Inversion solution to heavy-ion optical model potential at intermediate energies

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(Received 13 July 1995; revised manuscript received 4 October 1995)

In this paper we show by solving the inversion problem at high energies that the fundamental McIntyre parametrization of the *S* matrix, for heavy-ion collisions, will correspond to a Woods-Saxon-type optical model potential. The parameters of such a Woods-Saxon potential are directly related to the corresponding parameters of the McIntyre parametrization. The inversion solution results were tested for the available experimental data and were found to be in good agreement.

PACS number(s): 25.70.−z, 11.55.Bq, 24.10.Ht

I. INTRODUCTION

Several attempts have been made lately to describe elastic-scattering processes between heavy ions in terms of the optical limit to Glauber's model [1]. This simple model could provide a good description of nucleus-nucleus scattering over a large energy range since its only inputs are the experimental nucleon-nucleon amplitudes and the rms radii of the nuclei involved.

The optical limit can be obtained from the basic assumptions of Glauber theory which implies that $E \gg U$ and $\lambda \ll a$, with a as the distance on which the potential U exhibits significant variations, λ is the wavelength, and E is the energy of the incident particle [2].

For the use of the Glauber approach, the simultaneous fulfillments of two conditions are required [2]; namely, the validity of the eikonal approximation, in the framework of which the deviations from rectilinear motion of the incident particle are considered to be small and the validity of the adiabatic approximation where the positions of the nucleons in the nucleus are assumed to be fixed during the flight time of the incident particle through the nucleus. At high energy both conditions are satisfied simultaneously.

On the other hand, it has been shown that the use of the McIntyre parametrization of the *S* matrix for high-energy heavy-ion collisions gives excellent fits to the elastic-scattering data [3]. Actually, the McIntyre parametrization of the *S* matrix proved to give good fits to the heavy-ion data with as few as three parameters [4].

Moreover, the optical-model analysis of heavy-ion elastic-scattering experiments in the intermediate-energy range indicates that the data are sensitive to the real part of the nucleus-nucleus interaction for distances smaller than the strong absorption radius [5].

Thus based on the McIntyre parametrization of the *S* matrix and taking Glauber's high-energy approximation into account, we solve for the heavy-ion optical-model potential adopting a certain inversion procedure.

In Sec. II we give details of our inversion solution. Then in the next section we use the McIntyre parametrization of the *S* matrix to obtain the phase-shift parameters which are used to construct a Woods-Saxon-type optical-model potential.

Our calculations and results are given in Sec. IV and where a comparison of our potential and the optical-model potentials, which give best fits to the data in the case of ^{12}C-^{12}C elastic scattering at $E_{\rm lab}=1016$ MeV [6,7], is made.

In Sec. V we give a concluding discussion. It is also to be noted that an important related appendix is encountered at the end of this paper.

II. THE SOLUTION OF THE INVERSION PROBLEM

Within the framework of Glauber's eikonal approximation [8], we can relate the elastic-scattering *S*-matrix element $S(b)$ directly to the corresponding optical-model potential $U(r)$ in the following way:

$$S(b)=\exp[i\chi(b)], \tag{1}$$

where b is the corresponding impact parameter and $\chi(b)$ is the phase shift given by

$$\chi(b)=\frac{-k}{E}\int_b^\infty \frac{rU(r)dr}{\sqrt{r^2-b^2}}, \tag{2}$$

where $U(r)$ is the potential.

Further, writing $\chi(b)$ in the form $\chi(b)=\chi_R(b)+i\chi_I(b)$ and the corresponding optical potential as $U(r)=V(r)+iW(r)$ [$V(r)$ and $W(r)$ are real] we get

$$\chi_R(b)=-\frac{k}{E}\int_b^\infty \frac{rV(r)}{\sqrt{r^2-b^2}}\,dr \tag{3}$$

and

$$\chi_I(b)=-\frac{k}{E}\int_b^\infty \frac{rW(r)}{\sqrt{r^2-b^2}}\,dr. \tag{4}$$

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Equations (3) and (4) are of Abel's type and the inverse solution to these equation are given by [9]:

$$V(r)=\frac{2E}{k\pi}\frac{1}{r}\frac{d}{dr}\int_r^\infty \frac{\chi_R(b)bdb}{\sqrt{b^2-r^2}} \tag{5}$$

and

$$W(r)=\frac{2E}{k\pi}\frac{1}{r}\frac{d}{dr}\int_r^\infty \frac{\chi_I(b)bdb}{\sqrt{b^2-r^2}}. \tag{6}$$

Equations (5) and (6) give the potential in its general form.

To proceed further, we assume that our heavy-ion scattering regime is quasiclassical so that at high energies the above inversion solution may be adopted.

III. THE USE OF MCINTYRE PARAMETRIZATION

The McIntyre parametrization [3] for the elastic particle wave matrix element s_l is normally given by [4]:

$$s_l=|s_l|\exp(2i\delta_e), \tag{7}$$

where

$$|s_l|=\left[I+\exp\left(\frac{l_0-l}{a}\right)\right]^{-1} \tag{8}$$

and

$$\delta_l=\frac{\mu}{1+\exp[(l-l_0)/a]}. \tag{9}$$

As can be seen from these two formulas the grazing angular momentum l_0 and its related width a can be, in general, different; therefore one may be dealing with either three or five parameters for the S matrix. Moreover, the grazing angular momentum l_0 and the corresponding width a are related to the reduced radius r_0 and diffusivity d through the following semiclassical relationship:

$$l_0+\frac{1}{2}=kR_0 \tag{10}$$

and

$$a=kd \tag{11}$$

with

$$R_0=r_0(A_P^{1/3}+A_T^{1/3}), \tag{12}$$

where A_P and A_T are the mass numbers for the incident and target nuclei, respectively.

In the above relations Coulomb effects are neglected; this is because we are dealing with a high-energy scattering process and because we are mainly interested in the nuclear potential.

Adopting the McIntrye parametrization, writing the S matrix in the form of Eq. (7) and taking into account Eq. (1), we obtain

$$i\chi(b)=\ln|s_l|+2i\delta_l \tag{13}$$

from Eqs. (8), (9), and (13), we have

$$i\chi_R(b)-\chi_I(b)=-\ln\left[1+\exp\left(\frac{l_0-l}{a}\right)\right]+\frac{2i\mu}{1+\exp((l-l')_0/a')}. \tag{14}$$

With Eq. (14) in mind and using Eqs. (10) and (11), we obtain.

$$\chi_R(b)=\frac{2\mu}{1+\exp[(b-b_0')/d']} \tag{15}$$

and

$$\chi_I(b)=\ln\left[1+\exp\left(\frac{b_0-b}{d}\right)\right], \tag{16}$$

where b is the impact parameter normally given by $kb=\ell+1/2$ and $b_0=R_0$.

Analytical calculations of the optical-model potentials, given by Eqs. (5) and (6), with the above phases is practically very difficult; this is because the potentials cannot be well extracted at $r=0$. Therefore, we shall for the moment approximate the phases $\chi_R(b)$ and $\chi_I(b)$ by the formulas

$$\chi_R(b)=2\mu\sum_{n=1}^N c_n\exp\left(\frac{-nb^2}{\alpha^2}\right) \tag{17}$$

and

$$\chi_I(b)=\sum_{n=1}^N b_n\exp\left(\frac{-nb^2}{\beta^2}\right). \tag{18}$$

These formulas can be shown to reproduce χ_R and χ_I to a high degree of accuracy with a restricted number of terms and allow us to calculate the potential analytically [10].

To show that this is the situation we present such a fit, for the nuclear density in the case of ^{40}Ca, in Fig. 1. We note that 12 terms were taken into account.

Now, inserting Eqs. (17) and (18) into Eqs. (5) and (6) we get

$$V(r)=-\frac{4\mu E}{\pi k\alpha}\sum_{n=1}^N c_n\sqrt{\pi n}\,\exp\left(-\frac{nr^2}{\alpha^2}\right) \tag{19}$$

and

$$W(r)=-\frac{2E}{\pi k\beta}\sum_{n=1}^N b_n\sqrt{\pi n}\,\exp\left(-\frac{nr^2}{\beta^2}\right). \tag{20}$$

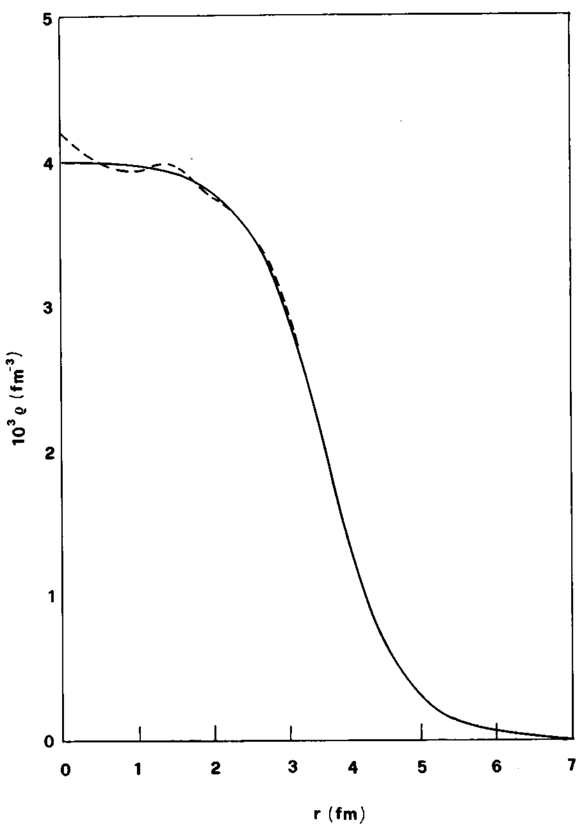

FIG. 1. A plot of the nuclear density $\rho(r)$ for ^{40}Ca (solid curve) with Woods-Saxon parameters taken as $\rho_0=4\times10^{-3}$ fm^{-3}, $R=1.07$ fm, and $a=0.545$ fm; the dashed curve corresponds to the approximate density with 12 terms taken from the expansion of the forms (21) and (22).

It may, now, be apparent that the use of Eqs. (19) and (20) greatly simplified the evaluation of our optical-model potential and thus ended our inversion procedure. Yet, in what follows and utilizing the above relations we develop a method which allows for $V(r)$ and $W(r)$ to be written in the familiar Woods-Saxon forms.

To see that, let us rewrite $V(r)$ and $W(r)$ in the forms

$$V(r)=-\left(\frac{4\mu E}{\pi k\alpha}\right)f(R',\Delta',r) \tag{21}$$

and

$$W(r)=-\left(\frac{2E}{\pi k\beta}\right)f(R,\Delta,r), \tag{22}$$

where

$$f(x_0,a_0,x)=\frac{1}{1+\exp[(x-x_0)/a_0]}. \tag{23}$$

With the above parametrization and as is shown in the Appendix, the following relations, between the parameters of the corresponding phase shifts, will hold. For the real part,

$$\frac{I_4(R',\Delta')}{I_2(R',\Delta')}=\frac{3}{2}\frac{I_3(b_0',d')}{I_1(b_0',d')} \tag{24}$$

and

$$\frac{I_6(R',\Delta')}{I_4(R',\Delta')}=\frac{5}{4}\frac{I_5(b_0',d')}{I_3(b_0',d')}. \tag{25}$$

For the imaginary part,

$$\frac{I_4(R,\Delta)}{I_2(R,\Delta)}=\frac{3}{4}\frac{I_4(b_0,d)}{I_2(b_0,d)} \tag{26}$$

and

$$\frac{I_6(R,\Delta)}{I_4(R,\Delta)}=\frac{5}{6}\frac{I_6(b_0,d)}{I_4(b_0,d)}, \tag{27}$$

where $I_\nu(x_0,a_0)$ is an integral given by

$$I_\nu(x_0,a_0)=\int_0^\infty\frac{x^\nu dx}{1+\exp[(x-x_0)/a_0]} \tag{28}$$

and which can be evaluated analytically [11] (see the Appendix).

Now, in a real application to any system we can evaluate the phase-shift parameters (b's and d's) from a fitting procedure and hence the right-hand side of Eqs. (24)–(27). Then, in each case we get two nonlinear simultaneous equations which we can solve by using a certain iteration procedure such as the Newton-Raphson method.

Once we get the R's and Δ's, then there remains only the task of evaluating the parameters α and β in Eqs. (21) and (22). For that purpose we insert Eqs. (21) and (22) into Eqs. (3) and (4) to get

$$\chi_R(b)=\frac{4\mu}{\pi\alpha}\int_b^\infty\frac{rdr}{\sqrt{r^2-b^2}\{1+\exp[(r-R)/\Delta]\}} \tag{29}$$

and

$$\chi_I(b)=\frac{2}{\pi\beta}\int_b^\infty\frac{rdr}{\sqrt{r^2-b^2}\{1+\exp[(r-R)/\Delta]\}}. \tag{30}$$

Taking into account Eqs. (15) and (16) and putting $b=0$ in the resulting expressions, we find that

$$\alpha=\frac{2}{\pi}[1+\exp(-b_0'/d')]I_0(R',\Delta') \tag{31}$$

and

$$\beta=\frac{2}{\pi}\frac{I_0(R,\Delta)}{\ln[1+\exp(b_0/d)]}, \tag{32}$$

where $I_0(R',\Delta')$ is given by

$$I_0(R',\Delta')=R'+\Delta'\ln[1+\exp(-R'/\Delta')] \qquad (33)$$

and $I_0(R,\Delta)$ is given by a similar expression in R and Δ.

Now, Eqs. (21) and (22) can be rewritten in form

$$V(r)=\frac{-V_0}{1+\exp[(r-R')/\Delta']} \qquad (34)$$

and

$$W(r)=\frac{-W_0}{1+\exp[(r-R)/\Delta]}, \qquad (35)$$

where $V_0(=4\mu E/\pi h\alpha)$ and $W_0(=2E/\pi h\beta)$ represent the depths of the optical-model potential.

Thus, we have succeeded, through our inversion procedure, in obtaining Woods-Saxon-type optical-model potentials with parameters that can be determined directly from the McIntyre parametrization of the phase shift.

IV. APPLICATION TO ^{12}C-^{12}C SYSTEM AT E_{lab}=1016 MeV

We shall apply, here, the preceding formalism to the symmetric system ^{12}C-^{12}C and as it can be seen, the present method yields a five-parameter fit to any data. These parameters are μ, R', R, Δ', and Δ, and for the ^{12}C-^{12}C system these are obtained on the basis of the phase-shift analysis carried out by Mermaz [4]. In doing that Coulomb effects on the parameters are to be substracted.

The reduced radius r_0 and diffusivity d are related to the grazing angular momentum and angular momentum width through the following semiclassical relationship [4]:

$$l_0=kR_0\left(1-\frac{2\eta}{kR_0}\right)^{1/2} \qquad (36)$$

and

$$a=kd\left(1-\frac{\eta}{kR_0}\right)\left(1-\frac{2\eta}{kR_0}\right)^{1/2}. \qquad (37)$$

Using the data from Mermaz [4] for l_0, l_0', a, and a' and calculating the other related parameters (such as η, k, etc.), we get the phase-shift parameters as b_0=4.981 87 fm, b_0' =3.075 95 fm, d=0.598 26 fm, and d'=0.922 54 fm. Note that Coulomb effects have been substracted off in evaluating the parameters cited above.

Now with these parameters at hand and with reference to the Appendix, we can evaluate $I_i(b_0',d')$ $(i=1,3,5)$ and $I_i(b_0,d)$ $(i=2,4,6)$ and use them with Eqs. (24)–(27) to write down two sets of nonlinear simultaneous equations for $\{R',\Delta'\}$ and $\{R,\Delta\}$, respectively.

Using an iteration procedure such as the modified Newton-Raphson method we can solve for R, Δ, R', and Δ'. Their values are

$$R=3.811\,417, \quad \Delta=0.6671, \quad R'=3.3558,$$

and

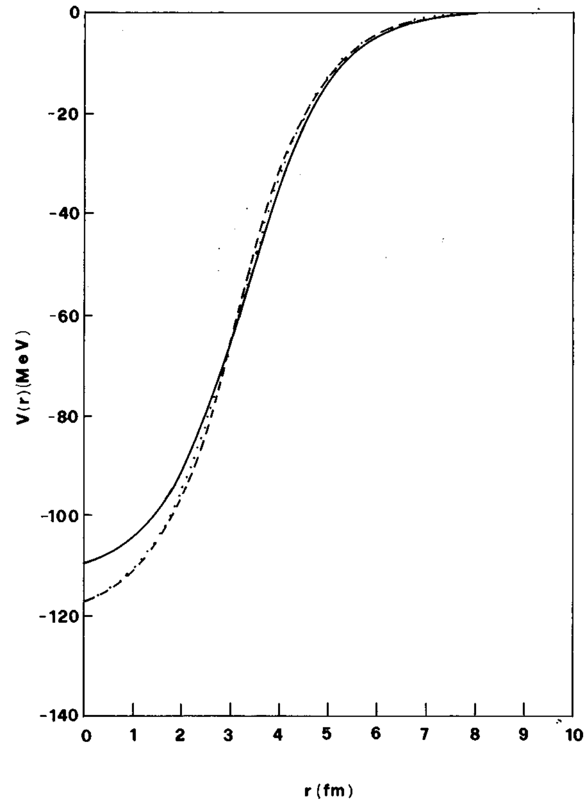

FIG. 2. The real part of the optical-model potential for ^{12}C-^{12}C scattering at E_{lab}=1016 MeV. The solid curve corresponds to the inversion solution; while the dashed and dotted curves refer to the experimental data from [6,7], respectively.

$$\Delta'=0.87.$$

Moreover we get from Eqs. (31) and (32) that α=2.224 51 and β=0.291 75. Accordingly we obtain

$$V_0=112.276\ \text{MeV} \quad \text{and} \quad W_0=91.745\ \text{MeV}.$$

V and W can be calculated now and the results are given in Figs. 2 and 3 with a comparison made with the best-fit Woods-Saxon geometry adopted for the same system [6,7].

From these figures we clearly see that our procedure is in good agreement with the best-fit Woods-Saxon potential as far as the real part is concerned; while it gives a much deeper potential for the imaginary one.

On the other hand, if we parametrize the absorption coefficient by $\eta(b)\equiv|S_l|^2$ rather than $|S_l|$ as is taken by many authors [5]. Then we have

$$\chi_R=\frac{2\mu}{1+\exp[(l-l_0')/a]} \qquad (38)$$

and

$$\chi_I(b)=\frac{1}{2}\ln\left[1+\exp\left(\frac{l_0-l}{a}\right)\right]. \qquad (39)$$

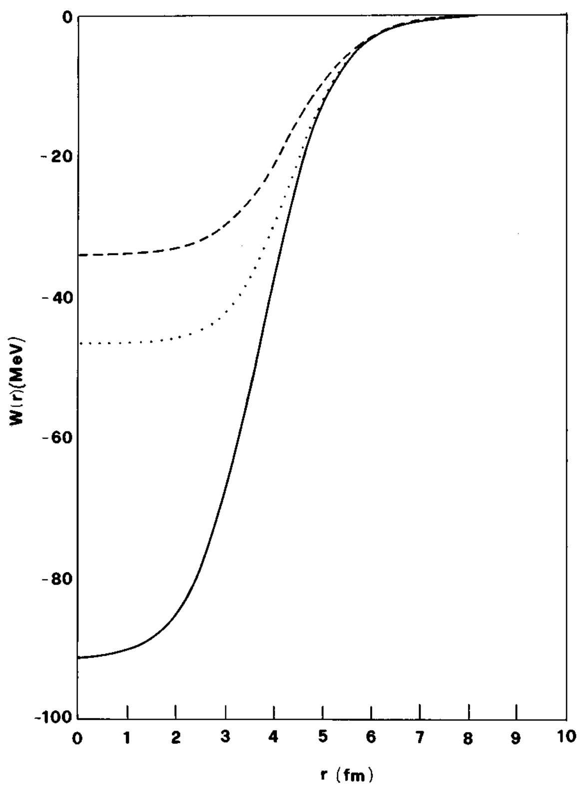

FIG. 3. The imaginary part of the optical-model potential for ^{12}C-^{12}C scattering at $E_{\mathrm{lab}}=1016$ MeV. The solid curve corresponds to the inversion solution; while the dashed and dotted curves refer to the experimental data from [6,7], respectively.

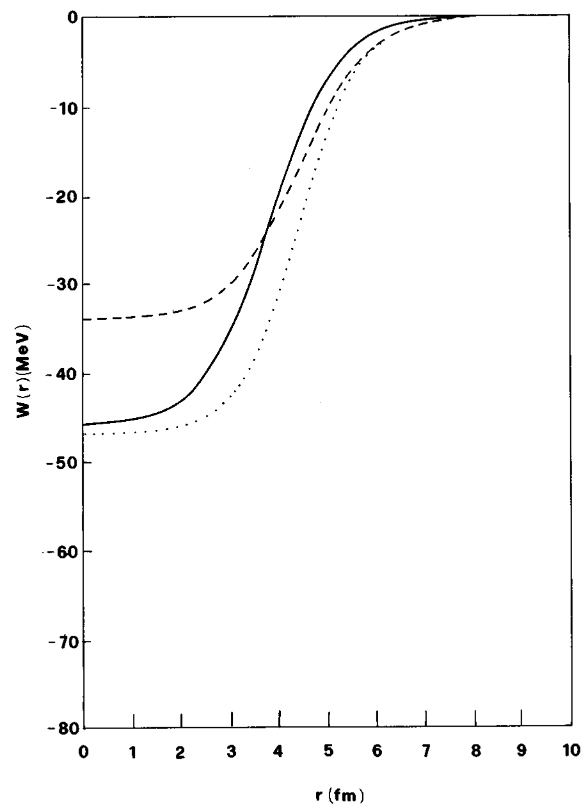

FIG. 4. The imaginary part of the optical-model potential for ^{12}C-^{12}C scattering at $E_{\mathrm{lab}}=1016$ MeV. The solid curve corresponds to the inversion solution (second parametrization); while the dashed and dotted curves refer to the experimental data from [6,7], respectively.

Thus, the only change we obtain, due to this new parametrization, is that Eq. (16) is multiplied by a factor of one-half so that all related equations must be multiplied by the same factor. The final expected result is that the depth of the imaginary potential W_0 will be reduced by a factor of one-half and hence becomes much closer to the best-fit potential. This is illustrated in Fig. 4 and where the curves have the same meanings as before.

V. CONCLUDING DISCUSSION

We have seen, through our inversion procedure, that we were able to relate the McIntyre parametrization of the S matrix to a Woods-Saxon-type optical-model potential.

The assumption that the ions have straight-line trajectories through the scattering processes made it possible to use the semiclassical approach and thus to make a correspondence between the high-energy approximation and the partial-wave expansion for the scattering amplitude.

It is to be emphasized that Coulomb effects contribute negligibly ($\cong$1%) to the various parameters of the McIntyre parametrization in our case of the ^{12}C-^{12}C scattering.

As to the result obtained for the depth of the imaginary part of the optical potential, it can be understood on the lights of the parametrization procedure we have followed. The multiplication factor present in W_0 can be reduced if we parametrize the absorption coefficient as $\eta(b)\equiv|S_l|^2$ rather than $|S_l|$ given by Eq. (1); this was shown to be the case and the depth of the imaginary potential got reduced by a factor of one-half and became much closer to the best-fit potential.

We should point out, also, that many relations between the I_ν's other than those obtained in Eqs. (24)–(27) are satisfied with our values of the potential parameters; for instance another set of equations between the I_ν's may be

$$\frac{I_8(R',\Delta')}{I_6(R',\Delta')}=\frac{7}{6}\,\frac{I_7(b_0',d')}{I_5(b_0',d')} \tag{40}$$

and

$$\frac{I_8(R,\Delta)}{I_6(R,\Delta)}=\frac{7}{8}\,\frac{I_8(b_0,d)}{I_6(b_0,d)}. \tag{41}$$

These two equations are exactly satisfied by the same values obtained previously for the optical potential parameters; more details are given in the Appendix.

In addition, we should note that within the WKB approximation the inverse scattering method was applied to obtain an optical potential from a strong absorption fit to data for ^{12}C-^{12}C scattering and that the S matrix, there, was represented by a rational function. The present results are consistent with those obtained previously [12–14]. Moreover, writing the S matrix as a sum of Gaussians enabled us to construct a Woods-Saxon-type optical-model potential and hence to get the potential down to $r=0$.

Finally, it is to be mentioned that, in principle, Coulomb effects can be incorporated in the present calculations; but the situation, then, will be much more involved [15].

ACKNOWLEDGMENTS

One of the authors (A.M.A) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX

In this appendix we give detailed calculations of the parameters R, Δ, R', and Δ' in terms of the impact parameters (b_0 and b_0') and diffusivities (d and d').

Comparing Eqs. (15) and (16) with Eqs. (17) and (18), we have

$$\frac{1}{1+\exp[(b-b_0')/d']}=\sum_n c_n \exp\left(\frac{-nb^2}{\alpha^2}\right) \tag{A1}$$

and

$$\ln\left[1+\exp\left(\frac{b_0-b}{d}\right)\right]=\sum_n b_n \exp\left(\frac{-nb^2}{\beta^2}\right). \tag{A2}$$

Now, introducing the integral

$$I_\nu(x_0,a_0)=\int_0^\infty \frac{x^\nu dx}{1+\exp[(x-x_0)/a_0]} \tag{A3}$$

and using Eqs. (A1) and (A2) we get

$$I_\nu(b_0',d')=\sum_n c_n \int_0^\infty b^\nu \exp\left(\frac{-nb^2}{\alpha^2}\right) db \tag{A4}$$

and

$$I_\nu(b_0,d)=\frac{2d}{\beta^2}\sum_n b_n n \int_0^\infty b^{\nu+1} \exp\left(\frac{-nb^2}{\alpha^2}\right) db. \tag{A5}$$

Accordingly we have, for $I_\nu(b_0',d')$,

$$I_\nu(b_0',d')=\frac{1}{2}\left(\frac{\nu-1}{2}\right)!\,\alpha^{\nu+1}\sum_n \frac{c_n}{n^{(\nu+1)/2}} \quad \nu=1,3,5,\ldots,11. \tag{A6}$$

Also, for $I_\nu(b_0,d)$, we find

$$I_\nu(b_0,d)=d\left(\frac{\nu}{2}\right)!\,\beta^\nu \sum_n \frac{b_n}{n^{\nu/2}}, \quad \nu=2,4,\ldots,12. \tag{A7}$$

Thus, from Eq. (A6), we obtain

$$\frac{I_{\nu+2}(b_0',d')}{I_\nu(b_0',d')}=\left(\frac{\nu+1}{2}\right)\alpha^2 \frac{\sum_n c_n/n^{(\nu+3)/2}}{\sum_n c_n/n^{(\nu+1)/2}}, \quad \nu=1,3,\ldots,9. \tag{A8}$$

Similarly, from Eq. (A7), we get

$$\frac{I_{\nu+2}(b_0,d)}{I_\nu(b_0,d)}=\left(\frac{\nu}{2}+1\right)\beta^2 \frac{\sum_n b_n/n^{\nu/2+1}}{\sum_n b_n/n^{\nu/2}}, \quad \nu=2,4,\ldots,10. \tag{A9}$$

On the other hand, from Eqs. (19) and (20) and Eqs. (21) and (22), we get

$$\frac{1}{1+\exp[(r-R')/\Delta']}=\sum_n c_n\sqrt{n\pi}\,\exp\left(\frac{-nr^2}{\alpha^2}\right) \tag{A10}$$

and

$$\frac{1}{1+\exp[(r-R)/\Delta]}=\sum_n b_n\sqrt{n\pi}\,\exp\left(\frac{-nr^2}{\beta^2}\right). \tag{A11}$$

Again, from these relations, we obtain

$$\frac{I_{\nu+2}(R',\Delta')}{I_\nu(R',\Delta')}=\left(\frac{\nu+1}{2}\right)\alpha^2 \frac{\sum_n c_n/n^{\nu/2+1}}{\sum_n c_n/n^{\nu/2}}, \quad \nu=2,4,\ldots,10. \tag{A12}$$

We, also, get

$$\frac{I_{\nu+2}(R,\Delta)}{I_\nu(R,\Delta)}=\left(\frac{\nu+1}{2}\right)\beta^2 \frac{\sum_n b_n/n^{\nu/2+1}}{\sum_n b_n/n^{\nu/2}}, \quad \nu=2,4,\ldots,10. \tag{A13}$$

Comparing Eq. (A8) with Eq. (A12), we find

$$\frac{I_{\nu+2}(R',\Delta')}{I_\nu(R',\Delta')}=\left(\frac{\nu+1}{\nu}\right)\frac{I_{\nu+1}(b_0',d')}{I_{\nu-1}(b_0',d')}, \quad \nu=2,4,\ldots,10. \tag{A14}$$

Again, comparing Eq. (A9) with Eq. (A13), we find

$$\frac{I_{\nu+2}(R,\Delta)}{I_\nu(R,\Delta)}=\left(\frac{\nu+1}{\nu+2}\right)\frac{I_{\nu+2}(b_0,d)}{I_\nu(b_0,d)}, \quad \nu=2,4,\ldots,10. \tag{A15}$$

Further, from the analytic relation for $I_\nu(x_0,a_0)$, we have [11]

$$I_\nu(x_0,a_0)=\frac{x_0^{\nu+1}}{\nu+1}+\sum_{m=0}^{\nu}[1-(-1)^m]\frac{\nu!x_0^{\nu-m}}{(\nu-m)!}a_0^{m+1}(1-2^{-m})\xi(m+1)-\sum_{k=1}^{\infty}(-1)^{\nu+k}\exp\left(-\frac{kx_0}{a_0}\right)\nu!\,\frac{a_0^{\nu+1}}{k^{\nu+1}}, \tag{A16}$$

where $\xi(z)$ is the Riemann zeta function.

Equation (A16) can be simplified somewhat [11]; since the infinite sum in the last term can be neglected. For the case of $\nu=2$, with typical values of $x_0=2.0608$ fm and $a_0=0.513$ fm for ^{16}O, the last term, if neglected would only introduce an error in I_2 which amounts to $\sim 0.02\%$.

From Eq. (A16), we find that

$$I_0(x_0,a_0)=x_0-\sum_{k=1}^{\infty}(-1)^k\exp\left(-\frac{kx_0}{a_0}\right)\frac{a_0}{k} \tag{A17}$$

or

$$I_0(x_0,a_0)=x_0+a_0\ln\left[1+\exp\left(-\frac{x_0}{a_0}\right)\right]. \tag{A18}$$

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