Three-body reaction theory in a model space

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(Received 24 January 1995)

A distorted waves Faddeev method for three-body systems is introduced, following Bencze and Redish and Birse and Redish, by using a multipole expansion to select parts of the interactions in rearrangement channels to serve as three-body distorting potentials in the first Faddeev equation. The consequence is a greatly reduced role for the second and third Faddeev equations. Truncating the multipole expansion and discarding the second and third equations reduces us back to the well-known continuum discretized coupled channels (CDCC) method. The relations among these methods are discussed. Despite the difficulties of the CDCC method, it is much easier than a full Faddeev calculation.

PACS number(s): 21.45.+v, 25.10.+s, 24.10.Eq

I. INTRODUCTION

Many physical systems are modeled as three-body systems, giving rise to the idea that they should properly be analyzed by some version of the Faddeev theory [1]. Unfortunately, Faddeev equation methods, with their coupled equations, are complicated [2], with the result that these methods have only been applied for a few, simple physical systems or for systems with simple, separable interactions. Important interactions tend to enter only as off-diagonal operators that connect the equations [1,2].

Meanwhile, the continuum discretized coupled channels (CDCC) method, introduced originally as an approximation for stripping [3–7], has gradually been improved, until it has become a competitor to the Faddeev approach [8–12]. We carry out the coupled channels method in configuration space, where the physicist's imagination tends to be most comfortable. However, we do not experience the boundary condition problems of a completely configuration-space formulation [15–17], because our use of a channel expansion enables the outgoing boundary condition to be applied in one channel at a time.

The new theory originated from coupled channels approximations of deuteron stripping, regarded as a three-body system. It came as a surprise that a modest reinterpretation of CDCC calculations converts them into variations of the Faddeev method [6,7]. This result ends the long controversy [9–12] about convergence of CDCC calculations.

The new theory is obtained by transforming all coordinates to the same degrees of freedom as in the entrance channel, in a kind of resonating group [13] approach. A flexible angular momentum truncation is then introduced. Transformation between the old and new methods is purely kinematic; it can be performed without solving any equations of motion.

Our method resembles closely the use of distorted waves in the Faddeev theory, advocated some years ago by Bencze and Redish and by Birse and Redish [14]. However, our distorted waves are constructed by a procedure of partial wave expansion.

Section II discusses the new method in detail. It shows how the modifications of ψ_0 and the reduction of ψ_1 and ψ_2 to unimportance take place. It discusses multipole expansions of the transformed potentials. Section III takes up the problem of practical coupled-channels solutions of the truncated equation of motion for the system. It also discusses the long tails of the transformed potentials and their relation to discretization of the breakup continuum. It discusses special values of the breakup momenta and the use of complex potentials in nuclear physics. Section IV is a summary and a discussion of the relation of our method to other methods of calculation.

Appendix A reviews the stationary phase method for the evaluation of integrals. Appendix B considers the properties of a projection operator used in this article. Appendix C considers the procedure to be used if the kinematics differs from the special case treated here. Appendix D reviews methods for obtaining practical results from this theory.

II. EXACT THEORY IN MODEL SPACE

A. Structure of the theory

In a discussion [6] of iterative procedures for the three-body model,

$$[E-K-V(r)-U_1(r_1)-U_2(r_2)]\psi=0, \qquad (2.1)$$

we pointed out that the exact Faddeev equations for this system could be put into a form related to the CDCC model-space approximation by regarding model-space projections of the potentials $U_1(r_1)$, $U_2(r_2)$ as "compact distorting potentials" in the Faddeev equations, as in the discussions by Bencze and Redish and by Birse and Redish [14]. In general, particles 1 and 2, which may be composite, have coordinates

$\vec{r}_1$ and $\vec{r}_2$ with respect to an infinitely massive nucleus A located at the origin. (Infinite mass for A only simplifies the notation; it is not a necessary part of the theory. See Appendix C for details.) The kinetic energy operator for the active particles is $K_1+K_2=K$, the interaction potential between them is V, and the interaction "potentials" between A and the active particles (allowed to be nonlocal) are U_1 and U_2. For simplicity, we assume that V is a short-ranged central potential. We assume that the incident wave function in the entrance channel contains a bound state of particles 1 and 2. Spins are ignored.

"Distorted-wave" Faddeev differential equations for the above system are

$$[E-K-V(r)-\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda]\psi_\lambda=V[\psi_{1\lambda}+\psi_{2\lambda}], \quad (2.2)$$

$$[E-K-U_1(r_1)]\psi_{1\lambda}=\mathscr{P}_\lambda[U_1-\mathscr{P}_\lambda U_1\mathscr{P}_\lambda]\psi_\lambda+\mathscr{P}_\lambda U_1\psi_{2\lambda}, \quad (2.3)$$

$$[E-K-U_2(r_2)]\psi_{2\lambda}=\mathscr{P}_\lambda[U_2-\mathscr{P}_\lambda U_2\mathscr{P}_\lambda]\psi_\lambda+\mathscr{P}_\lambda U_2\psi_{1\lambda}, \quad (2.4)$$

with the condition

$$\psi_\lambda+\psi_{1\lambda}+\psi_{2\lambda}=\psi. \quad (2.5)$$

The model space is defined by the kinematic projection operator $\mathscr{P}_\lambda$ that selects low angular momenta $l\leqslant\lambda$ that are associated with the relative coordinate $\vec{r}=\vec{r}_1-\vec{r}_2$. Properties of the projector $\mathscr{P}_\lambda$ are discussed in Appendix B. Model-space projections of U_1 and U_2 appear as distorting potentials in (2.2)–(2.4).

There are two sets of internal coordinates in the above equations: $\vec{r},\vec{R}$ for the relative and center-of-mass coordinates of the active particles and $\vec{r}_1,\vec{r}_2$ for their individual displacements from A. Explicit transformations between these coordinates depend on the particle masses and will be avoided as far as possible in this paper. Evidently, "model space" is defined in terms of the coordinates $\vec{r},\vec{R}$.

In the standard Faddeev theory, with no distorting potentials ($\mathscr{P}_\lambda=1$), the Green's function for each Faddeev equation produces associated asymptotic bound two-body channels, with the result that component ψ_0 contains all the 1-2 bound states, ψ_1 all the 1-A bound states, and ψ_2 all the 2-A bound states. Three-body breakup appears in all three Faddeev components. The usual analyses of convergence associated with Eqs. (2.2)–(2.4) emphasize "compactness" of the right-hand side (RHS) of these equations. With $\mathscr{P}_\lambda=1$, compactness is produced in (2.3) and (2.4) by the short range of U_1 or U_2 in one variable and of the wave functions, especially ψ_0, in another; these effects combine to guarantee that the coupling terms are short ranged in configuration space in all directions. An important ingredient in proofs of compactness is the well-known $[r_1^2+r_2^2]^{-5/4}$ asymptotic decrease of the three-body breakup continuum.

In the distorted Faddeev theory, the asymptotically bound two-body channels are distributed among the Faddeev components in the same manner as above. Compactness of the coupling terms is maintained, because the "distorting potentials" $\mathscr{P}_\lambda U_1\mathscr{P}_\lambda$, $\mathscr{P}_\lambda U_2\mathscr{P}_\lambda$ are themselves compact [18].

B. Projection on model space

Model-space calculations are invariably performed in the framework of an expansion in the set of basis states

$$|k[lL]JM)\equiv|\phi_l(k,r)[Y_l(\hat{r}),Y_L(\hat{R})]_{JM}), \quad (2.6)$$

which we define in terms of the coordinates r,R of the entrance channel. The quantum numbers that label this set are $klLJM$. Here $\phi_l'(k,r)Y_{lm}(\hat{r})$ is the $k'lm$ wave function for the relative motion of 1-2 under potential V, with asymptotic momentum $\vec{k}'$ and energy $\varepsilon(k')$. One can also use Sturmians for this expansion [15]. Finally, the spherical harmonic $Y_{LM}(\hat{R})$ refers to the motion of the center of mass of 1,2.

The above basis is used to expand (2.2) for the first Faddeev component, giving

$$[E-\epsilon(k)-K_R]g^J_{\lambda lL}[P(k),\vec{R}]=\int d^3r\ d\hat{R}\{\phi_1(k',r)[Y_1(r),Y_L(R)]_{JM}\}^*\{\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda'\psi_0'+V[\psi_{1\lambda}+\psi_{2\lambda}]\}. \quad (2.7)$$

The expansion coefficients $g^J_{\lambda lL}$ of ψ_λ in the basis (2.6) are "channel functions" in our calculation. The wave numbers $P(k)$ are computed from k by requiring that the relative and center-of-mass motions conserve energy. Some procedures for calculating directly with (2.7) are discussed in Sec. III.

Since we are primarily concerned about the relation between the first Faddeev equation and the combined effects of the other two equations, it is convenient to add (2.3) and (2.4), to obtain the simpler coupled system

$$[E-K-V-\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda]\psi_\lambda=V\xi_\lambda, \quad (2.8)$$

$$[E-K-U_1-U_2]\xi_\lambda=[U_1+U_2-\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda]\psi_\lambda, \quad (2.9)$$

in which $\xi_\lambda\equiv\psi_{1\lambda}+\psi_{2\lambda}$. The further notation $U_\lambda=\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda'$ is also used below. The $V\xi_\lambda$ term in (2.8) expresses the reaction on ψ_λ by the breakup and rearrangement parts of ξ_λ.

Under the assumption that A has infinite mass and does not recoil, the same rearranged coordinates $\vec{r}_1,\vec{r}_2$ appear in (2.3) and (2.4) and again in (2.9), and the system of equations (2.8), (2.9) is equivalent to (2.2)–(2.4). (Also see Appendix C.) Like the original Faddeev equations, (2.8), (2.9), and their solutions are valid throughout the configuration space. Coupled differential equations provide a flexible framework in which to introduce the interactions and the boundary conditions. The combination of U_1 with U_2 on the LHS of (2.9) allows one interaction to be a distorting potential for the outgoing particle in a rearrangement channel and

the other to be the binding potential for the particle that is left behind.

The most obvious effect of the distorting potentials in (2.8), (2.9) is to present a counterterm to the potentials U_1+U_2 in (2.9), which significantly weakens the solution ξ_λ for low values of l. If λ is large enough, ψ_λ alone can be a good approximation over a large volume of space. This suggests a procedure for managing (2.8), (2.9): *Omit the $V\xi_\lambda$ term from (2.8) and carry a large enough value of the parameter λ to produce the desired accuracy in the evaluation of ψ_λ*. This procedure is already followed in CDCC calculations [3–5].

C. Significance of ξ_λ

The $V\xi_\lambda$ term decreases as λ increases, and the rearrangement and breakup effects move more directly into ψ_λ. We exploit this effect to discuss convergence. Our procedure will be to demonstrate that $V\xi_\lambda$ tends to become negligible as λ increases indefinitely. Since all S matrix elements can be obtained from the condition $V\xi_\lambda=0$ (Appendix D), this suffices to produce the required result.

To begin we solve (2.9) for ξ_λ, which gives

$$\xi_\lambda=[E^+-K-U]^{-1}(U-U_\lambda)\psi_\lambda. \tag{2.10}$$

Outgoing boundary conditions are appropriate in this equation, because by definition ψ_λ contains all the incident flux.

Our interpretation of (2.10) depends in an important way on the properties of the Green's function. Nevertheless, we note first that the overall magnitude of the RHS of the above equation is determined by the cancellation of low multipoles in the expression $U-U_\lambda$; it is a consequence of the structure of the multipole expansion.

The Green's function in (2.10) acts on the source function $(U-U_\lambda)\psi_\lambda$. The resulting expression has a hole near the origin, where the low multipoles of U cancel with those of U_λ. In the calculation of ξ_λ this hole cannot fill in with flux transferred at larger radii, because ξ_λ is known to be a purely outgoing wave function.

The hole created in ξ_λ has the significance of pushing the nonzero region of this function out to large radii, at least several nuclear radii. At the same time, because of the source function in (2.10), $(U-U_\lambda)\psi_\lambda=[U_1+U_2-\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda]\psi_\lambda$, one particle is compelled to remain near the target nucleus, and therefore the particles become well separated. Furthermore, our Green's function is outgoing and the source function $(U-U_J)\psi_\lambda$ on which it operates also tends to be outgoing; hence, there is no stationary point involved in its operation [19]. These effects combine to make ξ_λ small within the range of V.

Equation (2.10) limits the range of ξ_λ either for any one partial wave or for any limited set of partial waves, because the source then has a maximum "radius," beyond which ξ_λ becomes an outgoing "spherical wave." Cancellation in the above expression is likely to be particularly close and durable, because the terms that cancel approximately are derived from the same source potential.

Finally, we note that the manipulations above are only possible with distorted waves, for which the potential difference $U-U_\lambda$ is nonvanishing. Only distorted waves make the manipulations nontrivial.

Let us examine the iteration of our theory. Consider ψ, the complete wave function of the system, which satisfies the equation

$$[E-K-V-U]\psi=0, \tag{2.11}$$

and the λth approximation ψ_λ, which satisfies the equation

$$[E-K-V-\mathscr{P}_\lambda U\mathscr{P}_\lambda]\psi_\lambda=0. \tag{2.12}$$

From (2.11) one gets exactly

$$[E-K-V-\mathscr{P}_\lambda U\mathscr{P}_\lambda]\psi=\mathscr{P}_\lambda U(1-\mathscr{P}_\lambda)\psi. \tag{2.13}$$

Now, however, by the analysis given before, we recognize that $\mathscr{P}_\lambda U(1-\mathscr{P}_\lambda)\psi$ must be negligible in a domain $\mathscr{D}$, defined by $R<R_c$. We assume that we can find R_c in such a way that it is small enough to ensure that $\mathscr{P}_\lambda U(1-\mathscr{P}_\lambda)\psi$ is small in $\mathscr{D}$, so that partial waves with $l<\lambda$ are not much affected in $\mathscr{D}$ by those with $l>\lambda$. At the same time, R_c is large enough to ensure that the truncated tail of $\mathscr{P}_\lambda U\mathscr{P}_\lambda$ beyond R_c is small and smooth, so that the reflection of $\mathscr{P}_\lambda\psi_\lambda$ from outside is negligible. Then $\mathscr{P}_\lambda\psi$ satisfies approximately the same equation in $\mathscr{D}$ and approximately the same boundary conditions at $R=R_c$ as ψ_λ, so that $\psi_\lambda\approx\mathscr{P}_\lambda\psi$ in $\mathscr{D}$. The larger λ is, the larger R_c can be and the closer ψ_λ is to ψ.

An extra factor $\mathscr{P}_\lambda$ is inserted on the left in Eqs. (2.12), (2.13). This factor establishes consistency in the use of matrix notation. It establishes that the meaning of matrix labels is the same on the left as on the right. However, the use of this factor does not change the solutions of the equations above.

We also use a sharp cutoff at large radii for integrations in model space, so that $\mathscr{O}_\lambda$ in the truncated Schrödinger equation is effectively treated as $\mathscr{O}_\lambda\approx\mathscr{O}_\lambda\theta(R_c-R)$. We also employ discretization, which replaces an integration over the continuum by a sum over bins. We easily find that the effective range of the continuum-continuum interactions is small enough so that an excessive number of bins is not required. The same remark applies to the truncation errors, with the additional feature that the adiabatic theorem must be invoked [24].

The individual terms in the partial wave series decrease with l', both because the Clebsch-Gordan coefficients decrease with l' and because of the rapid convergence of the multipole expansion for the potential.

D. Expansion of potentials

A major step in the solution of (2.8) is the introduction of entrance channel coordinates in the potentials $U_1(r_1)$ and $U_2(r_2)$, giving $U_1(|\vec{R}-\vec{r}/2|)$ and $U_2(|\vec{R}+\vec{r}/2|)$. (Again, the notation is simplified by assuming particles 1 and 2 have equal masses.) This is followed by the expansion of U_1, U_2 in multipoles of the new coordinates. The expansion coefficient is

$$U_l(r,R)\equiv\int_{-1}^{1}d\mu\ P_l(\mu)U(\rho); \tag{2.14}$$

here, $\mu\equiv(\hat{r}\cdot\hat{R})$.

Equation (2.14) is convenient as it stands for low multipoles, for which it can be evaluated numerically. The low multipoles of U_J depend on details of the shape of the potential. However, for potentials of not too bizarre shapes, the convergence to the forms that apply for large l is rapid.

To treat the higher multipoles, it is convenient to take $\rho \equiv [R^2 + r^2/4 - \mu r R]^{1/2}$ as the independent variable, with $x = |r/2 - R|$, giving

$$U_l(r,R) = (R^2 + xR)^{-1} \int_x^{2R+x} \rho \, d\rho \; U(p) P_l \times \left(1 + \frac{x^2 - \rho^2}{2R^2 + 2Rx} \right). \tag{2.15}$$

We note that the principal role of x is to cut off the lower end of the integral. The integral vanishes rapidly as this lower limit becomes comparable to or exceeds the range of $U(\rho)$. Keeping this in mind, we examine the integral for the special case $x = 0$, along the "center line" of a contour graph of $U_l(r,R)$:

$$\bar{U}_l(r,R) \equiv R^{-2} \int_0^{2R} \rho \, d\rho \; U(\rho) P_l(1 - \rho^2/2R^2). \tag{2.16}$$

Aside from the overall decrease with R^{-2}, the magnitude of the integral (2.16) is determined by the interplay between the short-range function $U(\rho)$ and the oscillatory function P_l.

There are three cases to consider.

(i) For R less than or comparable to the range of U, several cycles of P_l lie within U, and even for moderate values of l the integral averages to a very small value.

(ii) When R becomes larger, the range of U limits the range of integration and the range of the argument of P_l becomes increasingly restricted. In turn, this restricts the oscillations of P_l and the integral reaches a broad maximum near $R = l \times$(range of U). At this maximum the full range of U just extends over the full first peak of the Legendre function.

(iii) Beyond the maximum, R becomes so large that further variations of the argument of P_l and of the integral are suppressed and the function $U_l(r,R)$ decreases toward zero as R^{-2}.

We can now restore the $|\vec{R} - \vec{r}/2|$ lower limit of the integral in (2.16). Overall, we see that as l increases, the peak of the Legendre transform moves outwards along the center line of a contour graph in the r,R plane and the peak stretches out along this line; the transverse width of the peak remains unchanged, independent of l or R. Peaks for successive multipoles overlap radially to a considerable extent.

It is interesting to apply the analysis of $\mathscr{P}_\lambda$ that is given in Appendix B. There it is shown that applying the operator $\mathscr{P}_\lambda$ to a function of ρ is equivalent to multiplying that function by an angle-averaging operator. Hence, wherever $\mathscr{P}_\lambda$ appears, we can just as well insert the averaging operation. Furthermore, the averaging operation is localized in the "forward" direction; thus, if $\vec{r}'$ is localized in a given direction, then $\vec{r}$ will tend to be localized in that same direction. This localization is more precise the larger λ is.

III. ASPECTS OF THE CALCULATION

Techniques for the practical calculation of component ψ_λ in model space as λ becomes large are more important than its coupling to the secondary component ξ_λ. We therefore omit the $V\xi_\lambda$ term from (2.7) and look for numerical procedures for solving the truncated equation.

A. Basic calculations in model space

The outgoing-wave Green's function for the LHS of (2.7) is applied to the source term on the RHS. We get

$$[E - \epsilon(k) - K_R] g^J_{\lambda lL}(P(k),R) \approx \int d^3r d\hat{R} \{\phi_l k, r)[Y_l(\hat{r}), Y_L(\hat{R})]_{JM}\}^* \times \mathscr{P}_\lambda (U_1 + U_2) \mathscr{P}'_\lambda \psi^J_{\lambda' lL} \, . \tag{3.1}$$

The $g^J_{\lambda lL}(P(k),R)$ are the *expansion coefficients* of ψ_λ in the basis chosen in Sec. II. Because the basis states are energy eigenstates governed by $V(r)$, the $g^J_{\lambda lL}$ can also be interpreted as model space *channel wave functions* for elastic and inelastic scattering from the entrance channel, and they can be required to satisfy outgoing wave boundary conditions in the coordinate R. The wave numbers $P(k)$ are computed from k by the requirement that the relative plus center-of-mass motions conserve energy.

Accurate calculation with (3.1) demands a coupled channels expansion of ψ_λ, in which this undetermined function is expressed in terms of a basis of states that have definite values of J, l', L', definite states of relative motion of particles 1,2, and unspecified relative motion of particle A and the center of mass of 1,2. In general, particles 1,2 will have continuous spectra, and so this step of going to a coupled channels expansion will also involve discretizing those continua.

In the lowest approximation, we take (3.1) in monopole order, with $\lambda' = 0$. While this does not quite reduce (3.1) to an expression involving only local potentials, it does convert (3.1) to a form that has a convenient expansion in two-body eigenfunctions. Such eigenfunction expansions are known to give excellent descriptions of elastic scattering.

The basis states for (3.1) use energy eigenstates for the $\vec{r}$ degree of freedom and configuration space for the $\vec{R}$ degree of freedom. Transformation to a configuration space basis for both degrees of freedom would require an integration over the channel momentum k. This integral is traditionally performed approximately by the method of stationary phase. (This stationary phase integration produces the well-known radial asymptotic form $\mathscr{R}^{-5/4} e^{i\mu \mathscr{R}}$ of the wave function for three-body breakup.) Unfortunately, it has been shown by accurate numerical studies of integrations over k that it may be necessary to go to $r_1, r_2 \sim$thousands of fm to come near the asymptotic result. This is a major difficulty for numerical calculations performed entirely in configuration space [15–18].

B. Special values of continuum momenta

Certain classes of basis states might be thought to give difficulties in the calculation of the associated channel wave

functions. First, we consider basis states whose energy $\epsilon(k)$ is greater than the total energy E. The channel energy becomes $E(k)<0$, the channel momentum becomes $P(k)=i|P(k)|$, imaginary, and the channel wave functions decrease exponentially at large R. The principal comment about such "closed channels" is that their inclusion in the analysis is *necessary* for the set of basis states to be complete. Beyond this, experience with model-space calculations for deuteron-induced reactions shows that for total energies below about 100 MeV the excitation of closed channels significantly affects the numerical results [20,21].

The threshold case $P(k)=0$ is more interesting. This $P(k)$ vanishes when the relative energy equals the total energy. It might be thought that $\mathscr{I}^J_{lL}(P,R)$ would vanish when $P=0$. However, $P(k)$ labels a Green's function, rather than a basis function, and the channel wave function need not vanish. Instead, Eq. (3.1) becomes

$$-K_R\mathscr{I}^J_{lL}(0,R)=\mathscr{F}(R), \tag{3.2}$$

with

$$\mathscr{F}(R)\equiv\int d^3r\ d\hat{R}\{\phi_l(k,r)[Y_l(\hat{r}),Y_L(\hat{R})]_{JM}\}^* \times\mathscr{P}_\lambda(U_1+U_2)\mathscr{P}_\lambda\psi_0(\vec{r},\vec{R}), \tag{3.3}$$

or

$$\left[\frac{d^2}{dR^2}-\frac{L(L+1)}{R^2}\right]R\mathscr{I}^J_{lL}(0,R)=-(2M/\hbar^2)R\mathscr{F}(R). \tag{3.4}$$

The solution is

$$R\mathscr{I}^J_{lL}(0,R)=-(2M/\hbar^2)(2L+1)^{-1}\times\int_0^R(R_<^{L+1}/R_>^L)\cdot R'\mathscr{F}(R')dR', \tag{3.5}$$

a straightforward continuation of the results for $P\neq0$. For large values of L, the Green's function becomes a very narrow function of R. On the other hand, for $L=0,1$, the convergence of (3.5) must be affected by the properties of $\mathscr{F}(R')$.

The $\mathscr{I}^J_{lL}(0,R)$ functions produced by (3.5) are standing waves, for which the outgoing wave boundary condition, emphasized in earlier sections of this paper, is best reinterpreted as a *no-incoming-wave* condition. The kinetic energy becomes zero in the external region, an effective discontinuity, and there is 100% reflection of the wave function. The standing wave is confined near $R=0$. The solutions of (3.5) are also rather confined if the channel energy is only slightly positive.

Finally, we consider relative motion states with $k\approx0$. In this paper they are defined to have the asymptotic form

$$\phi_l(k,r)\rightarrow r^{-1}\sin(kr+\delta_l-l\pi/2).$$

Then,

$$\langle\phi_l(k)|\phi_l(k')\rangle=\int_0^\infty r^2\ dr\ \phi_l(k,r)\phi_l(k',r)=(\pi/2)\delta(k-k').$$

These equations define the radial dependence and normalization of the overlap integrals. It is significant that k does not appear in them as a coefficient. We see that for any finite radius r the basis functions all vanish in the limit $k\rightarrow0$, and therefore the channel function for the $k=0$ channel vanishes.

C. Complex potentials

In a previous publication [22], we argued that the validity of the CDCC approximate solution of the three-body Schrödinger equation could be regarded as a result of the large imaginary parts in nuclear single-particle potentials. (For an opposing analysis, see [12].) We remarked that derivations of nuclear three-body models from more fundamental many-body theories show that optical potentials are strongly state dependent and their imaginary parts should vanish for bound states. However, this fact is largely ignored in CDCC model-space calculations. We further showed that retaining the imaginary potentials in rearrangement bound states damps the wave function of the outgoing particle. Thus the (spurious) bound state absorptive potentials truncate the rearrangement channels, somewhat like the kinematic truncations emphasized in this paper.

Effects of the imaginary potentials are more obvious in the distorted Faddeev formulation. Consider the simplified equations (2.8), (2.9), specialized to a channel determined by $\phi(1)$, a particle-1 bound state. Let the complex single-particle potential be $U(x)=V(x)+iW(x)$. Let the two active particles be in a stretched configuration, in which they are too far apart to interact, so that ξ_λ can be used in the form

$$\xi_\lambda=\int\phi(1)\chi(2),$$

where the integral is over energy-conserving products. Equation (2.10) becomes

$$[E-K_2-\varepsilon_1-1\langle\phi|W|\phi\rangle]\chi(2)=(\phi(1)|U_1-\mathscr{O}_1|\psi_\lambda(1,2)\rangle. \tag{3.6}$$

The expectation of $W(x)$ contributes a large positive imaginary part to the net energy of the outgoing particle, and this damps $\chi(2)$. For nucleons of about 100 MeV, we estimate $\langle W\rangle\approx10$ MeV, which implies a damping length for $\chi(2)$ of about 6 fm.

Evidently, a model-space calculation is subject to two truncations, both of which suppress long-range rearrangement channels: (1) The damping of outgoing waves described above and (2) truncation of relative angular momentum. The simultaneous presence of these two effects probably reduces the sensitivity of a calculation to either one alone. This may be a major reason for the well-known [20,23] insensitivity of CDCC calculations to more than $\lambda=4$ or 6 multipoles.

Breakup in the nuclear interior is particularly sensitive to the imaginary potentials. This situation was revealed accidentally in comparisons of recent model-space calculations

[23] with an early calculation [5] that used schematic optical potentials that had very small imaginary parts. The matrix elements of U_1, U_2 for interior breakup actually are found to be large, but absorption by the imaginary parts of U_1, U_2 prevents the breakup particles from getting out.

Although the spurious imaginary potentials discussed above can be excluded artificially when constructing exit channel wave functions for particular reactions, they still remain in the model-space calculation of ψ_λ. Perhaps the most realistic observation is that such physical effects cannot be entirely separated from the techniques for solving the three-body Schrödinger equation.

IV. SUMMARY

A method for solving the Faddeev equations for a three-body system is described, based on ordinary distorted-wave Faddeev components. The distorted-wave components are introduced in such a way as to be completely connected. This requires the use of three-body distorting potentials, which are derived by a process of multipole expansion. A truncated version of our chosen procedure is equivalent to CDCC.

Perhaps the most important step of our analysis is the demonstration that when the procedure is iterated the three-body Green's function is applied on an asymptotically outgoing three-body source function. Since the Green's function is outgoing, the natural consequence is that the integral averages to a small value.

In a review several years ago, Kuruoğlu [25] proposed that the coupled reaction channels (CRC) method should be applied for reaction calculations. This CRC approach uses the channels that actually occur in a reaction to expand the wave function for the system. At best, this set of channel functions is overcomplete and considerable effort is required to extract a meaningful set of functions from the chosen starting point.

Kaneko, Kanada, and Tang [25] have circulated a resonating group study of $p+d$ scattering.

Recent work by Kersting and Sandhas [13] resembles the method given in this paper. They apply a ''resonating group'' expansion of the wave function for the three-nucleon system, and they find excellent convergence for one of the cases treated by Payne and co-workers [15]. However, they do not point out the relation to the Faddeev theory, as we do here, and they do not employ distorted waves.

Payne and co-workers have described their calculations of the ^{3}H, ^{3}He systems as ''benchmarks of excellence'' for Faddeev calculations [15]. They take the first Faddeev equation and express it in the form of a differential operator times a Faddeev component equals a potential times a sum of other Faddeev components. They use symmetry arguments to get the other components in terms of the first one. In our procedure the differential operator on the LHS is first changed by adding to it some multipole pieces of the interactions on the RHS. By removing these pieces from the RHS, the roles of the other two components are considerably weakened and they can be omitted from the (modified) Faddeev equations.

ACKNOWLEDGMENTS

We are happy to acknowledge Professor Joe Redish, who first told us about the distorted-wave idea. We are grateful to Professor C. M. Vincent, who provided valuable criticisms of the manuscript.

APPENDIX A: STATIONARY PHASE

This topic concerns the approximate evaluation of integrals of the form

$$I=\int_0^\mu dk\ f(k)e^{ip(k)}, \tag{A1}$$

with

$$p(k)\equiv x_1k+x_2(\mu^2-k^2)^{1/2}, \tag{A2}$$

in the limit $x_1, x_2\rightarrow$asymptotically large. Such integrals appear, for example, in the transformation [26] of the asymptotic ψ_0 from a channel representation to a configuration representation, using the coordinates of both active particles. In the limit the integral is dominated by the momentum at the point $p'(k_0)=0$.

The Taylor expansion of $p(k)$ about k_0 is

$$p(k)=p_0-(q^2/2d^2)+\cdots \tag{A3}$$

with $q\equiv k-k_0$ and

$$k_0/x_1=(\mu^2-k_0^2)^{1/2}/x_2=\mu/\mathscr{R}, \tag{A4}$$

$$\mathscr{R}\equiv(x_1^2+x_2^2)^{1/2},\quad p_0=\mu.\mathscr{R},\quad d\equiv(x^2/\mathscr{R})(\mu/\mathscr{R})^{1/2}. \tag{A5}$$

In terms of the quantities just defined, the original integral becomes

$$I=e^{ip_0}f(k_0+q)\int_{-k_0}^{\mu-k_0}dq\ e^{-iq^2/2d^2}, \tag{A6}$$

where the higher terms of the Taylor series in (A3) have been omitted.

The usual discussion of (A6) points out that in the limit $x_1, x_2\rightarrow\infty$ the integrand oscillates rapidly and it averages to zero, everywhere except in an interval of width d centered at $q=0$. In this limit the integration is extended to $\pm\infty$ and $f(k_0+q)\approx f(k_0)$ is removed from the integrand, so that

$$I=e^{ip_0}f(k_0)\int_{-\infty}^{\infty}dq\ e^{-iq^2/2d^2}=(i2\pi)^{1/2}de^{i\mu R}f(k_0). \tag{A7}$$

The stationary phase evaluation of (A1) produces the asymptotic (semiclassical) limit of a wave mechanical expression. We see especially in (A4) that at the point of stationary phase the particles have traveled distances that are proportional to their velocities.

It is mentioned in Sec. III A that several authors [14–17] find that convergence to the stationary phase limit (A7) as x_1, x_2 become large is very slow. This should not be a surprise. Let us examine (A6), the result of a Taylor expansion in the original integral, without subsequent approximations. Consider the special case $x_1=x_2=x$. The Taylor expansion gives

$$I=e^{i\mu R}\int_{-k_0}^{\mu-k_0}dq\ f(k_0+q)e^{-i4q^2x/\mu}, \tag{A8}$$

which becomes

$$I=\mu e^{i1.414N}\int_{-0.707}^{0.273}f(0.707+\nu)e^{-i4N\nu^2}d\nu, \tag{A9}$$

where now all momenta are measured in units of μ, with $\nu\equiv q/\mu$ and with $N\equiv\mu x$ the number of wavelengths in the distance x.

It is obvious in the form (A9) that N must be very large if the region of the stationary phase is to be much smaller than the entire available range of integration or smaller than reasonable ranges of variation of the function f. In cases of practical interest, it might easily be necessary to go to $N\sim10^4$.

The cause of this slow convergence [15–18] is the slow variation of the original phase factor $p(k)$ in (A2). In dimensionless form, again taking the special case $x_1=x_2=x$,

$$p(k)=N[k/\mu+(1-k^2/\mu^2)]. \tag{A10}$$

The function in brackets has the value 1 at both end points of integration; at its maximum, it only rises to 1.414. It takes a very large coefficient N to sharpen up this maximum.

APPENDIX B: SUMMATION OF $\mathscr{P}_\lambda$

By definition, $\mathscr{P}_\lambda$ selects all angular momentum states that have $l\leqslant\lambda$,

$$\mathscr{P}_\lambda=\sum_{i,m=0}^{\lambda}Y_{l,m}(\hat r)Y^*_{l,m}(\hat r'). \tag{B1}$$

An integration over $\hat r'$ is understood when this expression operates on some given function of $\hat r'$. The spherical harmonic addition theorem gives

$$\mathscr{P}_\lambda=(4\pi)^{-1}\sum_{i=0}^{\lambda}(2l+1)P_l(\hat r\cdot\hat r'), \tag{B2}$$

and this is summed to produce

$$\mathscr{P}_\lambda=(4\pi)^{-1}\frac{d}{d\mu}[P_{\lambda+1}+P_\lambda]. \tag{B3}$$

This closed form for the projection operator is obtained from the derivative formula for Legendre polynomials, as in Blair [27]. Especially for λ large, $\mathscr{P}_\lambda$ is a somewhat broadened δ function of the angle between $\hat r$ and $\hat r'$, of half width $\sim\lambda^{-1}$. We obtain $\int d\Omega\,\mathscr{P}_\lambda=1$, as required by this interpretation. In a rough sense, the operator $\mathscr{P}_\lambda$ *averages the function of* $\hat r'$ *on which it operates*, over a range of angles $\delta\hat r'\sim\lambda^{-1}$. Averaging to this extent has little effect on $Y_{lm}(\hat r)$ functions with $l<\lambda$, but it tends to destroy $Y_{lm}(\hat r)$ functions with $l>\lambda$. We also note that the mixture of parities in $\mathscr{P}_\lambda$ eliminates contributions from the backward hemisphere.

APPENDIX C: UNEQUAL MASSES, RECOIL

Although the second and third Faddeev equations can always be added, giving a system of two equations, as in the text; these are not of much use unless the variables on the LHS of (2.9) can be separated. The kinetic energy operator K separates in any system of Jacobi variables, consisting of one arbitrary vector coordinate and the coordinate conjugate to it. However, $\vec r_{1A}$ and $\vec r_{2A}$ in the two potential functions in (2.9) are not usually conjugate to each other, and so the flexibility of K is of no use.

The distorted Faddeev equations

$$[E-K-V-\mathscr{O}_1-\mathscr{O}_2]\psi_\lambda=V(\psi_{1\lambda}+\psi_{2\lambda}), \tag{2.2′}$$

$$[E-K-U_1(r_1)]\psi_{1\lambda}-U_1\psi_{2\lambda}=[U_1-\mathscr{O}_1]\psi_\lambda, \tag{2.3′}$$

$$[E-K-U_2(r_2)]\psi_{2\lambda}-U_2\psi_{1\lambda}=[U_2-\mathscr{O}_2]\psi_\lambda \tag{2.4′}$$

offer another approach to the problem of coordinates. If the cutoff λ is large enough, the RHS's of (2.3) and (2.4) become negligible: hence, the RHS of (2.2) also vanishes, leaving a soluble equation to solve.

APPENDIX D: S-MATRIX ELEMENTS

In this appendix we reduce the S-matrix elements to more calculable forms, under the assumption $V\xi=0$. Standard expressions for the S-matrix elements are

$$T_{\rm el}=\langle\phi_0(r)\exp(|\vec p_0\cdot\vec R)|U|\Psi\rangle, \tag{D1}$$

$$T_{\rm re}=\langle\phi_1(r_1)\phi_{k_2}(r_2)|V|\Psi\rangle, \tag{D2}$$

$$T_{\rm br}=\langle\phi_{k_1}(r_1)\phi_{k_2}(r_2)|V|\Psi\rangle, \tag{D3}$$

respectively, where $\phi_0(r)$ and $\phi_1(r_1)$ are bound state wave functions for the 1-2 and 1-A subsystems and ϕ_{k_1/k_2} is the scattering wave function for the 1-A subsystem with the momentum k_1, k_2. In the limit $V\xi=0$, (D2) and (D3) obviously reduce to

$$T_{\rm re}\approx\langle\phi_1(r_1)\phi_{k_2}(r_2)|V|\psi_{\rm CDCC}\rangle, \tag{D2′}$$

$$T_{\rm br}\approx\langle\phi_{k_1}(r_1)\phi_{k_2}(r_2)|V|\psi_{\rm CDCC}\rangle. \tag{D3′}$$

Reduction of the elastic amplitude is a little more involved. We first apply the equation $\psi=\psi_\lambda+\xi_\lambda$, to get

$$\begin{aligned}U\psi&=U(\psi_\lambda+\xi_\lambda)=\{\mathscr{O}\psi_\lambda+(E-K-U)\xi_\lambda\}+U\xi_\lambda\\&=\mathscr{O}\psi_\lambda+(E-K)\xi_\lambda,\end{aligned} \tag{D4}$$

using (2.9). We now use the assumption $V\xi_\lambda\approx0$ to write (D4) as

$$U\psi\approx\mathscr{O}\psi_\lambda+(E-K-V)\xi_\lambda. \tag{D5}$$

If we now substitute (D5) in (D1), the second term vanishes because ξ has no asymptotic, elastic amplitude, and the elastic amplitude as a whole has reduced to

$$T_{\rm el}=\langle\phi_0(r)\exp(i\vec p_0\cdot\vec R)|\mathscr{O}|\psi_{\rm CDCC}\rangle, \tag{D1′}$$

which is no surprise.

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