Analysis of parity violation in neutron resonances

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The analysis used to determine the rms parity violating matrix element M from the longitudinal asymmetries measured by the TRIPLE Collaboration is described. The likelihood method is used to analyze the limited number of experimental data points available for each target nuclide. Much spectroscopic information is required to determine M, including resonance parameters (spins, neutron widths, resonance energies) of the s- and p-wave resonances which mix to cause the parity violation, and (for targets with $I\neq 0$) p-wave neutron partial decay amplitudes. We have developed statistical techniques to determine M in situations where incomplete spectroscopic information is available. Methods are described that can be applied when different amounts of partial spectroscopic information are available. The use of the methods is illustrated by examples.

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I. INTRODUCTION

The observation by the Dubna group [1] of large parity violating (PV) longitudinal asymmetries in compound nuclear resonances excited by polarized neutrons demonstrated that weak interaction effects are enhanced by many orders of magnitude in complex nuclei. Following these pioneering measurements, the TRIPLE Collaboration used the pulsed epithermal neutron beam from the spallation source at the Los Alamos Neutron Scattering Center (LANSCE) to measure many PV asymmetries in a single nuclide [2–5]. This group developed a statistical method to extract M, the root-mean-squared matrix element of the PV interaction between compound nuclear states, from the measured asymmetries. In this new approach the symmetry breaking matrix elements are assumed to be random variables. The status of the earlier measurements was summarized by Bowman *et al.* [6]. More recent results are discussed by Yen *et al.* [7]. The rms matrix element M can be related to the coupling strength of the weak nucleon-nucleon interaction. This relationship is described in several papers which present theories of symmetry breaking in the compound nucleus [8–10].

In this paper we describe the statistical procedures used to extract values and uncertainties for the rms PV matrix element M from the experimental data. In practice, the precision with which this rms matrix element can be determined is governed by the status of the relevant nuclear spectroscopic information. With all spectroscopic information available, the fractional error in M is determined by the number of resonances for which asymmetries are measured with small statistical errors. However, in most cases there is incomplete information available for some or all of the relevant spectroscopic quantities, and the error in M is larger than the sampling error discussed above. We discuss the methods of analysis appropriate under various circumstances, and attempt to clarify and codify the general analysis approach that our collaboration has adopted.

In these experiments the longitudinal asymmetry $p=(\sigma_+-\sigma_-)/\sigma_++\sigma_-)$ is measured for p-wave neutron resonances (σ_+ and σ_- are the resonance parts of the total cross section for positive and negative helicity neutrons). Since the parity violation in (weak) p-wave resonances is caused by mixing with (strong) s-wave resonances, clearly one needs the spectroscopic parameters for both the s- and p-wave states, including the total angular momentum J, the resonance energy E_0, and the neutron width Γ_n. We derive probability density functions for a number of circumstances, corresponding to the wide range of spectroscopic knowledge available for different nuclides. Given the probability density functions, the likelihood method can be used to determine the relevant parameters from the experimental results.

It is convenient to consider target spin $I=0$ and $I\neq 0$ separately. The $I=0$ case is much simpler. For $I=0$, all s-wave resonances must have $J=1/2$, the p-wave resonances have $J=1/2$ or 3/2, and the corresponding p-wave neutron amplitudes are pure $j=1/2$ or 3/2 (where j is the neutron total angular momentum). Usually the s-wave and p-wave neutron widths and resonance energies are known, but the spins (particularly of the much weaker p-wave resonances) often are not known. Bowman *et al.* [2] showed that for $I=0$ target nuclides, knowledge of the resonance energies and neutron widths of the s- and p-wave resonances is sufficient to extract values of M from a set of measured PV asymmetries.

More spectroscopic information is needed to characterize p-wave resonances for targets with $I\neq 0$. The p-wave states with $J=I\pm 1/2$ can be formed with both $p_{1/2}$ and $p_{3/2}$ neutrons, and thus there are two entrance channel neutron amplitudes. We use the following coupling scheme: the neutron orbital angular momentum ℓ and the neutron spin **i** are first coupled to form the projectile spin $\mathbf{j}=\mathbf{l}+\mathbf{i}$, and **j** is then coupled to the target spin **I** to form the total spin $\mathbf{J}=\mathbf{I}+\mathbf{j}$ of the target-projectile system. For s-wave resonances the projectile spin is $j=1/2$, while for p-wave resonances $j=1/2$ or 3/2. Very few projectile-spin neutron decay amplitudes have been measured. In addition, the spins of the resonances often are unknown. For these cases we derive expressions for the probability density function of the longitudinal asymmetry by averaging over unknown spectroscopic parameters. The assumptions used to determine the *a priori* distribution of these spectroscopic parameters are discussed.

The outline of the paper is as follows: in Sec. II we derive the probability density functions and in Sec. III the appropriate likelihood expressions. In Sec. IV level densities and strength functions are considered. Section V provides examples of the application of these methods to experimental data. In Sec. VI we discuss the current status of the relevant spectroscopic information, summarize the physical motivation for our analysis approach, and briefly review other work on the analysis of parity violation data. The last section provides a brief summary.

II. PROBABILITY DENSITY FUNCTIONS

A. Target spin $I=0$

For a target nucleus with $I^\pi=0^+$, the s-wave resonances have $I^\pi=1/2^+$ and the p-wave resonances have $I^\pi=1/2^-$ or $3/2^-$. The $3/2^-$ p-wave levels cannot mix with the $1/2^+$ s-wave levels through a $J=0$ PV interaction and therefore show no parity violation. The PV asymmetry was obtained in the two-level approximation by a number of authors [1,11–13]. In general the observed PV asymmetry for a given p-wave level μ has contributions from many s-wave levels ν. The PV asymmetry in this case is

$$p_\mu=2\sum_\nu \frac{V_{\nu\mu}}{E_\nu-E_\mu}\frac{g_{\nu_{1/2}}g_{\mu_{1/2}}}{\Gamma_{\mu_n}}, \tag{1}$$

where $g_{\mu_{1/2}}$ and $g_{\nu_{1/2}}$ are the neutron decay amplitudes of levels μ and ν ($g_\mu^2=\Gamma_{\mu_n}$ and $g_\nu^2=\Gamma_{\nu_n}$), E_μ and E_ν are the corresponding resonance energies, and $V_{\nu\mu}$ is the matrix element of the PV interaction between levels μ and ν. According to the statistical model of the compound nucleus, the (signed) quantities $V_{\nu\mu}$, g_μ, and g_ν are statistically independent random variables and have mean-zero Gaussian distributions. The common variance M^2 of the PV matrix elements is the mean square matrix element of the PV interaction. In most cases the neutron decay widths (Γ_{μ_n} and Γ_{ν_n}) and the resonance energies (E_μ and E_ν) are known from previous experiments. We assume hereafter that the total neutron widths and resonance energies are known.

The quantity p_μ is a sum of Gaussian random variables, the $V_{\nu\mu}$'s, and is itself a Gaussian random variable [14]. The variance of p_μ is $A_\mu^2M^2$, where

$$A_\mu^2=\sum_\nu A_{\nu\mu}^2 \quad \text{and} \quad A_{\nu\mu}^2=\left(\frac{2}{E_\nu-E_\mu}\right)^2\frac{\Gamma_\nu}{\Gamma_\mu}. \tag{2}$$

The probability density function (PDF) of p_μ is given by (unless confusion results we suppress the index μ)

$$P_p^0(p|MA)=G(p,M^2A^2), \tag{3}$$

where $G(x,\zeta^2)$ is a mean-zero Gaussian distribution of the variable x with variance ζ^2. If there is an experimental error σ in the measurement of p, the convolution theorem for Gaussian probability density functions can be used. The PDF for the asymmetry is still Gaussian, but with variance $M^2A^2+\sigma^2$:

$$P_p^0(p|MA,\sigma)=G(p,M^2A^2+\sigma^2). \tag{4}$$

B. Target spin $I\neq 0$

For a target with spin and parity I^π ($I\neq 0$) the s-wave levels can have $(I\pm 1/2)^\pi$, while the p-wave levels can have $(I\pm 1/2)^{-\pi}$ and $(I\pm 3/2)^{-\pi}$. The expression for p_μ is now

$$p_\mu=\sum_{\nu:J_\nu=J_\mu}\frac{V_{\nu\mu}}{E_\nu-E_\mu}\frac{g_{\nu_{1/2}}}{\sqrt{\Gamma_{\mu_n}}}\frac{g_{\mu_{1/2}}}{\sqrt{g_{\mu_{1/2}}^2+g_{\mu_{3/2}}^2}}, \tag{5}$$

where $g_{\mu_{1/2}}$ and $g_{\mu_{3/2}}$ are the projectile-spin ($j=1/2$ and 3/2) neutron amplitudes. Note that only the $g_{1/2}$ amplitude contributes to the parity violation. An important difference between this expression and that for $I=0$ is that the sum extends only over those s-wave levels ν having the same spin as the p-wave level μ. This difference leads to complications when the spins of the levels are not known.

1. All level spins and projectile-spin amplitudes known

If there is complete knowledge of the spectroscopic properties of the p-wave level and the nearby s-wave levels, then the PDF is similar to that for the $I=0$ case. Assume that one knows the total neutron widths and spins for the p-wave resonance and the s-wave resonances, and the projectile-spin amplitudes $g_{1/2}$ and $g_{3/2}$. Then the asymmetry p has a Gaussian distribution with variance $M^2A^2R^2$

$$P_p^I(p|MAR)=G(p,M^2A^2R^2), \tag{6}$$

where

$$A_\mu^2=\sum_{\nu:J_\nu=J_\mu}\left(\frac{2}{E_\nu-E_\mu}\right)^2\frac{\Gamma_\nu}{\Gamma_\mu} \quad \text{and} \quad R=\frac{g_{\mu_{1/2}}}{\sqrt{g_{\mu_{1/2}}^2+g_{\mu_{3/2}}^2}}. \tag{7}$$

If the asymmetry is measured with an experimental error σ, then

$$P_p^I(p|MAR,\sigma)=G(p,M^2A^2R^2+\sigma^2). \tag{8}$$

2. All level spins known, projectile-spin amplitudes not known

The quantity p is the product of R and Q, where R is given by Eq. (7) and the Gaussian random variable Q is

$$Q=\sum_{\nu:J_\nu=J_\mu}\frac{V_{\nu\mu}}{E_\nu-E_\mu}\frac{g_\nu}{\sqrt{\Gamma_{\mu_n}}}. \tag{9}$$

The quantity R also must be treated as a random variable, and the probability density function of p is no longer Gaussian. In order to obtain the PDF of p, one needs the PDF of R, which is a function of the projectile-spin amplitudes. According to the extreme statistical model [15] of the compound nucleus, the projectile-spin amplitudes $g_{1/2}$ and $g_{3/2}$ are statistically independent Gaussian random variables. (In

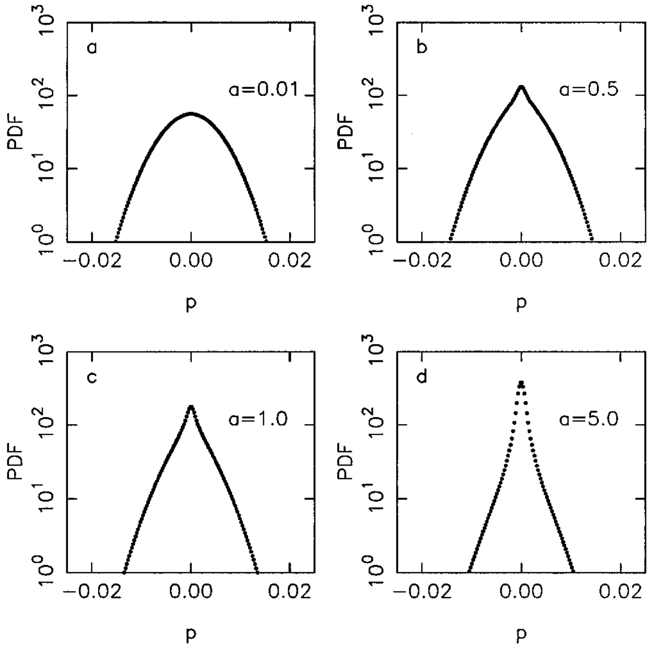

FIG. 1. Plots of $P_p^I(p|MA,a,\sigma)$ for several values of the parameter a, where $a^2 = Y^2/X^2$, the ratio of the $p_{3/2}$ and $p_{1/2}$ strength functions. (a) $a = 0.01$, (b) $a = 0.5$, (c) $a = 1.0$, (d) $a = 5.0$. See text for discussion.

fact the extreme statistical model does not always hold — see the review on amplitude correlations by Mitchell *et al.* [16]. We first assume that the g's are independent and obtain the PDF under this assumption. Then we consider the case where the projectile-spin amplitudes are correlated.)

The standard definition for the neutron strength function for spin J is [17]

$$S^{\ell}=\frac{\langle g(J)\Gamma_n^{\ell}\rangle}{(2\ell+1)D^{\ell}(J)},\quad g(J)=\frac{2J+1}{2(2I+1)},\tag{10}$$

where $g(J)$ is the spin statistical weight factor, Γ_n^{ℓ} is the average reduced neutron width, and $D^l(J)$ is the average level spacing. It is conventional in neutron physics to normalize the widths to 1 eV, $\Gamma_n^{\ell}=\sqrt{1\text{eV}/E}\,\Gamma_n/P_{\ell}$, where P_{ℓ} is the neutron penetrability.

In Sec. IV B we explicitly consider the dependence of the strength function S on the projectile-spin j. Here we need only the relative value of the p-wave strength functions (for a given resonance spin J) for the projectile-spins $j=1/2$ and $j=3/2$. After canceling common terms, the ratio of the $j=3/2$ to $j=1/2$ strength functions is simply the ratio of the average values of the reduced neutron widths. If we call X^2 the variance of $g_{1/2}$ and Y^2 the variance of $g_{3/2}$, then this ratio is simply Y^2/X^2. The PDF's for the projectile-spin amplitudes are

$$P(g_{1/2})=G(g_{1/2},X^2)\quad\text{and}\quad P(g_{3/2})=G(g_{3/2},Y^2).\tag{11}$$

It is convenient to convert the expression for R to polar coordinates: $g_{1/2}=r\,\sin\theta$ and $g_{3/2}=r\,\cos\theta$. Then

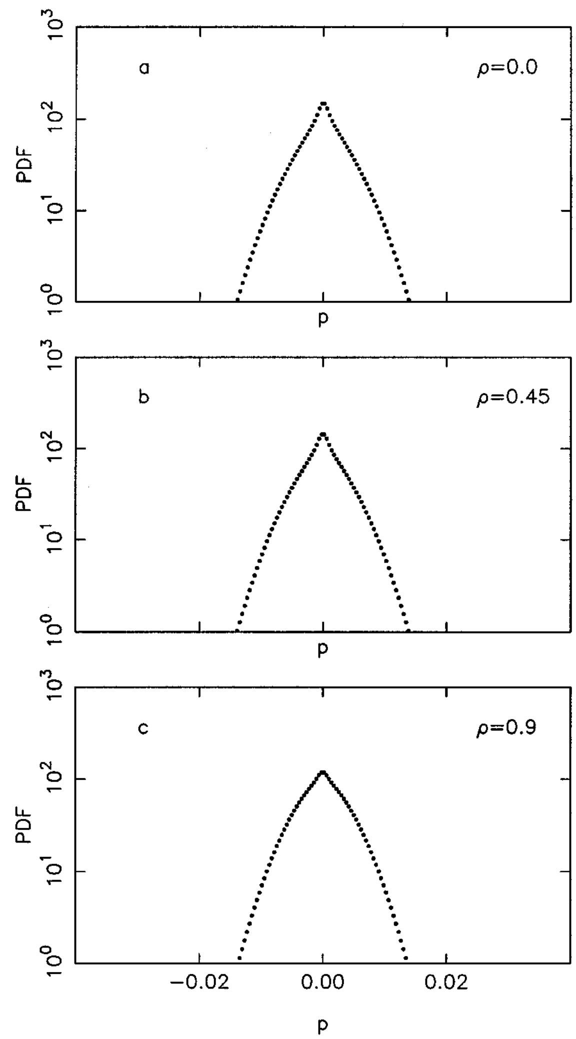

FIG. 2. Plots of $P_p^I(p|MA,a,\rho,\sigma)$ for the 29.67-eV resonance in ${}^{116}_{49}$In for several values of the linear correlation coefficient ρ, with the strength function ratio a fixed at 0.7. (a) $\rho = 0.0$, (b) $\rho = 0.45$, (c) $\rho = 0.9$. Even for extremely large values of the correlation coefficient, the maximum likelihood estimate for M is essentially unchanged. See text for discussion.

$$R=\frac{g_{1/2}}{\sqrt{g_{1/2}^2+g_{3/2}^2}}=\sin\theta\tag{12}$$

and

$$P_{r\theta}(r,\theta)rdrd\theta=\frac{1}{2\pi}\frac{rdrd\theta}{XY}\times\exp\left[\frac{-r^2}{2}\left(\frac{\sin^2\theta}{X^2}+\frac{\cos^2\theta}{Y^2}\right)\right].\tag{13}$$

Integrating with respect to r yields

$$P_{\theta}(\theta)=\frac{1}{2\pi}\frac{a}{a^2\sin^2\theta+\cos^2\theta},\tag{14}$$

where $a^2=Y^2/X^2$, the ratio of the $p_{3/2}$ and $p_{1/2}$ strength functions. Strength functions (and their experimental determination) are discussed in Sec. IV. The PDF of the product of the two independent random variables R and Q is

$$P_p^I(p|MA,a)=\frac{2}{\pi}\int_0^{\pi/2}\frac{a}{a^2\sin^2\theta+\cos^2\theta}\times G(p,M^2A^2\sin^2\theta)d\theta. \qquad (15)$$

The experimental error σ in the asymmetry p can be included by using the convolution theorem for Gaussian probability density functions, yielding

$$P_p^I(p|MA,a,\sigma)=\frac{2}{\pi}\int_0^{\pi/2}\frac{a}{a^2\sin^2\theta+\cos^2\theta}\times G(p,M^2A^2\sin^2\theta+\sigma^2)d\theta. \qquad (16)$$

Plots of $P_p^I(p|MA,a,\sigma)$ for $a=0.01$, 0.5, 1, and 5 are shown in Fig. 1. The plot for $a=0.01$ is approximately Gaussian and is essentially the same as the $I=0$ case. For $a=0.5$ the distribution is much sharper than a Gaussian, with a significant spike at $p=0$. For $a=1$ there is an even larger spike near $p=0$, while for $a=5$ the distribution is dominated by the spike at $p=0$. The effect of increasing the parameter a (and therefore the relative magnitude of the $p_{3/2}$ amplitude versus the $p_{1/2}$ amplitude) is to produce a peak near $p=0$ at the expense of large values of p. The shape of $P_p^I(p|MA,a,\sigma)$ for large values of p is sharper than Gaussian whenever the contribution of the $p_{3/2}$ amplitude is important, but is still similar (for large values of p) to the distribution for $I=0$.

As noted above, the extreme statistical model does not always hold: amplitudes (and widths) in different channels may be correlated. In fact this is predicted for doorway states [18] and for direct reactions. For example, for a fragmented isobaric analog state, there is a definite phase relation between the fine structure states μ. If the doorway (analog) is common to two or more channels, say c and c$'$, then the μc and μc$'$ amplitudes will be correlated. These predictions for doorway states were confirmed by Mitchell *et al.* [16]. They also observed large correlations under circumstances where the statistical model worked well for other observables. Since there is a very limited amount of data available on such correlations, specific predictions for the size or frequency of these correlations cannot be made reliably. However, formally the problem is straightforward — on the basis of the central limit theorem one expects the joint probability density function of $g_{\mu_{1/2}}$ and $g_{\mu_{3/2}}$ to be a bivariate Gaussian form with some linear correlation coefficient ρ. The result is

$$P_p^I(p|MA,a,\rho)=\frac{1}{2\pi}\int_0^{2\pi}\frac{a(1-\rho^2)^{1/2}}{a^2\sin^2\theta-2a\rho\sin\theta\cos\theta+\cos^2\theta}G(p,M^2A^2\sin^2\theta)d\theta, \qquad (17)$$

which reduces to Eq. (15) for $\rho=0$. The effect of the correlation is to distort the shape of the PDF, as illustrated in Fig. 2. However, what is important for the analysis is the effect of the correlation on the maximum likelihood estimate for M, not the effect on the shape of the PDF. Our calculations indicate that M is insensitive even to large correlations, i.e., to the shape of the distribution for small values of p. For example, changing ρ from 0.0 to 0.9, as done in Fig. 2, only changes M from 1.41 to 1.38 meV. In any event large correlations are expected only for special cases such as fragmented common doorways or when direct reactions are important. The additional uncertainty due to lack of information about channel correlations seems small compared to the uncertainty arising from lack of information about the ratio of the projectile-spin amplitudes. Therefore we assume in the following that the projectile-spin amplitudes are uncorrelated.

C. Level spins not known

Assume that all of the resonance parameters are known for the p-wave levels, and that the resonance parameters of the s-wave levels are known except for the spins. Our approach is to deal explicitly with the uncertain level spins in formulating the likelihood function. For large N, this gives many terms in the sum in the likelihood expression. However, in the specific applications that we are considering, there is always at least partial information on the spins of the s-wave resonances, and the total number of resonances with uncertain spin is not very large. For example, for one nuclide which we have studied, $^{115}_{49}$In, the spins of all but seven of the s-wave resonances are known from threshold to 400 eV. In practice we add terms in the likelihood expressions to account for these uncertainties.

Next consider the case where the spins of the p-wave states are unknown. The PV experimental results provide information concerning the spin of the resonance. This is simplest for $I=0$. Clearly a resonance with a large PV effect must with very high probability have $J=1/2$. A resonance with zero (within error) PV effect is more likely to have spin $J=3/2$. The argument is similar for $I\neq 0$, except there are now two allowed spins and two spins (probably) disallowed. For example, a resonance with a strong parity violation must have $J=I\pm 1/2$, and not $J=I\pm 3/2$. As we spell out below, these issues are dealt with by the choice of likelihood function.

III. LIKELIHOOD FUNCTIONS

A. Review of likelihood analysis

The next step is to develop expressions for the likelihood functions for M [14]. Likelihood formalism can be summarized as follows: assume that one has a set of theoretical expressions that predict the outcomes of experiments given some theoretical parameters m. These theoretical expressions are combined with the known statistical errors σ in the experiments to develop a joint probability density function $p_x(x|m,\sigma)$ for the outcome of an experiment to measure quantities x with errors σ involving parameters m. Assume that the actual outcome of an experiment yields values y for the quantities x. Then the likelihood function $L(m)$ is defined as

$$L(m)=P_x(y|\sigma,m)P_m(m), \tag{18}$$

where the *a priori* probability density $P_m(m)$ describes our knowledge of the parameters before the new experimental information y was available. $L(m)$ can be considered the (un-normalized) probability density that describes our knowledge of the parameters m. In most cases of interest we wish to determine some but not all of the parameters involved in $L(m)$. Suppose that the parameters are divided into two sets m_1 and m_2. Then the likelihood function for the parameters m_1, $L_1(m_1)$, can be obtained by integrating (summing) $L(m)$ over the parameters m_2.

Having constructed the appropriate likelihood function, an estimate of m, the maximum likelihood estimate (MLE), or m_L, is obtained by finding the value of m that maximizes $L(m)$. The maximum likelihood estimate m_L is a random variable in the sense that if the experiment that gave the values y is repeated, a different value of m_L will be obtained. As the number of measurements used to form $L(m)$ increases the distribution of m_L tends to a Gaussian. The estimation of confidence intervals for m_L is discussed by Eadie *et al.* [14]. A confidence interval for m_L can be estimated by solving the equation

$$\ln\left[\frac{L(m_\pm)}{L(m_L)}\right]=\frac{1}{2}. \tag{19}$$

For a Gaussian distribution of m_L this error estimate corresponds to the standard deviation of the Gaussian distribution. Bowman and Sharapov [19] demonstrated by numerical simulation that this approach accurately estimates M and its error. Note that in this approach the normalizability of $L(m)$ is not an issue.

Of course the whole notion of a confidence interval, particularly for a strongly non-normal distribution, is considered by many to be outdated. As Hall [20] notes ‘‘more sophisticated techniques...convey more information in an equally palatable form. One such device is a confidence picture, by means of which one may present empirical evidence about the relative likelihood of the true parameter value lying in different regions.’’ Since our present focus is on formulating the appropriate likelihood function, here we simply quote the standard interval given by Eq. (19).

B. Target spin $I=0$

1. p-wave spins known

First we consider the likelihood function for the situation where $I=0$ and the spins of the p-wave resonances are known. Then for one $p_{1/2}$ level

$$P_p^0(p|MA,\sigma)=G(p,M^2A^2+\sigma^2). \tag{20}$$

If the experimental asymmetry is q, then

$$L(M)=G(q,M^2A^2+\sigma^2)P_M(M). \tag{21}$$

[In order to obtain a normalizable function, we might assume that $P_M(M)$ is constant from $M = 0$ to $M_{\max}$, and zero elsewhere.]

Here and in the following we assume that the values of asymmetries measured for different p-wave resonances have mean zero and are statistically independent. It follows that the likelihood function for several resonances is the product of the likelihood functions for the individual resonances. We note that the resonance strengths are assumed to be known, and that this information is included explicitly in the formulation of the likelihood function. For this reason, the value of M is not systematically altered if weak p-wave resonances are excluded from the analysis. This result holds for $I\neq 0$ as well as for $I=0$ targets.

2. p-wave spins not known

We next develop expressions for the likelihood functions when there is incomplete information on the spins of the resonances. As noted in the review of the likelihood method above, when one wishes to determine only a subset of the parameters involved, the complete likelihood function is summed over the unwanted parameters. Clearly the p-wave resonances have $J = 1/2$ or 3/2. The p-wave resonances with $J = 1/2$ can display parity violation, with the PDF for the asymmetry p given by Eq. (20), and the likelihood function by Eq. (21). For a p-wave resonance with $J = 3/2$, the PDF for the asymmetry also is Gaussian, but does not involve the rms matrix element M. Since the $J = 0$ PV interaction can only mix states of the same J, the matrix element between $J = 1/2$ s-wave states and $J = 3/2$ p-wave states vanishes identically. Therefore a measurement of the PV asymmetry for $J = 3/2$ p-wave resonances cannot yield any information on the matrix element M.

If the spins of the p-wave levels are not known, then the likelihood function for a single level also depends on the spin of the p-wave level. The PDF of the asymmetry p is the sum of two terms, and the likelihood function is

$$L(M,J)=[p(1/2)\delta(J,1/2)G(q,M^2A^2+\sigma^2) + p(3/2)\delta(J,3/2)G(q,\sigma^2)]P_M(M), \tag{22}$$

where $p(1/2)$ and $p(3/2)$ are the probabilities that $J=1/2$ and $J=3/2$, respectively. The evaluation of the relative probabilities of resonances with different spins is discussed in Sec. IV. For several resonances, the likelihood function is the product of the individual likelihood functions. If only M is to be determined, then one integrates (sums) $L(M,J)$ over J to obtain

$$L(M)=[p(1/2)G(q,M^2A^2+\sigma^2) + p(3/2)G(q,\sigma^2)]P_M(M). \tag{23}$$

Note that for $N\geqslant 2$, the likelihood function for $J = 1/2$ states is normalizable, but the likelihood function in Eq. (23) is not unless $P_M(M)$ tends to zero for large M. This difference is due to the $J = 3/2$ terms, which are independent of M and lead to a divergent normalization integral

$$\int_0^\infty L(M)dM > \int_0^\infty \prod_{\mu=1}^{N} p(3/2)G(p_\mu,\sigma_\mu^2)dM = \infty. \tag{24}$$

In practice we resolve the normalization issue by setting $P_M(M)$ equal to a constant up to some value of $M_{\max}$ and zero otherwise. In the Bayesian spirit, one could adopt the view that all resonances that display a parity violation with greater than some given statistical significance, say $n\sigma$, must have $J=1/2$. One could then analyze this set of resonances. The resulting likelihood curve could then be used as the *a priori* distribution $P_M(M)$. This new function is normalizable.

C. Target spin $I \neq 0$

1. All level spins and projectile-spin amplitudes known

More parameters are involved in the expressions for the likelihood functions for target spin $I \neq 0$, and there are more cases to consider. First assume that the spin of the p-wave level, the spins of all s-wave levels, and all of the projectile-spin neutron amplitudes are known. Then the asymmetry has the distribution of $p = QR$, where Q is a Gaussian random variable with standard deviation MA and is given by Eq. (9), and R is a known constant given by Eq. (7). The result is the same as for $I=0$ except that the PDF for p has the variance $Q^2R^2 = M^2A^2R^2$. Including the experimental uncertainty σ yields a Gaussian with variance $M^2A^2R^2+\sigma^2$:

$$P_p^I(p) = G(p, M^2A^2R^2+\sigma^2). \tag{25}$$

If q is the experimental outcome, then

$$L(M) = G(q, M^2A^2R^2+\sigma^2)P_M(M). \tag{26}$$

2. All level spins known, projectile-spin amplitudes not known

Next suppose that the projectile-spin mixing amplitudes are not known, but that the spins of the p-wave resonance and of all s-wave resonances are known. The factor A_μ is then known, but R is not. The likelihood function is given by

$$L(M) = P_p^I(q|MA,a,\sigma)P_M(M). \tag{27}$$

Recall that a^2 is the ratio of $p_{3/2}$ to $p_{1/2}$ strength functions for levels of the same total angular momentum J as the p-wave resonance. As discussed in Sec. IV.B., we use average experimental properties (the strength functions) to obtain a value for a.

3. s-wave spins known, p-wave spins and projectile-spin amplitudes not known

Next suppose that neither the spin of the p-wave level nor its projectile-spin amplitudes are known, but that the spins of all s-wave resonances are known. If the spin of the p-wave level is assumed, then the factor A_μ can be evaluated, but $A_\mu = A_\mu(J)$ depends on the spin sequence assumed because only s-wave levels with the same spin as the p-wave level mix to produce parity violation. The likelihood function is then obtained by summing over p-wave level spins as in the corresponding situation when $I=0$.

The rms PV matrix element may be different for $J=I\pm 1/2$ states. Clearly the average size of the matrix element depends on the level density, and thus one should include this effect. The spreading width of the parity violating interaction is defined by $\Gamma_W = 2\pi M^2/D(J)$, which approximately removes the density dependence. It is unlikely that there is any other dependence of the parity violation on J; we assume that Γ_W is independent of J. We also assume that the level spacing $D(J)$ has the J dependence given by Eq. (33). The likelihood function can be expressed as a function of the weak spreading width through the relation $M(J) = (\Gamma_W D(J)/2\pi)^{1/2}$,

$$L(\Gamma_W) = P_M(M)\left(\sum_{J=I\pm 1/2} p(J)P_p^I[q|M(J)A_\mu(J),a,\sigma] + \sum_{J=I\pm 3/2} p(J)G(q,\sigma^2)\right). \tag{28}$$

4. s-wave spins, p-wave spins, and projectile-spin amplitudes not known

It is very difficult to evaluate the likelihood function if only the level energies and neutron widths (and their s- or p-wave character) are known. The quantities $A_\mu(J)$ depend on the J values of all nearby s-wave levels. To emphasize this fact we write $A_\mu(J,\Sigma)$, where Σ denotes which possible sequence of spin assignments is assumed for some number N of the nearby s-wave levels. There are two possible spin assignments $(I\pm 1/2)$ for each s-wave level and $2N$ spin sequences (Σ). The likelihood function is an average over the likelihood functions for the different spin sequences,

$$L(\Gamma_W) = \sum_{\Sigma} W(\Sigma)P_M(M)\left(\sum_{J=I\pm 3/2} p(J)G(q,\sigma^2) + \sum_{J=I\pm 1/2} p(J)P_p^I[q|M(J)A_\mu(J,\Sigma),a,\sigma]\right). \tag{29}$$

The quantity $W(\Sigma)$ is a weighting factor which gives the probability of different s-wave spin sequences occurring.

[An alternate approach is to adopt a Monte Carlo procedure. Consider the case of unknown s-wave spins. From the statistical model one knows the relative density of states with $I\pm 1/2$. Therefore one can sample the set of N s-wave levels at random and arbitrarily designate the appropriate fraction of the levels with the proper spin. Then (for each spin) one can obtain the quantity A defined above. Repeating this process many times yields a distribution of A values for each J value. These distributions provide mean values of A as well as the appropriate (numerical) PDF for A for each spin. Of course a more probable circumstance is that some of the s-wave spins are known, but not all of them. Then one follows the same procedure, but the sampling is only applied to the states with unknown spin.]

For more than one p-wave level $L(\Gamma_W)$ is a product of the likelihood functions for the individual resonances. The PV data for a target nucleus typically consists of PV asymmetries for over twenty p-wave levels and the $A_\mu(J,\Sigma)$ terms involve a larger number of s-wave resonances. The numeri-

TABLE I. Spin cutoff parameters for target nuclei measured in the TRIPLE parity violation studies.

Nucleus	Spin cutoff parameter σ_c
$^{107}_{47}$Ag	3.80
$^{109}_{47}$Ag	3.82
$^{113}_{48}$Cd	3.86
$^{115}_{49}$In	3.88
$^{127}_{53}$I	3.99
$^{232}_{90}$Th	4.76
$^{238}_{92}$U	4.79

cal evaluation of $L(\Gamma_W)$ in this circumstance seems to be intractable. This emphasizes the need for spectroscopic information in order to evaluate M (or Γ_W).

IV. LEVEL DENSITIES AND STRENGTH FUNCTIONS

A. Level densities

The spin-dependent level density is $\rho(E,J)=f(J)\rho(E)$, where $\rho(E)$ is the nuclear level density and E is the excitation energy. The spin distribution depends on the spin cutoff parameter σ_c through [21,22]

$$f(J)=e^{-J^2/2\sigma_c^2}-e^{-(J+1)^2/2\sigma_c^2}. \tag{30}$$

The spin cutoff parameter was determined by von Egidy *et al.* [23] by counting low-lying levels with given spins and fitting the spin distribution to experimental data. They assumed that for low energies σ_c depends only on the mass number A, and adopted the empirical form $\sigma_c=xA^y$. The parameters x and y were determined from a least squares fit of the calculated number $n_{\rm calc}(J)$ of levels with spin J in each nucleus to the corresponding experimental number $n_{\rm exp}(J)$. If the spins range from J_1 to J_2, the fraction of levels with spin J can be calculated for each nucleus from

$$n_{\rm calc}(J)=f(J)\frac{\Sigma_{J_1}^{J_2}n_{\rm exp}(J)}{\Sigma_{J_1}^{J_2}f(J)}. \tag{31}$$

This fit was repeated for 75 selected nuclides (A = 20 – 250) with the result

$$\sigma_c=(0.98\pm0.23)A^{(0.29\pm0.06)}. \tag{32}$$

This formula gives σ_c values of 3.1 for Fe and 3.9 for Sb, in agreement with previous experimental values [24] for excitation energies 4 to 6 MeV in this mass region. This agreement for the spin cutoff parameter determined at different energies suggests that σ_c does not depend strongly on excitation energy. The σ_c-values for specific nuclei that we have studied experimentally are listed in Table I.

TABLE III. Spin densities $f(J)$ and relative probabilities $p(J)$ of p-wave resonances with spins J for $^{115}_{49}$In ($I=9/2$).

J	$e^{-(J+1/2)^2/\sigma_c^2}$	$(2J+1)/2\sigma_c^2$	$f(J)$	$p(J)$
3	0.666	0.232	0.155	0.283
4	0.510	0.299	0.152	0.278
5	0.366	0.365	0.134	0.245
6	0.246	0.432	0.106	0.194

The spin distribution can be approximated by [21]

$$f(J)=e^{-J^2/2\sigma_c^2}-e^{-(J+1)^2/2\sigma_c^2}\cong\frac{2J+1}{2\sigma_c^2}e^{-(J+1/2)^2/2\sigma_c^2}. \tag{33}$$

For the p-wave resonances of $I=0$ targets (such as $^{238}_{92}$U and $^{232}_{90}$Th), the ratio of $f(J=3/2)$ to $f(J=1/2)$ is close to two because the exponential factor is approximately the same for small J (see Table II). The exponential term must be included in order to obtain the correct ratio of spin densities for resonances with large spins. As an example, the values of $f(J)$ and the relative probabilities $p(J)$ are listed in Table III for the $I=9/2$ target nucleus $^{115}_{49}$In.

These considerations assume that all resonances are observed. In practice, resonances weaker than some experimental lower limit are not observed. Since the strength functions and the level densities are not the same for resonances with different spins J, the resulting average strengths are not equal in general. Therefore with a finite threshold for observation, resonances (such as $J=1/2^-$ and $J=3/2^-$) will not be observed with the ratio of densities predicted by Eq. (33). For example (see [5]), in the $^{232}_{90}$Th experiment the observed fraction of densities for the $1/2^-$ and $3/2^-$ states was 0.45 and 0.55, instead of the values of 0.35 and 0.65 listed in Table II.

B. $p_{1/2}$ and $p_{3/2}$ strength functions

The standard strength function definition for spin 1/2 projectiles [17], given in Eq. (10), with D^l the observed spacing for the p-wave resonances with all spins J, does not consider the possible dependence of the strength function on spin J, or on the projectile-spin j. For p-wave neutrons the spin-orbit coupling clearly leads to a j dependence of S^1. Therefore a more suitable strength function definition [25] for our purposes is

TABLE II. Spin densities $f(J)$ and relative probabilities $p(J)$ of p-wave resonances with spins J for $I=0$ targets.

Nucleus	J	$e^{-(J+1/2)^2/\sigma_c^2}$	$(2J+1)/2\sigma_c^2$	$f(J)$	$p(J)$
$^{232}_{90}$Th	1/2	0.987	0.0441	0.0435	0.35
$^{232}_{90}$Th	3/2	0.916	0.0882	0.0809	0.65
$^{238}_{92}$U	1/2	0.978	0.0436	0.0426	0.35
$^{238}_{92}$U	3/2	0.917	0.0872	0.0800	0.65

$$S^1=\frac{1}{3}\sum_{j=1/2}^{3/2}\sum_{|J=I-3/2|}^{I+3/2}\frac{\langle g(J)\Gamma^1_{nj}(J)\rangle}{D^1(J)}. \tag{34}$$

For target nuclei with $I\geqslant 3/2$ there are four terms in the sum over J, corresponding to $J_1=I-3/2$, $J_2=I-1/2$, $J_3=I+1/2$, and $J_4=I+3/2$. Both $j=1/2$ and $j=3/2$ can contribute to the neutron widths of J_2 and J_3 resonances, while only $j=1/2$ contributes to J_1 and only $j=3/2$ contributes to J_4. If the strength function $S^1_j=\langle\Gamma^1_{nj}(J)\ \rangle/D^1(J)$ does not depend on J, then

$$\begin{aligned}S^1&=\frac{1}{3}\left\{\frac{\langle\Gamma^1_{nj=1/2}(J)\rangle}{D^1(J)}[g(J_1)+g(J_2)+g(J_3)]\right.\\&\quad\left.+\frac{\langle\Gamma^1_{nj=3/2}(J)\rangle}{D^1(J)}[g(J_2)+g(J_3)+g(J_4)]\right\}\\&=\frac{I}{2I+1}S^1_{j=1/2}+\frac{I+1}{2I+1}S^1_{j=3/2}.\end{aligned} \tag{35}$$

For the $I=1/2$ case there are three possible J values $(J=0,1,2)$, and there is a contribution from two values of j only for $J=1$. Therefore

$$S^1(I=1/2)=\tfrac{1}{3}S^1_{j=1/2}+\tfrac{2}{3}S^1_{j=3/2}. \tag{36}$$

The strength functions $S_{j=1/2}$ and $S_{j=3/2}$ have been determined in the mass region $A\approx 100$ from measurements of the angular dependence of the average differential elastic scattering cross section [26]. These measurements were performed at the Dubna pulsed reactor; results for many samples have been reported [27,28]. For some nuclides of interest the $S_{j=1/2}$ and $S_{j=3/2}$ strength functions have been measured. For these nuclides the value of the parameter $a^2=(Y^2/X^2)$ can be taken directly from the data.

For other nuclides the value of a can be obtained by the following procedure. The $3p$ strength function is fragmented into two peaks with a spin-orbit splitting of $\Delta A=13\pm 4$ mass units: the $p_{1/2}$ peak is located near $A = 94$ and the $p_{3/2}$ peak near $A = 107$. Lorentzian fits of the form

$$X^2=S_{1/2}\frac{\Gamma^2_{1/2}/4}{(A-A_{1/2})^2+\Gamma^2_{1/2}/4}$$

and

$$Y^2=S_{3/2}\frac{\Gamma^2_{3/2}/4}{(A-A_{3/2})^2+\Gamma^2_{3/2}/4} \tag{37}$$

to the measured strength functions yield $S_{1/2}=6.0\times 10^{-4}$, $A_{1/2}=107$, $\Gamma_{1/2}=45$, and $S_{3/2}=5.3\times 10^{-4}$, $A_{3/2}=94$, $\Gamma_{3/2}=40$. The quantities a, X^2, and Y^2 can be estimated for nuclei in this mass region using these parameters.

Values of the parameter a [either obtained from the direct measurements or estimated with Eq. (37)] were then used to characterize the projectile-spin amplitudes in the likelihood analysis.

V. RESULTS

A. $I=0$ targets

The likelihood curve $L(M)$ versus M for $^{238}_{92}$U is shown in Fig. 3. The data consist of 22 p-wave resonances [29]; in this calculation the spins of the resonances are assumed to be unknown. The ratio of observed 1/2 and 3/2 resonances is estimated to be 0.45 to 0.55. The likelihood estimate for $M=0.55\,^{+0.19}_{-0.12}$ meV. Next we consider only the seven resonances with $J = 1/2$. (Corvi *et al.* [30] recently obtained spin assignments for many of the p-wave resonances in $^{239}_{92}$U.) The resulting likelihood curve is shown in Fig. 4, with $M = 0.56\,^{+0.20}_{-0.12}$ meV. This result illustrates that knowledge of the spins of the p-wave resonances is not very important for the $I=0$ case. This was also demonstrated by Corvi *et al.* [30], who used these spin assignments in an analysis of the earlier TRIPLE data set on $^{238}_{92}$U and found that M was essentially unchanged, and that the range of uncertainty was only slightly reduced from the values obtained assuming all spins unknown. On the other hand, information about the s-wave resonances is crucial — one cannot perform the analysis without the s-wave resonance parameters. Thus the key piece of spectroscopic information for $I=0$ targets is the parity of the resonances. Of course a major limitation on the precision of the determination of M is the number of $p_{1/2}$ resonances. Bowman and Sharapov [19] show that in the

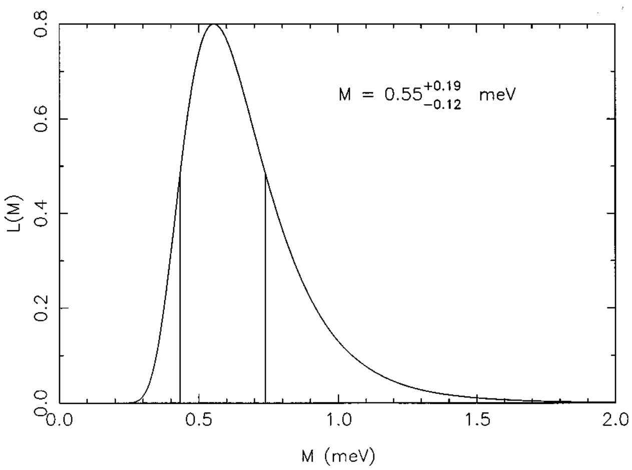

FIG. 3. The likelihood curve $L(M)$ versus M for $^{238}_{92}$U, with the p-wave resonances all assumed to have unknown spins. The values of the relative probabilities $p(J)$ are $p(1/2) = 0.45$ and $p(3/2) = 0.55$.

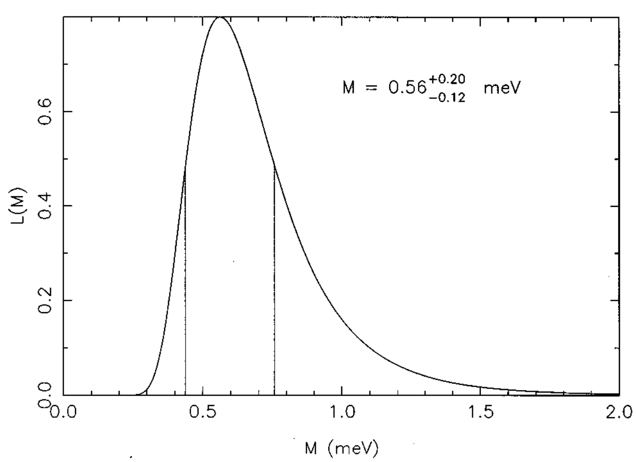

FIG. 4. The likelihood curve $L(M)$ versus M for $^{238}_{92}$U considering only the seven known $p_{1/2}$ resonances.

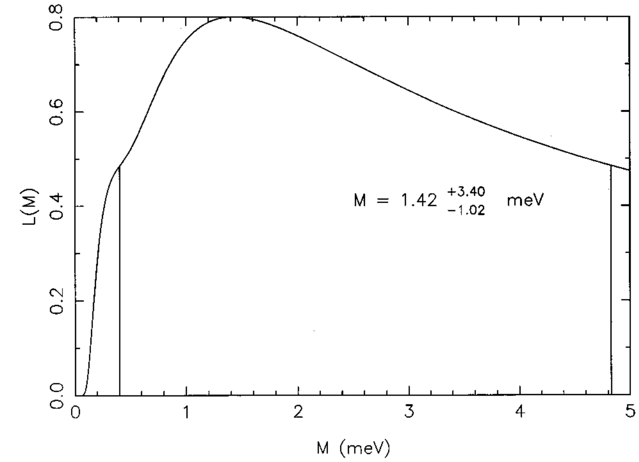

FIG. 5. The likelihood curve $L(M)$ versus M for the 29.67-eV resonance in $^{116}_{49}$In. The spin of the resonance is assumed to be unknown, and the ratio of the neutron projectile-spin amplitudes is assumed to have an average value of $a = 0.7$.

ideal case when all spins are known and with errors $\sigma^2 \ll M^2$, the relative uncertainty in M is given by $\Delta M/M = (2N)^{-1/2}$, where N is the number of resonances.

As discussed in Sec. IV, the predicted ratio of $3/2^-$ to $1/2^-$ resonances for $^{239}_{92}$U is close to two (the $2J+1$ dependence). Since the $J=3/2$ and $J=1/2$ resonance strengths are not equal (the $J=1/2$ resonances are twice as strong if the two strength functions are equal), and we do not observe all of the resonances (there is some experimental threshold for observation), the values for $p(3/2)$ and $p(1/2)$ to be used in Eq. (23) must be determined empirically. The procedure followed to determine this ratio is described in Sec. VI of Ref. [5]. (One scales from the s-wave density to obtain the fraction of all p-wave levels observed, and determines the weakest level observed from the data. Assuming a Porter-Thomas distribution for the widths, one can determine the probability that the p-wave resonances observed have $J = 1/2$.) In practice M shows only weak dependence on the values of $p(1/2)$ and $p(3/2)$. For example, when the data set used to obtain the results shown in Fig. 3 was reanalyzed with $p(1/2) = 0.33$ and $p(3/2) = 0.67$, the result was $M = 0.58^{+0.21}_{-0.13}$ meV. This is almost the same as the value of M obtained for $p(1/2) = 0.45$ and $p(3/2) = 0.55$.

B. $I \neq 0$ targets

As an example consider $^{115}_{49}$In. The likelihood function is shown in Fig. 5 for a sample resonance (29.67 eV) in $^{116}_{49}$In [31]. The parities of the resonances are known (that is, one knows which resonances are s-wave and which are p-wave), but the spins of the p-wave resonances are unknown. The spins of the s-wave resonances are known. In addition, the ratio of the neutron projectile-spin amplitudes for $j = 1/2$ and $j = 3/2$ is assumed to have the value $a = 0.7$ (see Sec. IV B) with the PDF given by Eq. (16). The corresponding distribution is extremely broad, illustrating the impact of the lack of spectroscopic information.

In Fig. 6 the likelihood function is shown for the same resonance assuming that all of the spins are known. Although the same average value and distribution is assumed for the projectile-spin amplitudes, the resulting likelihood function is much narrower. The maximum likelihood estimate for M also has changed significantly. In Fig. 7 the likelihood function is shown for the same resonance assuming a fixed value of $R = 0.82$; i.e., that Eq. (8) applies. (For values of the parameter R ranging from $\sim$ 0.1 to 1.0, M varies approximately as $1/R$, as does the relative uncertainty.) This again illustrates the need for the relevant spectroscopic information.

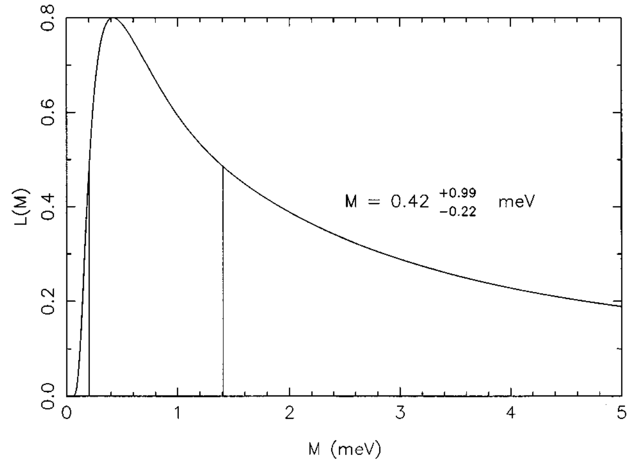

FIG. 6. The likelihood curve $L(M)$ versus M for the 29.67-eV resonance in $^{116}_{49}$In. All parameters are the same as for Fig. 5, except that the spin of the p-wave resonance is now assumed to be known.

C. Nonstatistical effects

Although the central theme of our entire analysis is the statistical model of the compound nucleus, there was an unexpected nonstatistical result observed in the experimental data on $^{232}_{90}$Th [3,5,32]. In the latest (improved) data set on $^{232}_{90}$Th [32], eight of eight statistically significant longitudinal asymmetries are positive. Although this raises a number of very interesting theoretical questions, here we deal only with the practical issue of extracting a value of M from the data when nonstatistical effects are present.

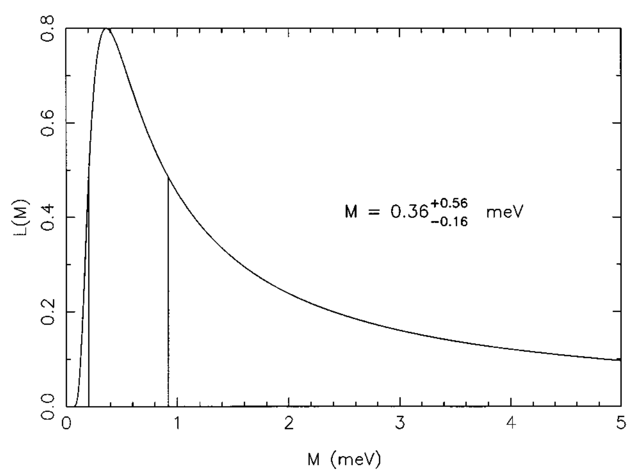

FIG. 7. The likelihood curve $L(M)$ versus M for the 29.67-eV resonance in $^{116}_{49}$In. All parameters are the same as for Fig. 6, except that the ratio of the projectile-spin amplitudes is fixed.

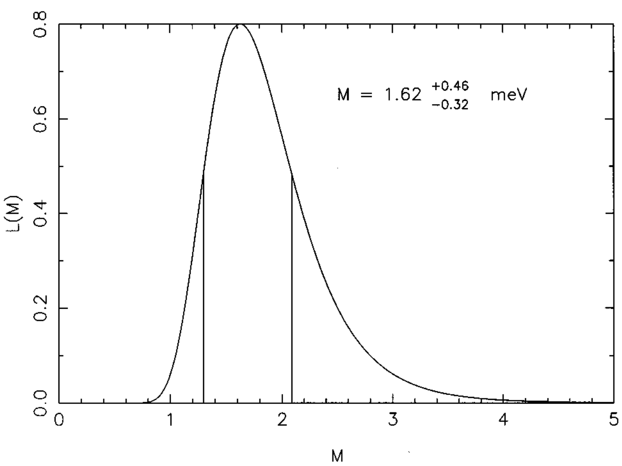

FIG. 8. The likelihood curve $L(M)$ versus M for $^{232}_{90}$Th, with the spins of the p-wave resonances all assumed to be unknown. The values of the parameters $p(1/2)\ =\ 0.44$ and $p(3/2)\ =\ 0.56$.

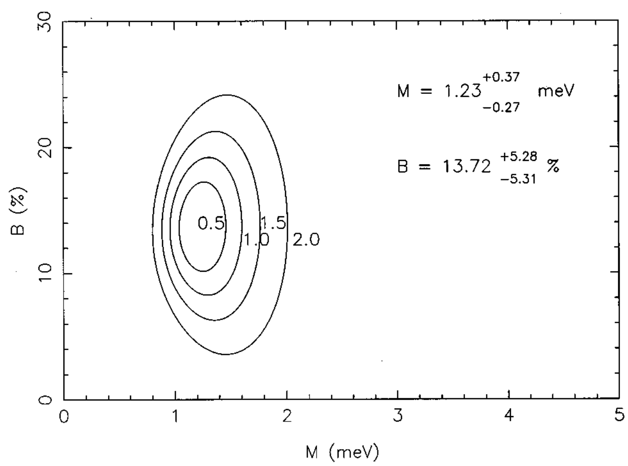

FIG. 9. The two-parameter analysis for the p-wave resonances in $^{232}_{90}$Th. The spins of the resonances are all assumed to be unknown and the values for the relative probabilities $p(J)$ are $p(1/2)\ =\ 0.44$ and $p(3/2)\ =\ 0.56$. The solid lines indicate constant probability contours, with the numbers in units of the standard deviation.

We assume that the longitudinal asymmetry has two terms, one statistical and one non-statistical (this latter term is sometimes called the regular or direct term). The following form was obtained by Bowman *et al.*, [33] Hussein *et al.* [34], and Auerbach *et al.* [35]:

$$p_{\mu}=2\sum_{\nu}\frac{V_{\mu\nu}}{E_{\nu}-E_{\mu}}\frac{g_{\nu}}{g_{\mu}}+B(1\ \mathrm{eV}/E)^{1/2}, \qquad (38)$$

where E is the energy of the neutron in eV. Note that both terms have the same energy dependence. Empirically one fits the longitudinal asymmetries to this functional form and determines M and B from the data. For example, when the $^{238}_{92}$U data used to construct the likelihood curve in Fig. 3 were fit to this two-parameter expression, the resulting values for M and B were $M\ =\ 0.58^{+0.21}_{-0.13}$ meV (unchanged) and $B\ =\ -0.04^{+1.61}_{-1.78}$ %. There is no evidence for a nonstatistical effect for $^{238}_{92}$U.

On the other hand, for $^{232}_{90}$Th the one- and two-parameter results are quite different, as shown in Figs. 8 and 9. The result for the one-parameter analysis (Fig. 8) is $M=1.62^{+0.46}_{-0.32}$ meV, while the two-parameter analysis (Fig. 9) with $p(1/2)=0.44$ and $p(3/2)=0.56$ yields the values $M=1.23^{+0.36}_{-0.27}$ meV and $B=+13.57^{+5.34}_{-5.23}$ %. Note that the value of M is relatively insensitive to changes in B; even for this huge offset (13.6%) the value of M changes only 24%. Repeating the latter calculation with $p(1/2)=0.33$ and $p(3/2)=0.67$ leads to $M=1.26^{+0.39}_{-0.26}$ meV and $B=+14.47^{+5.51}_{-5.43}$%. Thus the $p(1/2)/p(3/2)$ ratio has little effect on the maximum likelihood estimate for M.

VI. DISCUSSION

A. Comments on incomplete spectroscopic information

For $I=0$ targets the only missing spectroscopic information is the spin of the p-wave resonances. Although aesthetically it would be preferable to know the spins of these resonances, in practice this lack of information causes little additional uncertainty in M. Therefore additional spectroscopic information for $I=0$ targets is interesting, but not crucial.

The situation for $I\neq 0$ targets is quite different. There lack of knowledge (particularly of the resonance spins) leads to large increases in the uncertainty in M. Since it is important to know the spins of the s-wave resonances, it fortunate that the spins of these s-wave resonances are relatively well known. Determining the spins of the weak p-wave resonances is much harder; in general there is little information on the spins of p-wave resonances. Significant experimental efforts on obtaining the J values for the s- and p-wave resonances would be valuable and reduce the error on M.

The other missing spectroscopic information for $I\neq 0$ targets involves the projectile-spin amplitudes. This mixture is known only for a few resonances. At present the procedure that was described in Sec. II — estimating an average value for the ratio R from the experimental strength functions, and averaging over a range of possible R values — is the only practical approach. Measurement of the projectile-spin amplitudes would significantly reduce the uncertainties in M.

B. Comments on physical motivation for likelihood functions

The maximum likelihood approach seems the most appropriate method to determine the rms PV matrix element. This approach seems ideally suited for the limited data sets available, and provides a convenient framework for incorporating partial information. Most of the issues are illustrated by the $I=0$ case.

One issue is the normalizability of $L(M)$. As noted earlier, if we take the *a priori* distribution of M, $P_M(M)$, to be a constant, then $L(M)$ is not normalizable if the spins of the resonances are not known. However, we can make $L(M)$ normalizable if we take $P_M(M)$ to be a constant up to some large value of M (0.01 eV) and set $P_M(M)$ to zero for larger values of M. Our estimate for M is insensitive to the choice of *a priori* distribution.

Another issue is the appropriate expression for the likelihood function when there is incomplete information for the spins of the p-wave resonances. We have assumed that all of the spins are unknown, and that the likelihood function is given by Eq. (23). Bowman and Sharapov [19] and Bunakov [36] present contrasting views on the use of the likelihood method to extract M values. Bowman and Sharapov essentially give an earlier version of the views adopted in the present paper. They demonstrated the reliability of the likelihood method developed here using Monte Carlo techniques. Bunakov argues that one can only obtain an upper limit for M from data when the spins are unknown, and takes exception to the result by Corvi *et al.* [30] that determining the spins of the p-wave resonances led to very little change in M. As noted in Sec. V, we agree with Corvi's results.

The different conclusions result from different assumptions for the *a priori* probability density function for J and M, $P_{MJ}(M,J)$. Bowman and Sharapov assume that $P_{MJ}(M,J)$ is a product of a function of J and a function of M, equivalent to assuming that J and M are statistically independent. The measurement of J for a particular level does not affect the knowledge of the distribution of matrix elements for $J=1/2$ levels. Bunakov's assumptions are equivalent to assuming different *a priori* distributions for M for the $J=1/2$ and $J=3/2$ levels.

Reliable estimates for M can be obtained even when the spins are unknown: The levels with small values of parity violation (either spin 1/2 resonances with accidentally small PV asymmetries or resonances with spin 3/2 and consequently no PV asymmetry) do not strongly distort the distribution for those resonances with large asymmetries. The distinct maximum in the likelihood function, which arises from the resonances with statistically significant parity violations, remains even in the presence of many small or zero values. The additional information on the spins of the resonances has limited impact because of the relative statistical significance of the asymmetries for the different resonances. Some of the resonances are much more strongly weighted than others because they should be.

C. Other approaches

Other than in papers published by the TRIPLE collaboration, and those discussed in Sec. VI B [19,36] there has been little discussion about the effect of incomplete spectroscopic information. Bunakov, Davis, and Weidenmüller [37] discuss some of these issues both for parity violation and time reversal invariance violation tests. They consider the issue of unknown projectile-spin amplitudes. They derive an expression (when the amplitudes are unknown) for the corresponding distribution for $\Delta\sigma = \sigma_+ - \sigma_-$. Their results are similar to those obtained in the present work, but are not identical. The difference arises because we consider the distribution for the longitudinal asymmetry, or in their notation $\Delta\sigma/\sigma$, instead of $\Delta\sigma$. They do not consider unknown spins.

Davis [38] considers the parity violation case explicitly and focuses on the minimum sample size required to obtain statistically significant results. He also covers the issues of unknown projectile-spin amplitudes and unknown spins. However, for the practical analysis of experimental parity violation data it is crucial to incorporate partial information. Including this information minimizes the uncertainties in the determination of M. This incorporation of partial information is the focus of the present paper, which provides explicit prescriptions for the analysis of parity violation data with provision for inclusion of such information.

VII. SUMMARY

Analysis of the longitudinal asymmetries measured in the parity violation experiments by the TRIPLE Collaboration has been described. The likelihood method is well suited for analysis of these limited data sets. In practice one rarely has all of the relevant spectroscopic information, but normally has some of this information. Therefore a key consideration in choosing an analysis approach is the convenience of incorporating this partial information. For the $I=0$ case most of the required spectroscopic information is available, except for resonance spins. Fortunately the ansatz of treating the spin of each resonance as unknown (see Sec. III B 2) works well in practice.

For $I\neq 0$ targets there are often several unknown quantities: s- and p-wave resonance spins and projectile-spin neutron amplitudes. However, usually one has partial information concerning the spins; the analysis is formulated to facilitate incorporation of this partial knowledge. The projectile-spin neutron amplitudes at present are known only in an average sense (if the $p_{1/2}$ and $p_{3/2}$ strength functions are known); there is also the possibility of correlations between the entrance channel amplitudes. The analysis is formulated to permit explicit inclusion of these effects.

We conclude that given the relevant spectroscopic parameters, one can obtain reliable and fairly precise values for the rms parity violating matrix element M from longitudinal asymmetry data. With partial information, the rms matrix element M still can be obtained, but with increased uncertainty. Measurements to improve the level of spectroscopic information by determining the s- and p-wave resonance spins and the projectile-spin amplitudes would significantly reduce the uncertainties in the determination of the effective neutron-nucleus weak interaction. Such measurements are strongly encouraged.

ACKNOWLEDGMENTS

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