Transport coefficients of a hot pion gas

D. Davesne
Institut de Physique Nucléaire de Lyon, IN2P3-CNRS et Université Claude Bernard, 43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

(Received 25 July 1995)

General expressions for transport coefficients of a single-component gas (namely, thermal conductivity and shear and bulk viscosities) of bosons are derived from a Boltzmann-Uehling-Uhlenbeck transport equation by means of the Chapman-Enskog method to first order. These expressions are then used for the calculation of the associated transport relaxation times and applied to the pion gas produced in ultrarelativistic heavy-ion collisions. The influence of Bose enhancement factors on transport properties can be seen by comparison with previous calculations. [S0556-2813(96)02306-0]

PACS number(s): 25.75.−q, 05.60.+w

I. INTRODUCTION

The transport properties of a system out of equilibrium are governed by some phenomenological coefficients (transport coefficients) which relate flows to thermodynamic forces. In other words, these coefficients characterize the magnitude of the response of the system (flows) to a certain disturbance (thermodynamic forces). They have already been derived in the context of nonrelativistic [1,2] and relativistic kinetic theory [3,4] but only, to our knowledge, with the use of a Boltzmann collision term (i.e., without Bose enhancement factors). Our aim is to generalize the treatment given in [5–7] to a transport equation with a Boltzmann-Uehling-Uhlenbeck collision term:[1]

$$\begin{aligned} p_\mu \partial^\mu f(x,p) = & \int\int\int dw_1 dw' dw_1' [f' f_1' (1+A_0 f) \\ & \times (1+A_0 f_1) - f f_1 (1+A_0 f')(1+A_0 f_1')] \\ & \times W(p p_1 / p' p_1') \\ \equiv & I(f), \end{aligned} \tag{1}$$

where $f \equiv f(x,p), f' \equiv f(x,p')$, etc., $dw = d^3\mathbf{p}/p^0$, and the collision rate W is related to the differential cross section by

$$W = \sigma P^2 \delta^{(4)}(p^\alpha + p_1^\alpha - p'^\alpha - p_1'^\alpha) \tag{2}$$

with $P = [-(p^\alpha + p_1^\alpha)(p_\alpha + p_{1\alpha})]^{1/2}$.

Furthermore, an application of such calculations is the hadronic phase (which will be supposed here only constituted of pions) produced in ultrarelativistic heavy-ion collisions at CERN and in the near future at the BNL Relativistic Heavy Ion Collider (RHIC). The dissipative effects (controlled by transport coefficients) and the different time scales (characterized by relaxation times) over which particles produced are equilibrated are important in order to describe precisely the space-time evolution of these collisions.

The paper is organized as follows. In the second section we will recall some basic equations and give the general formulas for thermodynamic quantities at equilibrium. In the third section, we will develop the Chapman-Enskog formalism and apply it to our transport equation in order to obtain the expressions of the three transport coefficients (thermal conductivity and shear and bulk viscosities). In the fourth section we will derive the corresponding relaxation times. All the numerical results are given for a pion gas and compared with those previously obtained in [6] and [10].

II. BASIC EQUATIONS AND THERMODYNAMIC EQUILIBRIUM

From a formal exchange of the four variables $\mathbf{p},\mathbf{p}',\mathbf{p}_1$, and $\mathbf{p}_1'$, it can be easily proved from Eq. (1) that

$$\int \frac{d^3\mathbf{p}}{p^0} I(f) S_I = 0 \tag{3}$$

where S_I can be any linear combination of p^α and some constants (these are called summational invariants).

From this equation and Eq. (1) we can then deduce, as for a Boltzmann collision term, that the mass four-flow and the energy-momentum tensor defined as the first and second moments of the distribution function (m is the mass of the particle),

$$\begin{aligned} M^\mu(x) &= mc \int \frac{d^3\mathbf{p}}{p^0} p^\mu f(x,p), \\ T^{\mu\nu}(x) &= c \int \frac{d^3\mathbf{p}}{p^0} p^\mu p^\nu f(x,p), \end{aligned} \tag{4}$$

are conserved:

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= 0, \\ \partial_\mu M^\mu &= 0. \end{aligned} \tag{5}$$

Let us now introduce another fundamental quantity which will be used further: the hydrodynamic four-velocity U^μ. It is a timelike vector with norm c: $U^\mu U_\mu = -c^2$. From it, we can construct a projector on the spatial part:

[1] The notations are those of Ref. [7]. In particular, A_0 is a constant that makes $A_0 f$ dimensionless $[A_0 = (2\pi\hbar)^3]$ and we use the metric $g^{00} = -1, g^{11} = 1, g^{22} = 1, g^{33} = 1$, and 0 for the other components.

$\Delta^{\mu\nu} \equiv g^{\mu\nu} + c^{-2} U^{\mu} U^{\nu}$. Furthermore, the relation between U^{μ} and the other quantities will be chosen such that (Eckart choice [5])

$$M^{\mu} = \rho U^{\mu}. \tag{6}$$

At this point, one must note that this choice is only possible if there is a conserved charge in the system. In this paper, as in [6,7,10], this charge is the mass. This choice is actually of fundamental importance for the calculation of the thermal conductivity: in the absence of a conserved charge, this coefficient vanishes (this has already been noted in [9] and in the context of the pion gas in [10] where a physical interpretation based on a formal analogy with thermal conductivity in insulators is proposed).

The choice of the velocity made, we can determine the density uniquely. Effectively, by contraction with U_{μ}, the above equation reads

$$\rho = -c^{-2} M^{\mu} U_{\mu}. \tag{7}$$

An important remark concerning (7) is that the density is actually defined in a permanent rest frame [the frame in which $U^{\mu} = (c, \mathbf{0})$]. Similarly, we define the energy density in the same frame as

$$\rho e = c^{-2} U_{\mu} T^{\mu\nu} U_{\nu}. \tag{8}$$

With U^{μ}, we can also construct an energy four-flow (h is the enthalpy):

$$I_q^{\mu} \equiv (h M^{\sigma} + U_{\nu} T^{\nu\sigma}) \Delta^{\mu}{}_{\sigma}, \tag{9}$$

and a pressure tensor:

$$P^{\mu\nu} \equiv \Delta^{\mu}{}_{\sigma} T^{\sigma\tau} \Delta^{\nu}{}_{\tau} \equiv P \Delta^{\mu\nu} + \Pi^{\mu\nu}. \tag{10}$$

Let us now turn to the equilibrium properties and, as a consequence, to the entropy production. The four-entropy is equal to

$$S^{\mu}(x) = -k_B c \int \frac{d^3\mathbf{p}}{p^0} p^{\mu} [A_0 f \ln(A_0 f) - (1 + A_0 f) \times \ln(1 + A_0 f)] \tag{11}$$

so that the entropy density, also defined in the permanent rest frame, reads

$$\rho s = -\frac{S^{\mu} U_{\mu}}{c^2}. \tag{12}$$

By taking the derivative and using the mass conservation law (5), one can then write from Eq. (12) ($D \equiv U^{\nu} \partial_{\nu}$)

$$\rho D s = -\partial_{\mu} (\Delta^{\mu\nu} S_{\nu}) + \sigma. \tag{13}$$

This relation can be seen as a balance equation for the entropy. Thus, by definition, $\sigma = \partial_{\mu} S^{\mu}$ is the entropy production. When the equilibrium is reached, σ vanishes and the resolution of the equation $\sigma = 0$ leads to the equilibrium function f_{eq}. Of course, one finds the usual Bose-Einstein distribution

$$f_{\text{eq}}(p) = \frac{A_0^{-1}}{e^{-(p_{\mu} U^{\mu} + m\mu)/k_B T} - 1}. \tag{14}$$

By using the above definitions, we can then obtain the following equilibrium formulas

$$\rho = 4\pi m^3 c A_0^{-1} k_B T S_2^{-1}, \tag{15}$$

$$P = \frac{\rho k_B T}{m} \frac{S_2^{-2}}{S_2^{-1}}, \tag{16}$$

$$\rho e = -P + \rho \frac{S_3^{-1}}{S_2^{-1}} c^2, \tag{17}$$

$$h = \frac{S_3^{-1}}{S_2^{-1}} c^2 \equiv \hat{h} c^2, \tag{18}$$

$$T^{\mu\nu} = \rho e \frac{U^{\mu} U^{\nu}}{c^2} + P \Delta^{\mu\nu}, \tag{19}$$

$$I_q^{\mu} = 0,$$

$$\Pi^{\mu\nu} = 0 \tag{20}$$

with $z = mc^2/k_B T$, $S_n^{\alpha} = \Sigma_{k=1}^{\infty} e^{km\mu/k_B T} k^{\alpha} K_n(kz)$, and K_n the MacDonald function. Note that the sum S_n^{α} introduced above is such that the first term corresponds to the case of a Boltzmann equilibrium distribution function so that a comparison with the results of [6,7] is possible at each step of the calculation.

III. TRANSPORT COEFFICIENTS

A. Chapman-Enskog expansion

The Chapman-Enskog expansion is used to find a solution of transport equations in the so-called hydrodynamic regime. In this regime, a system tries to smooth spatial inhomogeneities to go from local to global equilibrium. Local equilibrium means that at each point the distribution function is equal to

$$f^{(0)}(x,p) = \frac{A_0^{-1}}{e^{-[p_{\mu} u^{\mu}(x) + m\mu(x)/k_B T(x)]} - 1} \tag{21}$$

(where μ, U^{μ}, and T are now *parameters* which depend on x).

Physically, "local" means that the mean free path λ is less than a distance L over which macroscopic quantities such as T can vary appreciably and that the time between two collisions is smaller than the time needed for a macroscopic parameter to change significantly. Actually, one assumes that a system can be divided into volumes large compared to the mean free path and small enough so that T, μ, or U^{μ} is uniform inside. The Chapman-Enskog expansion is performed with respect to ϵ ($\epsilon \equiv \lambda/L \ll 1$ in the hydrodynamic regime):

$$f(x,p) = f^{(0)}(x,p) + \epsilon f^{(1)}(x,p) + \epsilon^2 f^{(2)}(x,p) + \cdots. \tag{22}$$

We are going to restrict ourselves to the first order and write f, for convenience, as

$$f(x,p)=f^{(0)}(x,p)+f^{(0)}(x,p)[1+A_0f^{(0)}(x,p)]\phi(x,p) \tag{23}$$

where $f^{(0)}$ is given by (21). We have to remark that in this expression μ, T, and U^ν are *parameters*. In order to identify T with the temperature, we have previously used the pressure tensor. However, in a nonequilibrium system this tensor is not reduced to a scalar. Since it seems natural to associate $T(x)$ with the local temperature and $\mu(x)$ with the local chemical potential, we will impose that the density and the energy density must be equal to those calculated at equilibrium:

$$\rho(x)=4\pi m^3cA_0^{-1}k_BT(x)S_2^{-1}[z(x)],$$

$$\rho(x)e(x)=\frac{\rho k_BT(x)}{m}\frac{S_2^{-2}[z(x)]}{S_2^{-1}[z(x)]}+\rho(x)\frac{S_3^{-1}[z(x)]}{S_2^{-1}[z(x)]}c^2. \tag{24}$$

These equations have thus to be considered as the definitions for $T(x)$ and $\mu(x)$. This natural choice imposes some constraints on ϕ [see Eq. (7)]:

$$\int dw(p^\alpha U^\alpha)f^{(0)}(1+A_0f^{(0)})\phi=0,$$

$$\int dw(p_\alpha U^\alpha)^2f^{(0)}(1+A_0f^{(0)})\phi=0. \tag{25}$$

These constraints reflect that, by choice, only the local equilibrium distribution function contributes to the calculation of local densities. In the same spirit, in order to preserve the relation $M^\mu(x)=\rho(x)U^\mu(x)$, that is, to interpret $U^\mu(x)$ as the local velocity, one must impose

$$\int dwp_\alpha f^{(0)}(1+A_0f^{(0)})\phi=0. \tag{26}$$

Whereas ϕ plays no role for local densities, it does enter in the expression of the flows:

$$I_q^\mu=(U_\nu T^{\nu\sigma}+hM^\sigma)\Delta\ _\sigma{}^\mu$$

$$=c\int dw[(p_\nu U^\nu)+mh]p_\sigma\Delta^{\sigma\mu}f^{(0)}(1+A_0f^{(0)})\phi,$$

$$T^{\mu\nu}=T^{\mu\nu(0)}+c\int dwp^\mu p^\nu f^{(0)}(1+A_0f^{(0)})\phi. \tag{27}$$

One can also remark that the calculation of the entropy production with the formula (13) gives

$$\sigma=-k_Bc\int p_\nu\partial^\nu f^{(0)}\phi dw. \tag{28}$$

By replacing $f^{(0)}$ by its expression, one would obtain, as in the nonrelativistic case, a bilinear form between these flows and these thermodynamic forces. Moreover, since ϕ has to be considered as a perturbation, i.e., we are close to equilibrium, the relation between flows and thermodynamic forces can be approximated by a linear law:

$$I_q^\mu=-\lambda\Delta^{\mu\nu}(\partial_\nu T+c^{-2}TDU^\mu), \tag{29}$$

$$\Pi^{\mu\nu}=-2\eta_s\langle\partial^\mu U^\nu\rangle-\eta_v\Delta^{\mu\nu}\partial_\gamma U^\gamma \tag{30}$$

(if $t^{\mu\nu}$ is a tensor, $\langle t^{\mu\nu}\rangle\equiv[\frac{1}{2}(\Delta^{\mu\gamma}\Delta^{\nu\delta}+\Delta^{\nu\gamma}\Delta^{\mu\delta})-\frac{1}{3}\Delta^{\mu\nu}\Delta^{\gamma\delta}]t_{\gamma\delta}$; $\langle t^{\mu\nu}\rangle$ is thus space-space-like and traceless). Of course the decomposition (30) and (29), which defines the three transport coefficients of a single-component gas (λ, the thermal conductivity, η_s, the shear viscosity, and η_v, the bulk viscosity), is, although conventional, completely arbitrary.

We have two expressions at our disposal for the flows: one phenomenological [Eq. (30)], coming from the entropy production, and another one coming from the Chapman-Enskog expansion [Eq. (27)]. The consequence of considering only the first order in this expansion is thus now clear: the knowledge of ϕ is sufficient to obtain the expression of the transport coefficients. Moreover, to first order, the thermodynamic laws are still available. To illustrate this point, let us replace $f(x,p)$ by its expression (23) in (12):

$$s=\frac{k_B}{\rho c}\int dwp^\mu U_\mu\Bigg\{[A_0f^{(0)}\ln A_0f^{(0)}-(1+A_0f^{(0)})\times\ln(1+A_0f^{(0)})]-A_0f^{(0)}(1+A_0f^{(0)})\times\phi\ln\left(\frac{A_0f^{(0)}}{1+A_0f^{(0)}}\right)\Bigg\}. \tag{31}$$

The last term, describing the correction to the entropy due to ϕ, is proportional to $\int dwp^\mu U_\mu A_0f^{(0)}(1+A_0f^{(0)})\phi$ and $\int dw(p^\mu U_\mu)^2A_0f^{(0)}(1+A_0f^{(0)})\phi$ and is thus equal to zero by virtue of Eq. (25): to first order, the Chapman-Enskog expansion does not contribute to the entropy: $s=s^{(0)}$. Moreover, it is possible to check, with the definitions of M^μ and $T^{\mu\nu}$, that s can be written as

$$s=\frac{1}{T}\left(e+\frac{P}{\rho}-\mu\right)=\frac{1}{T}(h-\mu)\quad\text{(Euler formula)}. \tag{32}$$

By taking the derivative of (16), it is easy to show explicitly that

$$\partial_\nu P=\rho T\partial_\nu\left(\frac{\mu}{T}\right)+\rho hT^{-1}\partial_\nu T\quad\text{(Gibbs -Duhem)} \tag{33}$$

or, equivalently,

$$\rho^{-1}\partial_\nu P=s\partial_\nu T+\partial_\nu\mu. \tag{34}$$

By differentiation of (32), we then obtain

$$T\partial_\nu s=\partial_\nu e+P\partial_\nu\rho^{-1} \tag{35}$$

which is the *second thermodynamic law*. As a conclusion, we can say that all the relations of thermodynamics remain valid for a system out of equilibrium, to first order in the Chapman-Enskog expansion, *if* the quantities in these thermodynamic relations are defined in the local rest frame.

B. Linear Boltzmann equation

The comparison between (30) and (27) clearly shows that the calculation of transport coefficients reduces to the determination of the quantity ϕ. As a consequence, we now turn to the resolution of our transport equation (1).

If we replace f by its expression (23) in Eq. (1), we arrive at

$$p_\alpha \partial^\alpha f^{(0)} = f^{(0)} \int dw' dw_1 dw_1' f_1^{(0)} (1 + A_0 f'^{(0)})(1 + A_0 f_1'^{(0)}) \times [\phi + \phi_1 - \phi' - \phi_1'] W(\mathbf{p}\mathbf{p}_1/\mathbf{p}'\mathbf{p}_1') \equiv \mathcal{C}[\phi], \tag{36}$$

where $\mathcal{C}[\phi]$ represents *the collision integral*.

The aim is to calculate explicitly the left hand side in terms of the thermodynamic forces entering in Eq. (30). First of all, we can notice that

$$\partial^\alpha f^{(0)} = \partial^\alpha \left(\frac{p_\mu U^\mu + m\mu}{k_B T} \right) f^{(0)} (1 + A_0 f^{(0)}). \tag{37}$$

By writing $\partial_\mu = -c^{-2} U_\mu D + \Delta^{\mu\nu}\partial_\nu \equiv -c^{-2} U_\mu D + \nabla_\mu$ [i.e., by splitting time derivatives (D) and gradients (∇_μ)], we realize immediately that we are confronted with terms such as DT and $D(\mu/T)$. Such terms do not enter in the expression of the thermodynamic forces; thus they have to be eliminated. This is possible if we use the conservation laws $\partial^\mu M_\mu = 0$, $\Delta_{\nu\mu}\partial_\rho T^{\rho\nu} = 0$, and $U_\nu \partial_\mu T^{\mu\nu} = 0$. The result reads

$$D\rho = -\rho \partial_\mu U^\mu, \tag{38}$$

$$De = -\frac{P}{\rho} \partial_\mu U^\mu, \tag{39}$$

$$\Delta_{\mu\rho}\partial^\rho P = -\rho h c^{-2} D U_\mu. \tag{40}$$

Keeping Eq. (39) and transforming Eq. (40) with the help of the Gibbs-Duhem relation, we can write

$$De = -(P/\rho)\partial_\nu U^\nu,$$

$$Dh = TD(\mu/T) + hT^{-1} DT. \tag{41}$$

If we now expand De and Dh as

$$De = \left.\frac{\partial e}{\partial(\mu/T)}\right|_T D\left(\frac{\mu}{T}\right) + \left.\frac{\partial e}{\partial T}\right|_{\mu/T} DT,$$

$$Dh = \left.\frac{\partial h}{\partial(\mu/T)}\right|_T D\left(\frac{\mu}{T}\right) + \left.\frac{\partial h}{\partial T}\right|_{\mu/T} DT, \tag{42}$$

we obtain a system of equations for $D(\mu/T)$ and DT. Using (7) and (17), this system can be solved after a straightforward but tedious calculation:

$$T^{-1} DT = (1 - \gamma')\partial_\nu U^\nu, \tag{43}$$

$$TD\left(\frac{\mu}{T}\right) = \left\{ (\gamma'' - 1)h - \gamma''' \frac{k_B T}{m} \right\} \partial_\nu U^\nu, \tag{44}$$

with

$$\gamma' = \frac{(S_2^0/S_2^{-1})^2 - (S_3^0/S_2^{-1})^2 + 4z^{-1} S_2^0 S_3^{-1}/(S_2^{-1})^2 + z^{-1} S_3^0/S_2^{-1}}{(S_2^0/S_2^{-1})^2 - (S_3^0/S_2^{-1})^2 + 3z^{-1} S_2^0 S_3^{-1}/(S_2^{-1})^2 + 2z^{-1} S_3^0/S_2^{-1} - z^{-2}}, \tag{45}$$

$$\gamma'' = 1 + \frac{z^{-2}}{(S_2^0/S_2^{-1})^2 - (S_3^0/S_2^{-1})^2 + 3z^{-1} S_2^0 S_3^{-1}/(S_2^{-1})^2 + 2z^{-1} S_3^0/S_2^{-1} - z^{-2}}, \tag{46}$$

$$\gamma''' = \frac{S_2^0/S_2^{-1} + 5z^{-1} S_3^{-1}/S_2^{-1} - S_3^0 S_3^{-1}/(S_2^{-1})^2}{(S_2^0/S_2^{-1})^2 - (S_3^0/S_2^{-1})^2 + 3z^{-1} S_2^0 S_3^{-1}/(S_2^{-1})^2 + 2z^{-1} S_3^0/S_2^{-1} - z^{-2}}. \tag{47}$$

With these results it is then easy to obtain for (36)

$$[Q\partial_\nu U^\nu - p_\alpha \Delta^{\alpha\beta}(p_\mu U^\mu + mh)(T^{-1}\partial_\beta T + c^{-2} DU_\beta) + \langle p_\alpha p_\beta \rangle \langle \partial^\alpha U^\beta \rangle] f^{(0)}(1 + A_0 f^{(0)}) = k_B T \mathcal{C}[\phi] \tag{48}$$

with

$$Q = -\frac{1}{3}(mc)^2 + (p_\mu U^\mu)^2 c^{-2}\left(\frac{4}{3} - \gamma'\right) + c^{-2}[(\gamma'' - 1)mh - \gamma''' k_B T] p_\mu U^\mu,$$

which is the generalization of the usual linear equation (see [5,6]). In the left hand side of this equation thermodynamic forces with different tensorial rank appear. Because of the Curie principle, we can thus deduce that ϕ must be of the form

$$\phi = A\partial_\nu U^\nu + B_\alpha \Delta^{\alpha\beta}(T^{-1}\partial_\beta T + c^{-2} DU_\beta) + C_{\alpha\beta}\langle \partial^\alpha U^\beta \rangle. \tag{49}$$

The transport coefficients then read

$$\eta_v = -c \int AQ f^{(0)}(1 + A_0 f^{(0)}) dw, \tag{50}$$

$$\lambda = -\frac{1}{3} c T^{-1} \int dw f^{(0)} (1 + A_0 f^{(0)}) (p_\nu U^\nu + mh) B_\alpha p^\alpha, \tag{51}$$

$$\eta_s = -\frac{1}{10} c \int dw f^{(0)} (1 + A_0 f^{(0)}) C_{\alpha\beta} \langle p^\alpha p^\beta \rangle, \tag{52}$$

with A, B_α, and $C_{\alpha\beta}$ solutions of

$$\mathcal{C}[A] = (k_B T)^{-1} Q f^{(0)} (1 + A_0 f^{(0)}), \tag{53}$$

$$\mathcal{C}[B_\alpha] = (k_B T)^{-1} (p_\nu U^\nu + mh) \Delta_{\alpha\beta} p^\beta f^{(0)} (1 + A_0 f^{(0)}), \tag{54}$$

$$\mathcal{C}[C_{\alpha\beta}] = (k_B T)^{-1} \langle p_\alpha p_\beta \rangle f^{(0)} (1 + A_0 f^{(0)}). \tag{55}$$

Hence one just has to determine A, B_α, and $C_{\alpha\beta}$ to have access to the transport coefficients. The technical details for obtaining these quantities are given in Appendix A. At this point, one must notice that A, B_α, and $C_{\alpha\beta}$ are only determined perturbatively. That implies that we will obtain the corresponding transport coefficients perturbatively as well. Nevertheless, the numerical results shown in the next section exhibit clearly the fact that the convergence is fast.

C. Numerical results

As indicated in the introduction, all the numerical results will be given for the case of the hot pion gas produced in ultrarelativistic heavy-ion collisions. Several comments on the applicability of the formalism developed in the preceding section are therefore necessary. The first one concerns the chemical potential. Rigorously, the only conserved charge for the pion system is the electric charge. However, the experimental data clearly show that pion-pion scattering is elastic up to 1 GeV. Hence, in the energy range relevant for our problem, the number of pions is conserved with an excellent approximation. This allows the phenomenological introduction of a chemical potential associated with the total number of pions as pointed out by many authors (see Ref. [12], for instance). Moreover, this conservation of the total number of pions implies automatically the conservation of the mass (used in the preceding section).

If we now apply the formulas[2] given in Appendix A for the Boltzmann case, we recover the results given in Ref. [6] (Fig. 1) from which we can make two conclusions: first of all, the Chapman-Enskog expansion converges very rapidly for λ and η_s; secondly, η_v, the convergence of which is less rapid, is much smaller than the other transport coefficients. So only the first order will be necessary to discuss the order of magnitude of the effect of Bose-Einstein factors in our region of interest (say, 100 MeV $< T <$ 200 MeV for the hot pion gas).

On Fig. 2 is thus depicted, to first order, the comparison between the results obtained from a Boltzmann-like collision term (Ref. [6]) and from a Bose-Einstein-like collision term (this work) with various values of the chemical potential. When $\mu_\pi \to -\infty$ and more generally when $\mu_\pi < 0$ and $m/T > 1$, both cases are identical: the Bose enhancement factors in the collision integral play a negligible role. However, as soon as μ_π increases (say, $\mu_\pi > 0$ or $m/T > 1$), the results exhibit large differences. For example, at $T = 150$ MeV, the bulk viscosity varies by a factor of 2.5, the shear viscosity is decreased by more than 50%, and the thermal conductivity by 15%. This implies, for instance, that for a given thermal gradient, the energy flow, whose role is to smooth the effects of the gradient, will be decreased by 15%. As a consequence,

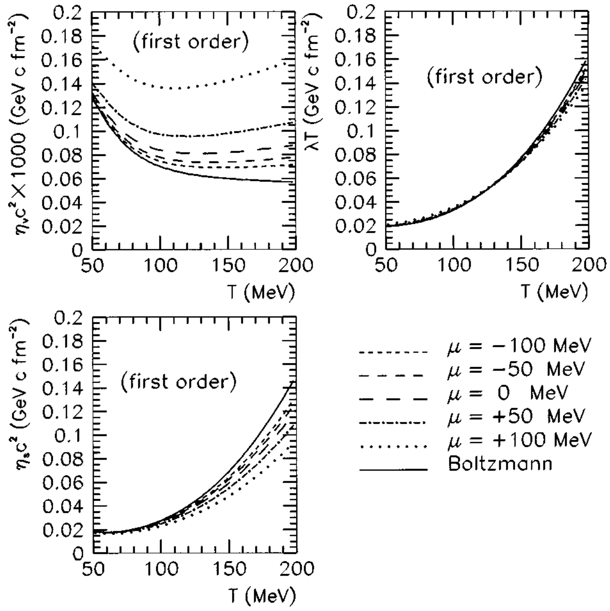

FIG. 1. Transport coefficients computed from a Bose-Einstein distribution and for different values of chemical potential.

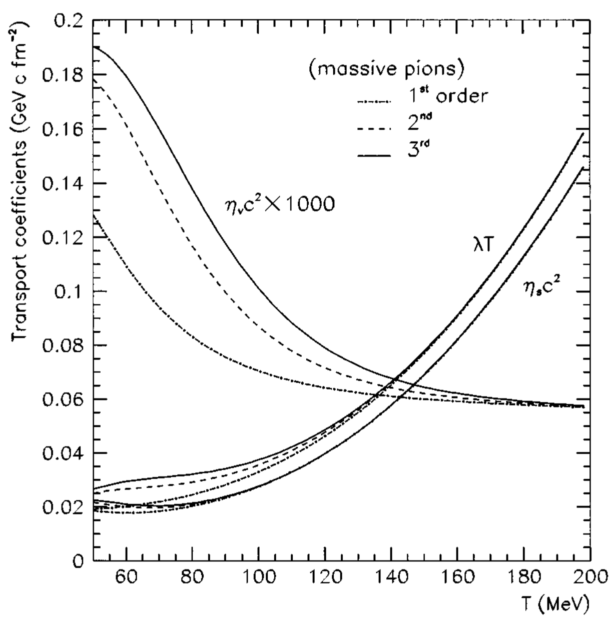

FIG. 2. Transport coefficients up to third order calculated with a Boltzmann distribution (cf. [6]).

[2]The numerical results are presented here with the commonly used units $h/2\pi = k_B = c = 1$, so that the temperature is expressed in energy units. Moreover, the cross section used in numerical calculations is the one described in [6].

one could think that the system will reach its equilibrium state in a longer time. Actually, this qualitative reasoning does not hold: a more precise study based on the quantities that really govern the time scales of a system, namely, the relaxation times, can show the opposite behavior. The calculation of these quantities is the subject of the next section.

IV. RELAXATION TIMES

It is impossible to have access to the relaxation times with the Chapman-Enskog formalism because it gives an infinite speed for the flows [mathematically, this comes from the fact that we neglect all the gradients of the flows in the conservation laws (see [5])]. Thus we will use in the following another method for solving our transport equation, the moment method ([5] and references therein). This method is more general than the Chapman-Enskog one in the sense that the validity is not limited to situations where the mean free path is smaller than the characteristic (macroscopic) length of the problem. However, it must be noticed that we do not use here this moment method in order to generalize the previous section or because we want to consider a different system with new physical hypotheses but only in order to have access to the relaxation times.

If we assume, as in the previous section, that we are close to equilibrium it is possible to look for a solution of the transport equation in the same form as Eq. (23):

$$f(x,p)=f^{(0)}(x,p)+f^{(0)}(x,p)[1+A_0 f^{(0)}(x,p)]\phi(x,p) \tag{56}$$

where $f^{(0)}$ is given by (21). Moreover, because f is a scalar function of x^μ and p^μ, ϕ can be expressed as a complete sum of scalar products between tensors depending on x^μ and tensors depending on p^μ (cf. [5]):

$$\phi=A(x,\tau)-B_\mu(x,\tau)\Pi^{\overline{\mu}}+C_{\mu\nu}(x,\tau)\langle\Pi^\mu\Pi^\nu\rangle+\cdots, \tag{57}$$

where $\Pi^{\overline{\mu}}=\Delta^{\mu\nu}\Pi_\nu$, $\Pi^\mu=cp^\mu/k_BT$, and $\tau=-p^\mu U_\mu/k_BT$.

Moreover, we will still assume that T,μ, and U^μ represent, respectively, the temperature, the chemical potential, and the hydrodynamic velocity, that is, we will still impose the conditions (25) and (26). ϕ is, on the other hand, a solution of the *complete* transport equation (we have reintroduced the term negligible in the Chapman-Enskog formalism, i.e., in the hydrodynamic regime):

$$p_\alpha\partial^\alpha f^{(0)}=-f^{(0)}(1+A_0 f^{(0)})p_\alpha\partial^\alpha\phi+\mathcal{C}[\phi]. \tag{58}$$

The convergence of the method is based on the fact that the distribution $f^{(0)}$ is sufficiently close to the equilibrium distribution, so that only the first terms of the expansion (57) are necessary to determine a solution of (58). Explicitly, we will truncate the sum (57) after $\langle\Pi^\mu\Pi^\nu\rangle$ and will expand A, B_μ, and $C_{\mu\nu}$ in powers of τ, the last power being the one which gives a nonzero contribution to the collision term. Under these restrictions, ϕ takes the form

$$\phi=\sum_{s=0}^{2}A^s(x)\tau^s-\sum_{s=0}^{1}B^s_\mu(x)\tau^s\Pi^{\overline{\mu}}+C^0_{\mu\nu}(x)\langle\Pi^\mu\Pi^\nu\rangle. \tag{59}$$

If the explicit dependence on flows is conserved, the conservations laws of mass and energy and the equation of motion can be generalized as

$$D\rho=-\rho\nabla_\mu U^\mu, \tag{60}$$

$$\rho De=-P\nabla_\nu U^\nu+c^{-2}\nabla_\mu I^\mu_q, \tag{61}$$

$$\rho hc^{-2}DU^\mu=-\nabla^\mu P-c^{-2}DI^\mu_q-\nabla_\nu\Pi^{\mu\nu}. \tag{62}$$

This implies that the formulas (43) and (44) giving DT and $D(\mu/T)$ in terms of $\nabla_\mu U^\mu$ $(=\partial_\mu U^\mu)$ are no longer valid. Following the same steps as in the previous paragraph, one finds instead

$$T^{-1}DT=(1-\gamma')\left(\nabla_\nu U^\nu+\frac{\delta}{P}\nabla_\nu I^\nu_q\right), \tag{63}$$

$$TD\left(\frac{\mu}{T}\right)=\left\{(\gamma''-1)h-\gamma'''\frac{k_BT}{m}\right\}\left(\nabla_\nu U^\nu+\frac{\delta}{P}\nabla_\nu I^\nu_q\right) -\frac{\delta'}{\rho}\nabla_\nu I^\nu_q, \tag{64}$$

with

$$\delta=\frac{S^0_2S^{-2}_2/(S^{-1}_2)^2}{1-(S^0_3S^{-1}_2-S^0_2S^{-1}_3)/(S^{-1}_2)^2}, \tag{65}$$

$$\delta'=\frac{-1}{1-(S^0_3S^{-1}_2-S^0_2S^{-1}_3)/(S^{-1}_2)^2.} \tag{66}$$

Following step by step the procedure of the last section, we can derive the new linear transport equation. The left hand side of (58) is, for instance,

$$\begin{aligned}c\Pi_\alpha\partial^\alpha f^{(0)}=f^{(0)}(1+A_0f^{(0)})\Bigg\{&c\left(\tau-\frac{mh}{k_BT}\right)\Pi_\alpha\frac{\nabla^\alpha T}{T}\\&+c\Pi_\alpha\frac{\nabla^\alpha P}{\rho k_BT/m}+\Pi_\alpha\Pi_\beta\langle\nabla^\alpha U^\beta\rangle+\hat{Q}\nabla_\alpha U^\alpha\\&+c^{-1}\tau\Pi_\beta DU^\beta+\tau\Bigg[\left(\tau(1-\gamma')+\frac{mh}{k_BT}(\gamma''-1)\right.\\&\left.-\gamma'''\right)\frac{\delta}{P}\nabla_\alpha I^\alpha_q-\frac{\delta'}{\rho k_BT/m}\nabla_\alpha I^\alpha_q\Bigg]\Bigg\},\end{aligned} \tag{67}$$

where

$$\hat{Q}\equiv[c^2/(k_BT)^2]Q. \tag{68}$$

This equation represents a generalization of the equation obtained in [5]. For the right-hand side of (58), one just has to replace ϕ by its expression (57).

The procedure is now standard and makes use of the variational approach proposed by Galerkin (see the reference given in [5]): one imposes arbitrary variations of ϕ of the form

$$\delta\phi(x,p)=\sum_{s=0}^{2}\tau^s\delta A^s(x), \tag{69}$$

$$\delta\phi(x,p)=-\Pi^{\overline{\mu}}\sum_{s=0}^{1}\tau^s\delta B^s_\mu(x), \tag{70}$$

$$\delta\phi(x,p)=\langle\Pi^\mu\Pi^\nu\rangle\delta C^0_{\mu\nu}(x) \tag{71}$$

to obtain a set of distinct equations satisfied by A^s, B^s_μ, and $C^0_{\mu\nu}$. Because of relations (25) and (26), on one hand, and thanks to the expressions of the coefficients given in Appendix B, on the other hand, one can determine some relations between flows and A^s, B^s_μ, and $C^0_{\mu\nu}$. From the resulting equations [5], one can extract the relaxation times with the results

$$\tau_v = \frac{[\eta_v]_1}{4\alpha_2^2}\,\frac{m}{\rho k_B T}\left[\frac{(v_3)^3 - 2v_2 v_3 v_4 + v_1 (v_4)^2}{(v_2)^2 - v_1 v_3} + v_5\right], \tag{72}$$

$$\tau_\lambda = \frac{[\lambda]_1 T}{\beta_1}\,\frac{mc^{-2}}{\rho k_B T}\left[\frac{l_2 k_B T}{mh} - \frac{l_1 l_2}{\beta_1} + 3 l_3\right], \tag{73}$$

$$\tau_s = 2[\eta_s]_1\,\frac{m}{\rho k_B T}\,\frac{5 s_1}{(s_0)^2}, \tag{74}$$

with

$$v_1 = \frac{S_2^0}{S_2^{-1}}, \tag{75}$$

$$v_2 = z\,\frac{S_3^0}{S_2^{-1}} - 1, \tag{76}$$

$$v_3 = z^2\left[\frac{S_2^0}{S_2^{-1}} + 3z^{-1}\frac{S_3^{-1}}{S_2^{-1}}\right], \tag{77}$$

$$v_4 = z^3\left[\frac{S_3^0}{S_2^{-1}} + 15 z^{-2}\frac{S_3^{-2}}{S_2^{-1}} + 2z^{-1}\right], \tag{78}$$

$$v_5 = z^4\left[\frac{S_2^0}{S_2^{-1}} + 15 z^{-2}\frac{S_2^{-2}}{S_2^{-1}} + 6z^{-1}\left(\frac{S_3^{-1}}{S_2^{-1}} + 15 z^{-2}\frac{S_3^{-3}}{S_2^{-1}}\right)\right], \tag{79}$$

$$l_1 = 3z\,\frac{S_3^{-1}}{S_2^{-1}}, \tag{80}$$

$$l_2 = 15 z\,\frac{S_3^{-2}}{S_2^{-1}} + 3z^2, \tag{81}$$

$$l_3 = 15 z^2\,\frac{S_2^{-2}}{S_2^{-1}} + 3z^3\,\frac{S_3^{-1}}{S_2^{-1}} + 90 z\,\frac{S_3^{-3}}{S_2^{-1}}, \tag{82}$$

$$s_0 = 10 z\,\frac{S_3^{-2}}{S_2^{-1}}, \tag{83}$$

$$s_1 = 10 z^2\left(\frac{S_2^{-2}}{S_2^{-1}} + 6z^{-1}\frac{S_3^{-3}}{S_2^{-1}}\right), \tag{84}$$

$$\alpha_2 = \frac{z^3}{2}\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}} - z^{-1}\right) + \left(\frac{S_2^0}{S_2^{-1}} + \frac{3}{z}\,\frac{S_3^{-1}}{S_2^{-1}}\right)[(1-\gamma'')\hat{h} + \gamma''' z^{-1}] - \left(\frac{4}{3} - \gamma'\right)\left(\frac{S_3^0}{S_2^{-1}} + 15 z^{-2}\frac{S_3^{-2}}{S_2^{-1}} + 2z^{-1}\right)\right\}, \tag{85}$$

$$\beta_1 = -3z^2\left[1 + 5z^{-1}\frac{S_3^{-2}}{S_2^{-1}} - \left(\frac{S_3^{-1}}{S_2^{-1}}\right)^2\right]. \tag{86}$$

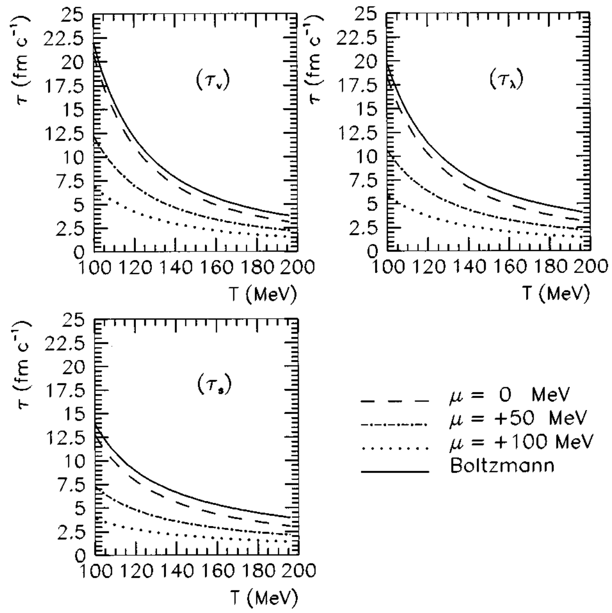

FIG. 3. Relaxation times associated with the different transport coefficients.

First of all, we can notice that the relaxation times are proportional to the corresponding transport coefficients. Thus the reduction found in the last section will imply a decrease of the relaxation times as well. Therefore the linear laws (30) must not be interpreted naively as in Sec. III C. They actually mean that the flows needed to reach equilibrium for a ''Bose-Einstein'' system are smaller than for a ''Boltzmann'' system; but they do not give any information on the velocity of the flow.

The relaxation times are depicted on Fig. 3 for several values of the chemical potential. One can see that these times are (for $\mu_\pi = 0$) smaller by 15% for $T = 150$ MeV than those obtained with a Boltzmann distribution (see Ref. [6]) or in a relaxation time approximation (see Ref. [10]). This is mainly due to the fact that the relaxation times are proportional to the transport coefficients (see the comment above) and inversely proportional to the density (we know that, for some given temperature and chemical potential, $\rho_{\mathrm{BE}} > \rho_{\mathrm{Boltz}}$). It is also interesting to look at the case where $\mu_\pi > 0$ because, as noticed in the litterature ([12], for example), the hot pion gas is thought to be out of chemical equilibrium. We remark that for $\mu_\pi = 100$ MeV, which represents a reasonable value for the hot pion gas [12], the relaxation times are divided by 3. For instance, at $T = 150$ MeV, the system needs aproximately 2.5 fm to reach its thermal equilibrium instead of 7.5 fm.

Other calculations qualitatively and quantitatively confirm our results. In Ref. [8], it has been shown that a characteristic time scale (the mean collision time $\overline{\tau_c}$) decreases when one incorporates Bose enhancement factors. Moreover, at $T = 150$ MeV and $\mu_\pi = 100$ MeV, these authors have obtained $\overline{\tau_c} \sim 1$ fm. In addition, it is known that only two or three collisions are needed to reach thermal equilibrium, that is, 2 or 3 fm, as our calculation shows.

V. CONCLUSION

We have derived the transport coefficients for a single-component gas of bosons. Our results exhibit a large difference from previous calculations. These coefficients have then been used for the determination of transport relaxation times. It has been found that these times are much smaller than those computed with a Boltzmann distribution. From this we can conclude that the pion gas is probably thermally equilibrated at early stage of its life (2.5 fm compared with about 10–20 fm). If we look at the other relaxation times, we can also say that the pion gas has enough time to dissipate all other gradients (velocity, etc.) before the freeze-out.

Several extensions of the present work are possible: a systematic study of the asymptotic values of the transport coefficients in the nonrelativistic and ultrarelativistic limits, a generalization to mixtures, or the incorporation of fermions [the only difference for fermions is the equation of state; one just has to replace S_α^n by $S'^{\,n}_\alpha=\Sigma_k(-1)^k k^\alpha e^{(km\mu/k_BT)}K_n(kz)$]. The latter point is motivated by the aim of taking into account the effect of the nucleons in the hadronic phase produced in ultrarelativistic heavy-ion collisions.

We must also notice that these results have been obtained with a free cross section although we know that, due to medium effects, the pion-pion interaction can be notably modified [13]. Finally, we can remark that chemical potentials of about 100 MeV correspond to systems with a relatively high density in which three-body interactions, not taken into account in our transport equation, can play a non-negligible role.

The author is indebted to J. Delorme and G. Chanfray for a critical reading of the manuscript.

APPENDIX A: CALCULATIONS OF TRANSPORT COEFFICIENTS AT FIRST ORDER

Let us start with the relations satisfied by A, B_α, and $C_{\alpha\beta}$:

$$\mathcal{C}[A]=(k_BT)^{-1}Qf^{(0)}(1+A_0f^{(0)}), \tag{A1}$$

$$\mathcal{C}[B_\alpha]=(k_BT)^{-1}(p_\nu U^\nu+mh)\Delta_{\alpha\beta}p^\beta f^{(0)}(1+A_0f^{(0)}), \tag{A2}$$

$$\mathcal{C}[C_{\alpha\beta}]=(k_BT)^{-1}\langle p_\alpha p_\beta\rangle f^{(0)}(1+A_0f^{(0)}). \tag{A3}$$

First of all, we can remark that from linearity and Lorentz invariance of the collision term, B_α and $C_{\alpha\beta}$ must be of the form $B_\alpha=B\Delta_{\alpha\beta}p^\beta$ and $C_{\alpha\beta}=C\langle p_\alpha p_\beta\rangle$. Considering the first equation, multiplying it by $L_n^{1/2}$, a Laguerre polynomial[3] of order $\frac{1}{2}$ and of degree n, and integrating on $\mathbf{p}$, we obtain

$$\begin{aligned}\int\int\int\int & f^{(0)}f_1^{(0)}(1+A_0f'^{(0)})(1+A_0f_1'^{(0)})\\ &\times(A_0+A_1-A'-A_1')L_n^{1/2}(\tau)Wdwdw'dw_1dw_1'\\ &=(k_BT)^{-1}\int dwL_n^{1/2}(\tau)f^{(0)}(1+A_0f^{(0)}),\end{aligned} \tag{A4}$$

where $\tau\equiv-(p_\mu U^\mu+mc^2)/k_BT$ is the kinetic energy in the permanent rest frame.

After symmetrization [i.e., a formal exchange between $\mathbf{p}\leftrightarrow\mathbf{p_1}$, $\mathbf{p}'\leftrightarrow\mathbf{p}_1'$, and $(\mathbf{p},\mathbf{p}_1)\leftrightarrow(\mathbf{p}',\mathbf{p}_1')$] the above equation reads

$$\underbrace{\frac{1}{4}\int\int\int\int f^{(0)}f_1^{(0)}(1+A_0f'^{(0)})(1+A_0f_1'^{(0)})\Delta(A)\Delta(L_n^{1/2})Wdwdw_1dw'dw_1'}_{\equiv\frac{\rho^2}{m^2c}[A,L_n^{1/2}(\tau)]}=\underbrace{(k_BT)^{-1}\int dwL_n^{1/2}(\tau)f^{(0)}(1+A_0f^{(0)})Q}_{\equiv\frac{\rho}{mc}\alpha_n} \tag{A5}$$

$$\Delta(F)\equiv F+F_1-F'-F_1'. \tag{A6}$$

where

Thus A satisfies the symbolic equation

$$[A,L_n^{1/2}(\tau)]=\frac{m}{\rho}\alpha_n \quad (n=0,1,\dots). \tag{A7}$$

The Laguerre polynomials are orthogonal polynomials so that we can expand A as

$$A=\sum_{m=0}^{+\infty}a_mL_m^{1/2}(\tau). \tag{A8}$$

Thus we obtain for a_m

$$\sum_{m=0}^{+\infty}a_ma_{mn}=\frac{m}{\rho}\alpha_n \quad (n=0,1,\dots) \quad \text{with}$$
$$a_{mn}=[L_m^{1/2}(\tau),L_n^{1/2}(\tau)]. \tag{A9}$$

Moreover, a_{mn} is symmetric ($a_{mn}=a_{nm}$) and $a_{m0}=a_{m1}=0$ because m and p^ν are summational invariants.

We can also show in the same manner, starting from the imposed conditions on ϕ [Eqs. (25) and (26)] that α_0 and α_1 are equal to zero. Equation (A9) can then be written as

[3] $L_n^\alpha(x)$ is defined by $L_n^\alpha(x)\equiv\Sigma_{m=0}^n\binom{n+\alpha}{n-m}(-x)^m/m!$ and has a property which we will use in the following: $L_n^\alpha(x+y)=\Sigma_{k=0}^nL_{n-k}^{\alpha+k}(x)(-y)^k/k!$. Moreover, $L_0^\alpha(x)=1$ and $L_1^\alpha(x)=\alpha+1-x$.

$$\sum_{m=2}^{+\infty} a_m a_{mn} = \frac{m}{\rho}\alpha_n, \quad n=2,3,\ldots . \tag{A10}$$

With these notations, the bulk viscosity η_v [Eq. (50)] becomes

$$\eta_v = \frac{\rho k_B T}{m}\sum_{n=2}^{+\infty} a_n \alpha_n . \tag{A11}$$

The complete sum gives, of course, the exact result for η_v but in practice only the first few terms contribute. For instance, the approximation of order 1 obtained when we keep only the term $n=2$ reads formally $[\eta_v]_1 = (\rho k_B T/m) a_2^{(1)}\alpha_2$ with[4] $a_2^{(1)} a_{22} = (m/\rho)\alpha_2$, i.e.,

$$[\eta_v]_1 = k_B T \frac{\alpha_2^2}{a_{22}}. \tag{A12}$$

In the same way,

$$[\eta_v]_2 = \frac{\rho k_B T}{m}(a_2^{(2)}\alpha_2 + a_3^{(2)}\alpha_3) \tag{A13}$$

with

$$a_{22}a_2^{(2)} + a_{32}a_3^{(2)} = \frac{m}{\rho}\alpha_2,$$

$$a_{23}a_2^{(2)} + a_{33}a_3^{(2)} = \frac{m}{\rho}\alpha_3. \tag{A14}$$

Then

$$[\eta_v]_2 = k_B T \frac{\alpha_2^2 a_{33} - 2\alpha_2\alpha_3 a_{23} + \alpha_3^2 a_{22}}{a_{22}a_{33} - a_{23}^2}, \quad \text{etc.} \tag{A15}$$

We now have to determine the coefficients a_{mn} and α_n.

We have for a_{mn}

$$a_{mn} = [L_m^{1/2}(\tau), L_n^{1/2}(\tau)]$$
$$= \frac{m^2 c}{4\rho^2}\int\int\int\int dw\, dw_1\, dw'\, dw_1'\, W f^{(0)} f_1^{(0)} (1 + A_0 f'^{(0)})$$
$$\times (1 + A_0 f_1'^{(0)}) \Delta\{L_m^{1/2}(\tau)\}\Delta\{L_n^{1/2}(\tau)\}. \tag{A16}$$

[4] $a_m^{(p)}$ represents the approximation of order p for the coefficient a_m.

We are going to simplify this expression by making the change of variables described in [4,11] which consists in introducing the relative and global variables g_α, g'_α, P_α, and P'_α defined as

$$g_\alpha = \tfrac{1}{2}(p_{1\alpha} - p_\alpha),$$

$$g'_\alpha = \tfrac{1}{2}(p'_{1\alpha} - p'_\alpha), \tag{A17}$$

$$P_\alpha = p_\alpha + p_{1\alpha} = p'_\alpha + p'_{1\alpha} = P'_\alpha .$$

With polar coordinates, P_α reads as $P^\alpha = P(\cosh\psi, \sinh\psi \sin\bar{\theta}\cos\bar{\varphi}, \sinh\psi \sin\bar{\theta}\sin\bar{\varphi}, \sinh\psi\cos\bar{\theta})$. Let us remark, then, that

$$e^\alpha_{(1)} = (0, \cos\bar{\theta}\cos\bar{\varphi}, \cos\bar{\theta}\sin\bar{\varphi}, -\sin\bar{\theta}),$$

$$e^\alpha_{(2)} = (0, -\sin\bar{\varphi}, \cos\bar{\varphi}, 0), \tag{A18}$$

$$e^\alpha_{(3)} = (\sinh\psi, \cosh\psi\sin\bar{\theta}\cos\bar{\varphi}, \cosh\psi$$
$$\times \sin\bar{\theta}\sin\bar{\varphi}, \cosh\psi\cos\bar{\theta})$$

form an orthonormal basis of spacelike four-vectors.

Moreover, in the center of mass frame, P^α reads $P^\alpha = (P, \mathbf{0})$ and $g^\alpha = (0, \mathbf{g})$ and $e^\alpha_{(i)} = (0, \mathbf{e}_i)$ have only space components. Therefore we can write

$$\mathbf{g} = g(\sin\theta\cos\varphi\,\mathbf{e}_1 + \sin\theta\sin\varphi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3). \tag{A19}$$

Let us call Θ the scattering angle in the center of mass frame ($\mathbf{g}\cdot\mathbf{g}' = g^2\cos\Theta$) and still define the variable ψ such as $\sinh\psi = g/mc$ (so that $P = 2mc\cosh\psi$).

With these definitions, we can easily show that [7]

$$P_0 = -2mc\,\cosh\psi\,\cosh\psi,$$

$$g_0 = -mc\,\sinh\psi\,\sinh\psi\,\cos\theta, \tag{A20}$$

$$g'_0 = -mc\sinh\psi\,\sinh\psi\,\cos\theta' \quad \text{with} \quad \cos\theta'$$
$$= \cos\theta\,\cos\Theta - \sin\theta\,\sin\Theta\,\cos\phi,$$

and, by introduction of the cross section σ,

$$a_{mn} = \frac{4m^2c}{\rho^2}(mc)^6\int f^{(0)} f_1^{(0)} (1 + A_0 f'^{(0)})(1$$
$$+ A_0 f_1'^{(0)})\Delta\{L_m^{1/2}(\tau)\}\Delta\{L_m^{1/2}(\tau)\}\sigma\,\sinh^2\psi\,\cosh^3\psi$$
$$\times \sinh^3\psi\,\sin\theta\,\sin\bar{\theta}\,\sin\Theta\; d\chi\, d\psi\, d\theta\, d\varphi\, d\bar{\theta}\, d\Theta. \tag{A21}$$

Moreover,

$$f^{(0)}f_1^{(0)}(1+A_0f'^{(0)})(1+A_0f_1'^{(0)})$$
$$=\frac{A_0^{-2}e^{(2m\mu/k_BT)}e^{2z\cosh\psi\cosh\psi}}{(e^{-[(P_0c-g_0c+m\mu)/2]/k_BT}-1)(e^{-[(P_0c+g_0c+m\mu)/2]/k_BT}-1)(e^{-[(P_0c-g_0'c+m\mu)/2]/k_BT}-1)(e^{-[(P_0c+g_0'c+m\mu)/2]/k_BT}-1)}. \quad \text{(A22)}$$

Because this expression depends on neither $\overline{\theta}$ nor ϕ and $\overline{\phi}$ we obtain

$$a_{mn}=\frac{3m^2c\pi^2}{\rho^2A_0^2}(mc)^6\int\frac{\sinh^2\chi(\cosh\psi\sinh\psi)^3e^{-(2z\cosh\psi\cosh\chi)}}{(e^{F+2z\sinh\psi\sinh\chi\cos\theta'}-1)(e^{E+2z\sinh\psi\sinh\chi\cos\theta}-1)}$$
$$\times\frac{\sigma\sin\theta\sin\Theta}{(e^E-1)(e^F-1)}\Delta\{L_m^{1/2}(\tau)\}\Delta\{L_n^{1/2}(\tau)\}d\chi\,d\psi\,d\theta\,d\Theta\,d\phi \quad \text{(A23)}$$

with

$$E=z(\cosh\psi\cosh\chi-\sinh\psi\sinh\chi\cos\theta)-m\mu/k_BT,$$
$$F=z(\cosh\psi\cosh\chi-\sinh\psi\sinh\chi\cos\theta')-m\mu/k_BT. \quad \text{(A24)}$$

We have now to calculate

$$a_{mn}^{**}\equiv\Delta\{L_m^{1/2}(\tau)\}\Delta\{L_n^{1/2}(\tau)\}. \quad \text{(A25)}$$

With the use of Laguerre polynomial properties, we arrive at

$$a_{mn}^{**}=\sum_{k,l=2}^{k=m,l=n}L_{m-k}^{1/2+k}(-z)L_{n-l}^{1/2+k}(-z)A_{kl}^{**} \quad \text{(A26)}$$

where

$$A_{kl}^{**}\equiv A_kA_l \quad \text{and} \quad A_k=\frac{z^k}{k!}u^{\alpha_1\cdots\alpha_k}\delta_{\alpha_1\cdots\alpha_k}, \quad \text{(A27)}$$

with the following short-hand notations

$$u^{\alpha_1\cdots\alpha_k}=c^{-k}U^{\alpha_1}\cdots U^{\alpha_k},$$
$$\delta_{\alpha_1\cdots\alpha_k}=(mc)^{-k}\delta(p_{\alpha_1}\cdots p_{\alpha_k}). \quad \text{(A28)}$$

The only unknown quantity, $\delta(p_{\alpha_1}\cdots p_{\alpha_k})$, can actually be expressed with products between g_α and P_α [7]:

$$\delta_{\alpha_1\cdots\alpha_k}=2\sum_{k=1}^{[n/2]}\binom{n}{2k}\hat{\hat{g}}_{(\alpha_1\cdots\alpha_{2k}}\hat{P}_{\alpha_{2k+1}\cdots\alpha_n)}, \quad \text{(A29)}$$

with

$$\hat{\hat{g}}=\frac{1}{(mc)^k}(g_{\alpha_1}\cdots g_{\alpha_k}-g'_{\alpha_1}\cdots g'_{\alpha_k}),$$
$$\hat{P}=\frac{1}{(2mc)^k}P_{\alpha_1}\cdots P_{\alpha_k}, \quad \text{(A30)}$$

and where the parentheses around indices indicate that we have to symmetrize the expression. To have access to a_{mn}^{**}, we just have to know A_{mn}^{**} as we can see by writing particular cases of the formula (A26):

$$a_{22}^{**}=A_{22}^{**},$$
$$a_{23}^{**}=A_{23}^{**}+L_1^{5/2}(-z)a_{22}^{**},$$
$$a_{33}^{**}=A_{33}^{**}+2L_1^{5/2}(-z)a_{23}^{**}-[L_1^{5/2}(-z)]^2a_{22}^{**}, \quad \text{etc.} \quad \text{(A31)}$$

And, by application of (A27),

$$A_{22}^{**}=z^4(\sinh\psi\sinh\chi)^4(\cos^2\theta-\cos^2\theta')^2,$$
$$A_{23}^{**}=-z^5(\sinh\psi\sinh\chi)^4(\cos^2\theta-\cos^2\theta')^2\cosh\psi\cosh\chi,$$
$$A_{33}^{**}=z^6(\sinh\psi\sinh\chi)^4(\cos^2\theta-\cos^2\theta')^2\times(\cosh\psi\cosh\chi)^2, \quad \text{etc.} \quad \text{(A32)}$$

Then, if we define

$$\chi_{ijkl}=\frac{2z^6ce^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty}d\psi(\cosh\psi\sinh\psi)^3\cosh^i\psi\sinh^j\psi$$
$$\times\int_0^\pi d\Theta\sin\Theta\,\sigma(\psi,\Theta)$$
$$\times\int_0^{+\infty}d\chi\sinh^2\chi\cosh^i\chi\sinh^j\chi\int_0^{2\pi}d\phi$$
$$\times\int_0^\pi d\theta\sin\theta\frac{e^{-2z\cosh\psi\cosh\chi}}{(e^E-1)(e^F-1)}$$
$$\times\frac{[\cos^2\theta-\cos^2\theta']^k[\cos^2\theta+\cos^2\theta']^l}{(e^{E+2z\sinh\psi\sinh\chi\cos\theta}-1)(e^{F+2z\sinh\psi\sinh\chi}-1)}, \quad \text{(A33)}$$

we obtain

$$a_{22}=\chi_{0420},$$
$$a_{23}=z(\chi_{0420}-\chi_{1420})+\tfrac{7}{2}\chi_{0420},$$

$$a_{33}=z^2\chi_{2420}+2\left(\frac{7}{2}+z\right)^2\left[z(\chi_{0420}-\chi_{1420})+\frac{7}{2}\chi_{0420}\right]$$
$$-\left[\frac{1}{2}z^2+\frac{9}{2}z+\frac{63}{8}\right]\chi_{0420},\quad \text{etc.} \tag{A34}$$

Let us now derive an expression for α_n. By using the same property as for (A26) one obtains

$$\alpha_n=-\frac{mc}{\rho k_B T}\int dw f^{(0)}(1+Af^{(0)})L_n^{1/2}(\tau)Q,$$
$$=-\frac{mc}{\rho k_B T}\sum_{i=2}^{n}\frac{z^i}{i!}L_{n-i}^{1/2+i}(-z)u_{\nu_1\cdots\nu_i}Q^{\nu_1\cdots\nu_i}, \tag{A35}$$

with

$$Q^{\nu_1\cdots\nu_i}=\frac{1}{(mc)^i}\int p^{\nu_1}\cdots p^{\nu_i}Qf^{(0)}(1+A_0f^{(0)})dw. \tag{A36}$$

Because of the form of Q, the knowledge of $Q^{\nu_1\cdots\nu_i}$ is governed by the knowledge of the moments of the distribution $f^{(0)}(1+A_0f^{(0)})$, or, in other words, by the tensor

$$F_{\nu_1\cdots\nu_n}\equiv\int p_{\nu_1}\cdots p_{\nu_n}f^{(0)}(1+Af^{(0)})dw. \tag{A37}$$

The result of the integral is a completely symmetric tensor of rank n. A complete basis of such tensors is given by

$$(\Delta u)_{nl}=\frac{1}{c^{n-2l}}\Delta_{(\alpha_1\alpha_2}\cdots\Delta_{\alpha_{2l-1}\alpha_{2l}}U_{\alpha_{2l+1}}U_{\alpha_{2l+2}}\cdots U_{\alpha_n)},$$
$$l\in\left[0,\left[\frac{n}{2}\right]\right]. \tag{A38}$$

Thus we can write [7]

$$F_{\nu_1\cdots\nu_n}=4\pi(mc)^{n+2}A_0^{-1}\sum_{l=0}^{[n/2]}a_{ln}(\Delta u)_{nl} \tag{A39}$$

with

$$a_{ln}=\frac{(-1)^n n!}{4\pi m^2c^2(2l+1)!(n-2l)!A_0^{-1}}$$
$$\times\int\frac{d^3\mathbf{p}}{\sqrt{\mathbf{p}^2+m^2c^2}}f^{(0)}(1+A_0f^{(0)})$$
$$\times\left(\frac{\Delta^{\beta\gamma}p_\beta p_\gamma}{m^2c^2}\right)^l\left(\frac{p_\delta U^\delta}{mc^2}\right)^{n-2l}. \tag{A40}$$

Because

$$f^{(0)}(1+A_0f^{(0)})=A_0^{-1}\sum_{k=1}ke^{kz\sqrt{p^2+1}+km\mu/k_BT}, \tag{A41}$$

we arrive at

$$a_{ln}=\sum_{k=1}^{+\infty}k\sum_{s=0}^{[n/2-l]}(-1)^s(2l+2s+1)!!\binom{l+s}{s}$$
$$\times\binom{n}{2l+2s}\frac{K_{n+l-s+1}(kz)}{(kz)^{l+s+1}}e^{k\mu/k_BT}. \tag{A42}$$

Therefore

$$F_{\nu_1\cdots\nu_n}=\frac{\rho c(mc)^n}{mk_BTS_2^{-1}}\sum_{l=0}^{[n/2]}a_{ln}(\Delta u)_{ln}. \tag{A43}$$

The various moments can be written in the form

$$F=\frac{\rho}{m^2c}\left(\frac{S_3^0}{S_2^{-1}}-\frac{4}{z}\right),$$

$$F_\alpha=\frac{\rho}{m}u_\alpha\frac{S_2^0}{S_2^{-1}},$$

$$F_{\alpha\beta}=\rho c\left(\frac{S_3^0}{S_2^{-1}}u_{\alpha\beta}+\frac{1}{z}(\Delta_{\alpha\beta}-u_{\alpha\beta})\right),$$

$$F_{\alpha\beta\gamma}=m\rho c^2\left(u_{\alpha\beta\gamma}\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}(\Delta_{(\alpha\beta}u_{\gamma)}+u_{\alpha\beta\gamma})\frac{S_3^{-1}}{S_2^{-1}}\right),$$

$$F_{\alpha\beta\gamma\delta}=m^2\rho c^3\left[\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)u_{\alpha\beta\gamma\delta}\right.$$
$$+\left(6z^{-1}+30z^{-2}\frac{S_3^{-2}}{S_2^{-1}}\right)\Delta_{(\alpha\beta}u_{\gamma\delta)}$$
$$\left.+3z^{-2}\frac{S_3^{-2}}{S_2^{-1}}\Delta_{(\alpha\beta}\Delta_{\gamma\delta)}\right]. \tag{A44}$$

Thus

$$\alpha_2=\frac{z^3}{2}\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}}-z^{-1}\right)+\left(\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}\frac{S_3^{-1}}{S_2^{-1}}\right)[(1-\gamma'')\hat{h}\right.$$
$$\left.+\gamma'''z^{-1}]-\left(\frac{4}{3}-\gamma'\right)\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)\right\}. \tag{A45}$$

This enables us to write explicitly $[\eta_v]_1$, that is, the lowest order approximation in the Chapman-Enskog expansion of first order.

For λ and η_s, the procedure is very similar: one expresses both coefficients in the form of an infinite sum,

$$\lambda=\frac{1}{3}\,\frac{\rho k_B^2T}{m}\sum_{n=1}^{+\infty} b_n\beta_n\,, \tag{A46}$$

where b_n is defined by $B=\Sigma_{n=0}^{+\infty}L_n^{3/2}(\tau)b_n$ and is a solution of $\Sigma_{m=1}^{+\infty}b_m b_{mn}=\rho^{-1}\beta_n$.

β_n, as far as it is concerned, reads

$$\beta_n=\frac{mc}{\rho k_B^2T^2}\int f^{(0)}(1+A_0f^{(0)})L_n^{3/2}(\tau)\times(p_\gamma U^\gamma+mh)\Delta_{\alpha\beta}p^\alpha p^\beta dw \tag{A47}$$

and b_{mn} is equal to

$$b_{mn}=\frac{1}{mk_BT}[L_m^{3/2}(\tau)p^\alpha,L_n^{3/2}(\tau)\Delta_{\alpha\beta}p^\beta]. \tag{A48}$$

In the same way, if we note that

$$\gamma_n=\frac{c}{\rho k_B^2T^2}\int f^{(0)}(1+A_0f^{(0)})L_n^{5/2}(\tau)\langle p_\alpha p_\beta\rangle\langle p^\alpha p^\beta\rangle dw \tag{A49}$$

and we look for C in the form $C=\Sigma_{m=0}^{+\infty}c_mL_m^{5/2}(\tau)$, we arrive at

$$\sum_{m=0}^{+\infty}c_mc_{mn}=\frac{1}{\rho k_BT}\gamma_n \quad \text{with} \quad c_{mn}=\frac{1}{(mk_BT)}\times[L_m^{5/2}(\tau)\langle p_\alpha p_\beta\rangle,L_n^{5/2}(\tau)\langle p^\alpha p^\beta\rangle] \tag{A50}$$

and η_v satisfying

$$\eta_v=\frac{1}{10}\rho(k_BT)^2\sum_{m=0}^{+\infty}c_m\gamma_m\,. \tag{A51}$$

The derivation is then very similar to that for η_v and, as a consequence, will not be detailed here.

APPENDIX B: EXPRESSIONS OF TRANSPORT COEFFICIENTS UP TO THIRD ORDER

In this appendix we give the complete expressions of the transport coefficients up to third order necessary for the numerical computation.

Let us start with the bulk viscosity. $[\eta_v]_1,[\eta_v]_2$, and $[\eta_v]_3$ are given with respect to a_{mn} $(m,n=2,3,4)$ and α_n $(n=2,3,4)$ (see Appendix A) by the following expressions:

$$[\eta_v]_1=k_BT\frac{\alpha_2^2}{a_{22}}, \tag{B1}$$

$$[\eta_v]_2=k_BT\frac{\alpha_2^2a_{33}-2\alpha_2\alpha_3a_{23}+\alpha_3^2a_{22}}{a_{22}a_{33}-a_{23}^2}, \tag{B2}$$

$$[\eta_v]_3=k_BT\left[\frac{(a_{33}a_{44}-a_{34}^2)\alpha_2^2+(a_{22}a_{44}-a_{24}^2)\alpha_3^2+(a_{22}a_{33}-a_{23}^2)\alpha_4^2+2(a_{24}a_{34}-a_{23}a_{44})\alpha_2\alpha_3}{a_{22}a_{33}a_{44}+2a_{23}a_{34}a_{24}-a_{33}a_{24}^2-a_{22}a_{34}^2-a_{44}a_{23}^2}+\frac{2(a_{23}a_{34}-a_{24}a_{33})\alpha_2\alpha_4+2(a_{23}a_{24}-a_{22}a_{34})\alpha_3\alpha_4}{a_{22}a_{33}a_{44}+2a_{23}a_{34}a_{24}-a_{33}a_{24}^2-a_{22}a_{34}^2-a_{44}a_{23}^2}\right]. \tag{B3}$$

The coefficients α_n, as far as they are concerned, can be expressed in terms of the sum S_n^α,z, and the quantities γ',γ'', and γ''' all defined in the main body. The result reads

$$\alpha_2=\frac{z^3}{2}\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}}-z^{-1}\right)+\left(\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}\,\frac{S_3^{-1}}{S_2^{-1}}\right)[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]-\left(\frac{4}{3}-\gamma'\right)\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)\right\}, \tag{B4}$$

$$\begin{aligned}\alpha_3=&\frac{z^3}{2}\left(\frac{7}{2}+z\right)\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}}-z^{-1}\right)+\left(\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}\,\frac{S_3^{-1}}{S_2^{-1}}\right)\right.\\&\left.\times[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]-\left(\frac{4}{3}-\gamma'\right)\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)\right\}-\frac{z^4}{6}\left\{\frac{1}{3}\left(\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}\,\frac{S_3^{-1}}{S_2^{-1}}\right)\right.\\&\left.+[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)-\left(\frac{4}{3}-\gamma'\right)\left(\frac{S_2^0}{S_2^{-1}}+15z^{-2}\frac{S_2^{-2}}{S_2^{-1}}+6z^{-1}\frac{S_3^{-1}}{S_2^{-1}}+90z^{-3}\frac{S_3^{-3}}{S_2^{-1}}\right)\right\},\end{aligned} \tag{B5}$$

$$\alpha_4=\frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)\frac{z^3}{2}\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}}-\frac{1}{z}\right)+\left(\frac{S_2^0}{S_2^{-1}}+3z^{-1}\frac{S_3^{-1}}{S_2^{-1}}\right)[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]-\left(\frac{4}{3}-\gamma'\right)\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)\right\}$$
$$+\left(\frac{9}{2}+z\right)\frac{z^4}{6}\left\{\frac{1}{3}\left(\frac{S_2^0}{S_2^{-1}}+\frac{3}{z}\frac{S_3^{-1}}{S_2^{-1}}\right)+[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)-\left(\frac{4}{3}-\gamma'\right)\right.$$
$$\left.\times\left[\frac{S_2^0}{S_2^{-1}}+15z^{-2}\frac{S_2^{-2}}{S_2^{-1}}+6z^{-1}\frac{S_3^{-1}}{S_2^{-1}}+90z^{-3}\frac{S_3^{-3}}{S_2^{-1}}\right]\right\}+\frac{z^5}{24}\left\{\frac{1}{3}\left(\frac{S_3^0}{S_2^{-1}}+15z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+2z^{-1}\right)-[(1-\gamma'')\hat{h}+\gamma'''z^{-1}]\right.$$
$$\left.\times\left[\frac{S_2^0}{S_2^{-1}}+15z^{-2}\frac{S_2^{-2}}{S_2^{-1}}+6z^{-1}\frac{S_3^{-1}}{S_2^{-1}}+90z^{-3}\frac{S_3^{-3}}{S_2^{-1}}\right]-\left(\frac{4}{3}-\gamma'\right)\left[\frac{S_3^0}{S_2^{-1}}+45z^{-2}\frac{S_3^{-2}}{S_2^{-1}}+630z^{-4}\frac{S_3^{-4}}{S_2^{-1}}+5z^{-1}\left(1+21z^{-2}\frac{S_2^{-3}}{S_2^{-1}}\right)\right]\right\}. \tag{B6}$$

The coefficients a_{mn} can be also calculated numerically thanks to the formulas

$$a_{22}=A_{22}, \tag{B7}$$

$$a_{23}=A_{23}+\left(\frac{7}{2}+z\right)A_{22}, \tag{B8}$$

$$a_{33}=A_{33}+2\left(\frac{7}{2}+z\right)A_{23}+\left(\frac{7}{2}+z\right)^2A_{22}, \tag{B9}$$

$$a_{24}=A_{24}+\left(\frac{9}{2}+z\right)A_{23}+\frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)A_{22}, \tag{B10}$$

$$a_{34}=A_{34}+\left(\frac{9}{2}+z\right)A_{33}+\left(\frac{3}{2}z^2+\frac{25}{2}z+\frac{189}{4}\right)A_{23}+\left(\frac{7}{2}+z\right)A_{24}+\frac{1}{2}\left(\frac{7}{2}+z\right)\left(z^2+9z+\frac{63}{4}\right)A_{22}, \tag{B11}$$

$$a_{44}=A_{44}+\left(\frac{9}{2}+z\right)^2A_{33}+\frac{1}{4}\left(z^2+9z+\frac{63}{4}\right)^2A_{22}+\left(z^2+9z+\frac{63}{4}\right)\left(\frac{9}{2}+z\right)A_{23}+\left(z^2+9z+\frac{63}{4}\right)A_{24}+2\left(\frac{9}{2}+z\right)A_{34}, \tag{B12}$$

where A_{mn} is expressed in terms of the integrals χ_{ijkl} defined in Appendix A:

$$A_{22}=\chi_{0420}, \tag{B13}$$

$$A_{23}=-z\chi_{1420}, \tag{B14}$$

$$A_{33}=z^2\chi_{2420}, \tag{B15}$$

$$A_{24}=\frac{z^2}{12}[6\chi_{2420}+\chi_{0621}], \tag{B16}$$

$$A_{34}=\frac{-z^3}{12}[6\chi_{3420}+\chi_{1621}], \tag{B17}$$

$$A_{44}=\frac{z^4}{144}[36\chi_{4420}+\chi_{0822}+12\chi_{2621}]. \tag{B18}$$

For the thermal conductivity one obtains some expressions comparable to (B1), (B2), and (B3) but with some new coefficients β_n $(n=1,2,3)$ and b_{mn} $(m,n=1,2,3)$:

$$[\lambda]_1=\frac{1}{3}\frac{k_B^2T}{m}\frac{\beta_1^2}{b_{11}}, \tag{B19}$$

$$[\lambda]_2=\frac{1}{3}\frac{k_B^2T}{m}\frac{\beta_1^2b_{22}-2\beta_1\beta_2b_{12}+\beta_2^2b_{11}}{b_{11}b_{22}-b_{12}^2}, \tag{B20}$$

$$[\lambda]_3=\frac{1}{3}\frac{k_B^2T}{m}\left[\frac{(b_{22}b_{33}-b_{23}^2)\beta_1^2+(b_{11}b_{33}-b_{13}^2)\beta_2^2+(b_{11}b_{22}-b_{12}^2)\beta_3^2+2(b_{13}b_{23}-b_{12}b_{33})\beta_1\beta_2}{b_{11}b_{22}b_{33}+2b_{12}b_{23}b_{13}-b_{22}b_{13}^2-b_{11}b_{23}^2-b_{33}b_{12}^2}+\frac{2(b_{12}b_{23}-b_{13}b_{22})\beta_1\beta_3+2(b_{12}b_{13}-b_{11}b_{23})\beta_2\beta_3}{b_{11}b_{22}b_{33}+2b_{12}b_{23}b_{13}-b_{22}b_{13}^2-b_{11}b_{23}^2-b_{33}b_{12}^2}\right]. \tag{B21}$$

The coefficients β_n read

$$\beta_1 = -3z^2\left[1+5z^{-1}\frac{S_3^{-2}}{S_2^{-1}} - \left(\frac{S_3^{-1}}{S_2^{-1}}\right)^2\right], \tag{B22}$$

$$\beta_2 = \left(\frac{7}{2}+z\right)\beta_1 + \frac{z^3}{2}\left[3\left(\frac{S_3^{-1}}{S_2^{-1}} + \frac{30}{z^2}\frac{S_3^{-3}}{S_2^{-1}} + \frac{5}{z}\frac{S_2^{-2}}{S_2^{-1}}\right) - 3\frac{S_3^{-1}}{S_2^{-1}}\left(1+\frac{5}{z}\frac{S_3^{-2}}{S_2^{-1}}\right)\right], \tag{B23}$$

$$\beta_3 = \frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)\beta_1 + \frac{3}{2}\left(\frac{9}{2}+z\right)z^3\left[\frac{S_3^{-1}}{S_2^{-1}} + \frac{30}{z^2}\frac{S_3^{-3}}{S_2^{-1}} + \frac{5}{z}\frac{S_2^{-2}}{S_2^{-1}} - \frac{S_3^{-1}}{S_2^{-1}}\left(1+\frac{5}{z}\frac{S_3^{-2}}{S_2^{-1}}\right)\right] - \frac{z^4}{2}\left[1+35z^{-2}\frac{S_2^{-3}}{S_2^{-1}} + \frac{10}{z}\left(\frac{S_3^{-2}}{S_2^{-1}} + \frac{21}{z^2}\frac{S_3^{-4}}{S_2^{-1}}\right) - \frac{S_3^{-1}}{S_2^{-1}}\left(\frac{S_3^{-1}}{S_2^{-1}} + \frac{30}{z^2}\frac{S_3^{-3}}{S_2^{-1}} + \frac{5}{z}\frac{S_2^{-2}}{S_2^{-1}}\right)\right]. \tag{B24}$$

And for b_{mn} one obtains

$$b_{11} = B'_{11}, \tag{B25}$$

$$b_{12} = B'_{12} + \left(\frac{7}{2}+z\right)B'_{11}, \tag{B26}$$

$$b_{22} = B'_{22} + 2\left(\frac{7}{2}+z\right)B'_{12} + \left(\frac{7}{2}+z\right)^2 B'_{11}, \tag{B27}$$

$$b_{13} = B'_{13} + \left(\frac{9}{2}+z\right)B'_{12} + \frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)B'_{11}, \tag{B28}$$

$$b_{23} = B'_{23} + \left(\frac{9}{2}+z\right)B'_{22} + \left(\frac{3}{2}z^2+\frac{25}{2}z+\frac{189}{4}\right)B'_{12} + \left(\frac{7}{2}+z\right)B'_{13} + \frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)\left(\frac{7}{2}+z\right)B'_{11}, \tag{B29}$$

$$b_{33} = B'_{33} + \left(\frac{9}{2}+z\right)^2 B'_{22} + \frac{1}{4}\left(z^2+9z+\frac{63}{4}\right)^2 B'_{11} + \left(z^2+9z+\frac{63}{4}\right)\left(\frac{9}{2}+z\right)B'_{12} + \left(z^2+9z+\frac{63}{4}\right)B'_{13} + 2\left(\frac{9}{2}+z\right)B'_{23}, \tag{B30}$$

where B'_{mn} are equal to

$$B'_{11} = B_{11} + 4\frac{A_{22}}{z^2}, \tag{B31}$$

$$B'_{12} = B_{12} + 6\frac{A_{23}}{z^2}, \tag{B32}$$

$$B'_{22} = B_{22} + 9\frac{A_{33}}{z^2} \tag{B33}$$

$$B'_{13} = B_{13} + 8\frac{A_{24}}{z^2}, \tag{B34}$$

$$B'_{23} = B_{23} + 12\frac{A_{34}}{z^2}, \tag{B35}$$

$$B'_{33} = B_{33} + 16\frac{A_{44}}{z^2} \tag{B36}$$

and where B_{mn} read

$$B_{11} = 4\chi'_{0402100}, \tag{B37}$$

$$B_{12} = -4z\chi'_{1412100}, \tag{B38}$$

$$B_{22} = -z^4\chi''_{24044} + 4z^2\chi'_{2422100}, \tag{B39}$$

$$B_{13} = \frac{2z^2}{3}[3\chi'_{2422100} + \chi'_{0604010}], \tag{B40}$$

$$B_{23} = \frac{z^3}{3}[3z^2\chi''_{34144} - 6\chi'_{3432100} - 2\chi'_{1614010}], \tag{B41}$$

$$B_{33} = \frac{z^4}{9}[-9z^2\chi''_{44244} + 9\chi'_{4442100} + 6\chi'_{2624010} + \chi'_{0806001}] \tag{B42}$$

(the integrals $\chi'_{ijklmno}$ and χ''_{ijklm} are defined at the end of this appendix).

Similarly, for η_s one obtains

$$[\eta_s]_1 = \frac{k_B T}{10}\frac{\gamma_0^2}{c_{00}}, \tag{B43}$$

$$[\eta_s]_2 = \frac{k_B T}{10}\frac{\gamma_0^2 c_{11} - 2\gamma_0\gamma_1 c_{01} + \gamma_1^2 c_{00}}{c_{00}c_{11} - c_{01}^2}, \tag{B44}$$

$$[\eta_s]_3 = \frac{k_B T}{10}\left[\frac{(c_{11}c_{22}-c_{12}^2)\gamma_0^2+(c_{00}c_{22}-c_{02}^2)\gamma_1^2+(c_{00}c_{11}-c_{01}^2)\gamma_2^2+2(c_{02}c_{34}-c_{23}c_{44})\gamma_0\gamma_1}{c_{00}c_{11}c_{22}+2c_{01}c_{12}c_{02}-c_{11}c_{02}^2-c_{00}c_{12}^2-c_{22}c_{01}^2}\right.$$
$$\left.+\frac{2(c_{01}c_{12}-c_{02}c_{11})\gamma_0\gamma_2+2(c_{01}c_{02}-c_{00}c_{12})\gamma_1\gamma_2}{c_{00}c_{11}c_{22}+2c_{01}c_{12}c_{02}-c_{11}c_{02}^2-c_{00}c_{12}^2-c_{22}c_{01}^2}\right], \tag{B45}$$

with

$$\gamma_0 = -\frac{S_3^{-2}}{S_2^{-1}}, \tag{B46}$$

$$\gamma_1 = -\left(\frac{7}{2}+z\right)\frac{S_3^{-2}}{S_2^{-1}}+10z\left(\frac{S_2^{-2}}{S_2^{-1}}+\frac{6}{z}\frac{S_3^{-3}}{S_2^{-1}}\right), \tag{B47}$$

$$\gamma_2 = -5\left(z^2+9z+\frac{63}{4}\right)\frac{S_3^{-2}}{S_2^{-1}}+10z\left(\frac{9}{2}+z\right)\left(\frac{S_2^{-2}}{S_2^{-1}}+\frac{6}{z}\frac{S_3^{-3}}{S_2^{-1}}\right)$$
$$-5z^2\left(\frac{S_3^{-2}}{S_2^{-1}}+\frac{42}{z^2}\frac{S_3^{-4}}{S_2^{-1}}+\frac{7}{z}\frac{S_2^{-3}}{S_2^{-1}}\right), \tag{B48}$$

and where

$$c_{00} = C'_{00}, \tag{B49}$$

$$c_{01} = C'_{01}+\left(\frac{7}{2}+z\right)C'_{00}, \tag{B50}$$

$$c_{11} = C'_{11}+2\left(\frac{7}{2}+z\right)C'_{01}+\left(\frac{7}{2}+z\right)^2 C'_{00}, \tag{B51}$$

$$c_{02} = C'_{02}+\left(\frac{9}{2}+z\right)C'_{01}+\frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)C'_{00}, \tag{B52}$$

$$c_{12} = C'_{12}+\left(\frac{9}{2}+z\right)C'_{11}+\left(\frac{3}{2}z^2+\frac{25}{2}z+\frac{189}{4}\right)C'_{01}$$
$$+\left(\frac{7}{2}+z\right)C'_{02}+\frac{1}{2}\left(z^2+9z+\frac{63}{4}\right)\left(\frac{7}{2}+z\right)C'_{00}, \tag{B53}$$

$$c_{22} = C'_{22}+\left(\frac{9}{2}+z\right)^2 C'_{11}+\frac{1}{4}\left(z^2+9z+\frac{63}{4}\right)^2 C'_{00}$$
$$+\left(z^2+9z+\frac{63}{4}\right)C'_{02}+\left(z^2+9z+\frac{63}{4}\right)\left(\frac{9}{2}+z\right)C'_{01}$$
$$+2\left(\frac{9}{2}+z\right)C'_{12}, \tag{B54}$$

with

$$C'_{00} = C_{00}+2z^{-1}B_{11}+\frac{8}{3}\frac{A_{22}}{z^2}, \tag{B55}$$

$$C'_{01} = C_{01}+4z^{-1}B_{12}+8\frac{A_{23}}{z^2}, \tag{B56}$$

$$C'_{11} = C_{11}+8z^{-1}B_{22}+24\frac{A_{33}}{z^2}, \tag{B57}$$

$$C'_{02} = C_{02}+6z^{-1}B_{13}+\frac{2}{3}\frac{A_{22}}{z^2}+16\frac{A_{24}}{z^2}, \tag{B58}$$

$$C'_{12} = C_{12}+12z^{-1}B_{23}+2\frac{A_{23}}{z^2}+48\frac{A_{34}}{z^2}, \tag{B59}$$

$$C'_{22} = C_{22}+18z^{-1}B_{33}+8\frac{A_{24}}{z^2}+96\frac{A_{44}}{z^2}, \tag{B60}$$

and

$$C_{00} = 8\chi'''_{040002}, \tag{B61}$$

$$C_{01} = -8\chi'''_{150001}, \tag{B62}$$

$$C_{11} = 4z^2(\chi''_{24022}+2\chi'''_{242001}), \tag{B63}$$

$$C_{02} = 2z^2(4\chi'''_{242001}+\chi'''_{060212}), \tag{B64}$$

$$C_{12} = 2z^3(2\chi''_{14122}-4\chi'''_{343001}-\chi'''_{151112}), \tag{B65}$$

$$C_{12} = 2z^3(4\chi''''_{440421}-2\chi''_{44222}+2\chi'''_{444001}+\chi''''_{440412}-2\chi'''''). \tag{B66}$$

Finally, the integrals used in the expressions of the different coefficients are listed below:

$$\chi'_{ijklmno} = \frac{2z^5 c e^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty} d\psi(\cosh\psi\sinh\psi)^3\cosh^i\psi$$
$$\times\sinh^j\psi\int_0^{\pi} d\Theta\sin\Theta\,\sigma(\psi,\Theta)$$
$$\times\int_0^{+\infty} d\chi\sinh^2\chi\,\cosh^k\chi\,\sinh^l\chi$$
$$\times\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\frac{e^{-2z\cosh\psi\cosh\chi}}{(e^E-1)(e^F-1)}$$
$$\times\frac{[\cos^2\theta+\cos^2\theta'-2\cos\theta\cos\theta'\cos\Theta]^m}{(e^{E+2z\sinh\psi\sinh\chi\cos\theta}-1)(e^{F+2z\sinh\psi\sinh\chi}-1)}$$
$$\times[\cos^4\theta+\cos^4\theta'-\cos^3\theta\cos\theta'\cos\Theta$$
$$-\cos\theta\cos^3\theta'\cos\Theta]^n\times[\cos^6\theta+\cos^6\theta'$$
$$-2\cos^3\theta\cos^3\theta'\cos\Theta]^o, \tag{B67}$$

$$\chi''_{ijklm}=\frac{2z^4ce^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty}d\psi(\cosh\psi\ \sinh\psi)^3\cosh^i\psi\ \sinh^j\psi$$
$$\times\int_0^{\pi}d\Theta\sin\Theta\,\sigma(\psi,\Theta)\int_0^{+\infty}d\chi\ \sinh^2\chi\cosh^k\chi\ \sinh^l\chi$$
$$\times\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\ \sin\theta\frac{e^{-2z\ \cosh\psi\ \cosh\chi}}{(e^E-1)(e^F-1)}$$
$$\times\frac{[\cos^m\theta-2\cos^{m/2}\theta\ \cos^{m/2}\theta'\cos^{m/2}\Theta+\cos^m\theta']}{(e^{E+2z\ \sinh\psi\ \sinh\chi\ \cos\theta}-1)(e^{F+2z\ \sinh\psi\ \sinh\chi}-1)},\qquad\text{(B68)}$$

$$\chi'''_{ijklmn}=\frac{2z^4ce^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty}d\psi(\cosh\psi\ \sinh\psi)^3\cosh^i\psi\ \sinh^j\psi\int_0^{\pi}d\Theta\sin\Theta\,\sigma(\psi,\Theta)\int_0^{+\infty}d\chi\ \sinh^2\chi\cosh^k\chi\ \sinh^l\chi$$
$$\times\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\ \sin\theta\frac{e^{-2z\ \cosh\psi\ \cosh\chi}}{(e^E-1)(e^F-1)}\frac{[\cos^2\theta+\cos^2\theta']^m(1-\cos^n\Theta)}{(e^{E+2z\ \sinh\psi\ \sinh\chi\cos\theta}-1)(e^{F+2z\ \sinh\psi\ \sinh\chi}-1)},\qquad\text{(B69)}$$

$$\chi'''_{ijklmn}=\frac{2z^4ce^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty}d\psi(\cosh\psi\ \sinh\psi)^3\cosh^i\psi\ \sinh^j\psi\int_0^{\pi}d\Theta\sin\Theta\,\sigma(\psi,\Theta)$$
$$\times\int_0^{+\infty}d\chi\ \sinh^2\chi\ \cosh^k\chi\ \sinh^l\chi\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\ \sin\theta\frac{e^{-2z\ \cosh\psi\ \cosh\chi}}{(e^E-1)(e^F-1)}$$
$$\times\frac{[\cos^{2n}\theta+(-1)^n\cos^{2n}\theta']^m}{(e^{E+2z\ \sinh\psi\ \sinh\chi\ \cos\theta}-1)(e^{F+2z\ \sinh\psi\ \sinh\chi}-1)},\qquad\text{(B70)}$$

$$\chi''''=\frac{2z^4ce^{2m\mu/k_BT}}{[S_2^{-1}]^2}\int_0^{+\infty}d\psi(\cosh\psi\ \sinh\psi)^3\cosh^8\psi\int_0^{\pi}d\Theta\ \sin\Theta\,\sigma(\psi,\Theta)\int_0^{+\infty}d\chi\ \sinh^2\chi\ \cosh^4\chi\int_0^{2\pi}d\phi$$
$$\times\int_0^{\pi}d\theta\ \sin\theta\frac{e^{-2z\ \cosh\psi\ \cosh\chi}}{(e^E-1)(e^F-1)}\frac{\cos^2\theta\ \cos^2\theta'\cos^2\Theta}{(e^{E+2z\ \sinh\psi\ \sinh\chi\ \cos\theta}-1)(e^{F+2z\ \sinh\psi\ \sinh\chi}-1)}.\qquad\text{(B71)}$$

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, England, 1970).

[2] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).

[3] M. Sasaki, in *Max Planck Festschrift*, edited by B. Kockel *et al.* (Veb. Deutscher Verlag der Wissenschaften, Berlin, 1958).

[4] W. Israel, J. Math. Phys. **4**, 1163 (1963).

[5] S. R. De Groot, W. A. Van Leeuwen, and Ch. G. Van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).

[6] M. Prakash, M. Prakash, R. Venugopalan, and G. M. Welke, Phys. Rep. **227**, 361 (1993).

[7] W. A. Van Leeuwen, P. H. Polak, and S.R. De Groot, Physica **63**, 65 (1973).

[8] G. M. Welke *et al.*, in *Proceedings of the 7th Winter Workshop on Nuclear Physics*, Key West, FL, 1991, edited by W. Bauer and J. I. Kapusta (World Scientific, Singapore, 1991), p. 189.

[9] P. Danielewicz and M. Gyulassy, Phys. Rev. D **31**, 53 (1985).

[10] S. Gavin, Nucl. Phys. **A435**, 826 (1985).

[11] W. A. Van Leeuwen and S. R. De Groot, Physica **51**, 1 (1971).

[12] M. Kataja and P. V. Ruuskanen, Phys. Lett. B **243**, 181 (1990).

[13] Z. Aouissat, G. Chanfray, P. Shuck, and G. M. Welke, Z. Phys. A **340**, 347 (1991).