find that, with parameters of Eq. (52), the right-hand side of Eq. (41) is well described by

$$m_\pi^2 N_\pi^2 = 2\,(0.45)^3 m_R + (2.6)^2 m_R^2 + 150\, m_R^3 \tag{53}$$

in the range $m_R \in [0,0.02]$ GeV, from which one may infer a value of $\langle \bar{q}q \rangle_\mu = -(0.45 \text{ GeV})^3$. At the value of m_R in Eq. (52) the term linear in m_R contributes almost 96% of the total. We see, therefore, that Eq. (41) entails $m_\pi^2 \propto m_R$, for small m_R, but that the constant of proportionality is not given by the usual definition of the vacuum quark condensate, Eq. (43).

Our one parameter model for the gluon propagator explicitly *excludes* the $\ln[k^2]$ corrections associated with the anomalous dimensions in QCD. It is therefore inappropriate to directly compare $m_R(\mu)$ in Eq. (52) with the QCD evolution of the commonly quoted value of $m_{\mu=1\text{ GeV}} \approx 7.5$ MeV [43]. (This entails that the same is true of $\langle \bar{q}q \rangle_\mu$.) We note that replacing $(\pi d)/k^2$ by $\alpha_S^{\text{two loop}}(k^2)/k^2$ in Eq. (4) would lead to a suppression of the tail of the quark mass function, thereby requiring a larger value of m_R to reproduce the pion mass and a commensurate change in m_t. This represents a quantitative improvement of our model but would not change its qualitative features.

A. ρ-meson observables

We have employed our model gluon propagator in a preliminary study of ρ-meson properties.

The regularized, generalized-ladder approximation to the ρ-meson BSE is

$$F'_\rho(p;P) = Z_2\, 3C_2(R) \int^\Lambda \frac{dq}{(2\pi)^4}\, g^2\, D_{\mu\nu}(p-q)\, \tfrac{1}{12} \mathrm{tr}[\gamma_\alpha i \gamma_\mu S(q_+) i T_\alpha(P) S(q_-) \gamma_\nu] F_\rho(q;P), \tag{54}$$

where $[T_\mu(P) = \gamma_\mu + \gamma \cdot P P_\mu / m_\rho^2]$. The subtractively renormalized amplitude is given by $F_\rho(p;P) = F'_\rho(p;P) - F'_\rho(\mu;P)$. We neglected the other Dirac structures allowed in the vector-meson Bethe-Salpeter amplitude. For the ρ meson the error introduced by this truncation is approximately 10% [36]. The ρ and ω mesons are degenerate at this level of approximation. As for the pion, we project this equation onto the lowest Tschebyshev moment and solve for $F_0(p^2,P^2)$, neglecting the coupling to the higher moments. This is a good approximation for the ρ meson [26].

In this preliminary study we have only solved the quark DSE at real p^2. For an on-shell ρ meson $P^2 < 0$ and hence Eq. (54) samples the quark propagator at complex values of p^2. To obtain an approximate solution of Eq. (54), without solving the quark DSE at complex p^2, we introduced an eigenvalue $\lambda(P^2)$ on the right-hand side of Eq. (54) and solved this equation at spacelike values of P^2, thereby obtaining $\lambda(P^2 > 0)$. For $0 < P^2 < 10\ \text{fm}^{-2}$ the results could be described by the quadratic (in P^2)

$$\lambda(P^2) = 0.44 - 0.021 P^2 + 0.000\,076 P^4, \tag{55}$$

with a standard deviation of 0.000 044. We compared this with both linear and cubic fitting forms: It provides a smaller standard deviation than the linear form and is monotonic, whereas the cubic is not. The value of P^2 for which this algebraic form of $\lambda(P^2)=1$ provides the mass estimate presented in Table I.

The calculated ρ-meson Bethe-Salpeter amplitude is much narrower in momentum space than that of the pion, in agreement with the results of Ref. [26].

The calculation of $g_{\rho\pi\pi}$ proceeds in a similar manner. In the generalized impulse approximation the $\rho\pi\pi$ coupling can be expressed in terms of a nonlocal coupling functional, $N_\mu(p,q)$, which is discussed in Ref. [47]. This expression is used to evaluate $g_{\rho\pi\pi}(P^2)$ at $0<P^2<10\ \mathrm{fm}^{-2}$. The results were fitted and extrapolated to the calculated mass-shell point. The best fit was obtained with

$$g_{\rho\pi\pi}(P^2)=1.15-0.076P^2+0.0013P^4-0.000\,022P^6, \tag{56}$$

giving a standard deviation of 0.000 23. This form provides a smaller value of the standard deviation than either a linear or quadratic form and is monotonic whereas the quartic is not. The value obtained at the calculated on-mass-shell point is given in Table I.

These calculations are only a first step. They serve merely to indicate that our one parameter model gluon propagator, which was fitted to pion observables, can reasonably be expected to provide a good description of other observables too.

B. Confinement

We have also solved the quark DSE for real $p^2<0$. There is no singularity on the real-p^2 axis. The solution therefore does not have a Lehmann representation and hence may be interpreted as describing a confined particle.

A plot of $1/[p^2+M(p^2)^2]$, which for a free particle would have a pole at the mass-shell point, has a broad resonance like peak centred at $p^2\approx-(0.55)\ \mathrm{GeV}^2$. This admits an interpretation as the ‘‘constituent-quark mass’’ in our model.

The form of our solution is suggestive of a pair of complex conjugate poles or branch points with timelike real parts and large magnitude imaginary parts. We have made no attempt to confirm this. A thorough study must identify whether this structure is an artifact of the rainbow approximation, which is known to be associated with unexpected behavior of the fermion propagator in the complex plane [48–52] that is modified when the vertex is dressed [53].

VI. SUMMARY AND CONCLUSIONS

Using a confining, one parameter model form for the gluon propagator, Eq. (4), which incorporates the essence of the solution of realistic, approximate gluon Dyson-Schwinger equations (DSE's), we solved the renormalized, rainbow approximation quark DSE and subsequently the renormalized, generalized-ladder approximation π- and ρ-meson Bethe-Salpeter equations (BSE's). We varied the parameter in the gluon propagator, m_t, which is a mass scale that marks the transition between the perturbative and nonperturbative domains, and the renormalized current quark mass, and obtained a good description of a range of π- and ρ-meson observables. The value of m_t was not known *a priori*. Good agreement with the data *required* $m_t\sim700$ MeV, which corresponds to a length of ~0.3 fm.

In studying the pion BSE we were led to a mass formula for the pion, Eq. (41), expressed solely in terms of the massive and massless quark propagators. This formula provides a very accurate estimate of the pion mass. It is valid to all orders in m_R, the renormalized current quark mass, and for $m_R<20$ MeV the nonlinear terms provide a contribution of no more than $\sim10\%$.

We obtained numerical solutions of the quark DSE on the timelike-p^2 axis, which showed the quark propagator to have no singularity on the real-p^2 axis in our model. We found evidence to suggest that, as a function of p^2, the quark propagator has a pair of complex conjugate poles or branch points with timelike real parts and large imaginary parts. Such a propagator does not have a Lehmann representation and admits the interpretation of describing a confined particle.

Our study illustrates the manner in which the DSE's can be used to develop a semiphenomenological approach to QCD that incorporates the perturbative, large spacelike-k^2 behavior known from renormalization group studies and, via an economical parametrization, extrapolates this into the nonperturbative, small spacelike-k^2 domain. This efficacious, nonperturbative approach allows for the correlation of a large range of observables via very few parameters, which it may be possible to relate to the fundamental parameters of QCD.

ACKNOWLEDGMENTS

The preparation of this manuscript has benefited from useful discussions with A. Bender, F. T. Hawes, and A. G. Williams. This work was supported by the U.S. Department of Energy, Nuclear Physics Division, that of M.R.F. under Contract No. DE-FG06-90ER40561, and that of C.D.R. under Contract No. W-31-109-ENG-38. The calculations described herein were carried out using a grant of computer time and the resources of the National Energy Research Supercomputer Center.

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