Modified three-body full-folding optical model potential

T. H. Rihan*

Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, Dubna (Moscow Region) 141980, Russia

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The optical model potential for nucleon elastic scattering is investigated within the context of a three-body model. It is shown that the standard full-folding optical model potential is obtained as a special case of an approximate solution to such a three-body problem. A new full-folding-type optical model potential together with its corresponding factorized $t\rho$ version are then obtained in the above three-body picture under some plausible approximations.

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I. INTRODUCTION

The basic ingredient in the multiple-scattering theory of the optical model potential is, as is well known, the scattering amplitude of the projectile on a bound-target nucleon. This amplitude is, however, a complicated many-body operator. Therefore, a common practice is to approximate it by another operator that does not contain the full complexity of the target Hamiltonian. Most of these approximations result in an optical potential which is calculated by folding the target density with an effective projectile-bound target nucleon interaction (the so-called folding model). Assuming, however a single-particle description of the target states, with the projectile interacting with only one target nucleon at a time while the remaining nucleons act as spectators (and/or a passive core), the above scattering amplitude becomes the solution to a three-body problem [1]. This three-body model for the projectile-bound target nucleon amplitude has the advantage that the reactive content of the corresponding optical model potential becomes better understood and recoil effects are included correctly. An approximate solution to the above three-body problem was then advanced [2], for the case when the projectile is very light as compared to the target nucleon. Further, an extension of such a solution to the case of arbitrary projectiles was subsequently developed by us [3]. In actual applications, however, these three-body solutions require (in principle) the knowledge of the off-shell projectile-bound target nucleon t-matrix element and the complete set of that bound nucleon-core states. Therefore, besides the need to introduce a plausible form (normally separable) for the projectile-bound target nucleon t matrix, the resulting optical model potential becomes largely dependent on the models used to generate the full target nucleon-core spectrum [2].

In the present work we shall show at first that the standard full-folding expression for the optical model potential [4] can be directly obtained from our extended three-body solution [3] adopting the same underlying assumptions. Moreover, we present a consistent treatment within our three-body approach to avoid the above-mentioned problem of target structure models. This then results in a new full-folding expression for the optical model potential. This full-folding expression, besides being more exact, has the advantage that the projectile-bound target nucleon t matrix appearing in it comes half-on-shell. Further, based on the above result, a new factorized $t\rho$ structure for the optical model potential which reflects the above three-body picture is also developed.

In Sec. II, we derive, starting from our three-body solution, the standard full-folding expression for the optical model potential. Then, we present the details of our treatment which leads to a new full-folding expression. Further, we develop our factorized $t\rho$ approximation. In Sec. III we draw the main conclusions of the present work.

II. THE OPTICAL MODEL POTENTIAL

In this section, let us consider the solution to the three-body optical model potential as given in [3]. The three bodies are the projectile p with mass m_p, an active target nucleon j (with which the projectile interacts) with mass m_j, and an inert core c comprising the rest of the target nucleus with mass m_c. Upon introducing a complete set of states $\{\phi_n\}$ that describes the active target nucleon-core relative motion together with its corresponding set of eigenvalues $\{\epsilon_n\}$, then to a very good approximation the first-order optical model potential can be given (in momentum space) by (cf. Eq. (2.15) of [3]):

$$\langle \mathbf{k}'|V_{\mathrm{opt}}(E)|\mathbf{k}\rangle = \frac{A-1}{A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}d\mathbf{Q}'\langle\overline{\phi_j}|\mathbf{Q}'-c\mathbf{k}'\rangle\langle\mathbf{Q}-c\mathbf{k}|\overline{\phi_j}\rangle\langle\mathbf{Q}'|\overline{\phi_n}\rangle\langle\overline{\phi_n}|\mathbf{Q}\rangle\Bigg[\langle\mathbf{k}'-a\mathbf{Q}'|V|\mathbf{k}-a\mathbf{Q}\rangle + \sum_{\lambda}\int d\mathbf{x}\frac{\langle\mathbf{k}'-a\mathbf{Q}'|V|\chi_\lambda\rangle\langle\chi_\lambda|\mathbf{x}\rangle\langle\mathbf{x}-a\Delta|V|\mathbf{k}-a\mathbf{Q}\rangle}{E^{+}+\epsilon_j-\epsilon_n-E_\lambda+(ac\hbar^2/2\overline{\mu})Q'^2}\Bigg]. \tag{1}$$

*On leave from Department of Mathematics and Theoretical Physics, Atomic Energy Authority, P. O. Code 13759, Cairo, Egypt.

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Here $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final projectile momenta, respectively, and A is the number of target nucleons while E is the projectile's energy. χ_λ is another complete set of states that describes the projectile's active target nucleon relative motion with its corresponding set of eigenvalues E_λ, while V is the interaction potential governing that motion. The summation over j in the above equation runs only over all occupied states in the target nucleus ground state while the summation over n will exhaust the whole spectrum of the active target nucleon-core states. In Eq. (1) $a=m_p/(m_p+m_j)$, $c=m_c/(m_j+m_c)$, and $\overline{\mu}=m_j\cdot m_c/(m_j+m_c)$, while $\Delta=\mathbf{Q}-\mathbf{Q}'$. One can infer, however, that for a local potential V, $\langle\mathbf{x}-a\Delta|V|\mathbf{k}-a\mathbf{Q}\rangle=\langle\mathbf{x}|V|\mathbf{x}-a\mathbf{Q}'\rangle$ (translation invariance), and henceforth Eq. (1) can be recast in the form

$$\langle\mathbf{k}'|V_{\rm opt}(E)|\mathbf{k}\rangle=\frac{A-1}{A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}d\mathbf{Q}'\langle\overline{\phi_j}|\mathbf{Q}'-c\mathbf{k}'\rangle\langle\mathbf{Q}-c\mathbf{k}|\overline{\phi_j}\rangle\langle\mathbf{Q}'|\overline{\phi_n}\rangle\langle\overline{\phi_n}|\mathbf{Q}\rangle\Bigg[\langle\mathbf{k}'-a\mathbf{Q}'|V|\mathbf{k}-a\mathbf{Q}\rangle + \sum_\lambda\frac{\langle\mathbf{k}'-a\mathbf{Q}'|V|\chi_\lambda\rangle\langle\chi_\lambda|V|\mathbf{k}-a\mathbf{Q}'\rangle}{E^+ +\epsilon_j-\epsilon_n-E_\lambda+(ac\hbar^2/2\overline{\mu})Q'^2}\Bigg]. \tag{2a}$$

But, by definition

$$t_{pj}(\varepsilon)=V+V\frac{1}{\varepsilon^+-h}V,$$

where t_{pj} is the free projectile-active target nucleon t matrix, and h is the corresponding Hamiltonian. Then, introducing the above-mentioned complete set of states $\{\chi_\lambda\}$ (which is such that $h|\chi_\lambda\rangle=E_\lambda|\chi_\lambda\rangle$) one can write (in obvious notations)

$$\langle\mathbf{p}'|t_{pj}(\varepsilon)|\mathbf{p}\rangle=\langle\mathbf{p}'|V|\mathbf{p}\rangle+\sum_\lambda\frac{\langle\mathbf{p}'|V|\chi_\lambda\rangle\langle\chi_\lambda|V|\mathbf{p}\rangle}{\varepsilon^+-E_\lambda}. \tag{2b}$$

Consequently, Eq. (2a) can be rewritten in the following equivalent form:

$$\langle\mathbf{k}'|V_{\rm opt}(E)|\mathbf{k}\rangle=\frac{A-1}{A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}d\mathbf{Q}'\,\phi_j^+(\mathbf{Q}'-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n(\mathbf{Q}')\,\phi_n^+(\mathbf{Q})\langle\mathbf{k}'-a\mathbf{Q}'|t_{pj}(\mathscr{E})|\mathbf{k}-a\mathbf{Q}'\rangle, \tag{3}$$

where $\mathscr{E}=E^+ +\epsilon_j-\epsilon_n+(ac\hbar^2/2\overline{\mu})Q'^2$ and $\langle\mathbf{x}|\overline{\phi_m}\rangle=\phi_m(\mathbf{x})$.

A. Standard full-folding model

Starting from the above equation, one can readily explore the details of the approximations underlying various models for the first-order optical model potential. For example, the so-called full-folding model, for nucleon-nucleus scattering, is usually derived under the following assumptions [4]: (a) setting the single-particle energy ϵ_n equal to its corresponding free energy (this amounts to assuming plane waves for the intermediate set of states $\{\phi_n\}$). Adopting such an assumption, Eq. (3) will read (invoking completeness):

$$\langle\mathbf{k}'|V_{\rm opt}(E)|\mathbf{k}\rangle=\frac{A-1}{A}\sum_{j=1}^{A}\int d\mathbf{Q}\phi_j^+(\mathbf{Q}-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\langle\mathbf{k}'-a\mathbf{Q}|t_{pj}(E_0)|\mathbf{k}-a\mathbf{Q}\rangle, \tag{4}$$

where $E_0=E^+ +\epsilon_j-[\hbar^2Q^2/2(m_p+m_j)]\{1+(m_p+m_j)/m_c)\}$. (b) Neglecting the recoil of the core, which amounts to assuming that m_c is very large as compared to m_j (and/or m_p). Noting further that for nucleon scattering $a=1/2$, Eq. (4) will take exactly the usual form of the full-folding approximation, which reads [4]

$$\langle\mathbf{k}'|V_{\rm opt}(E)|\mathbf{k}\rangle=\frac{A-1}{A}\sum_{j=1}^{A}\int d\mathbf{Q}\phi_j^+(\mathbf{Q}-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\left\langle\mathbf{k}'-\mathbf{Q}/2\left|t_{pj}\left(E^+ +\epsilon_j-\frac{\hbar^2Q^2}{4m_p}\right)\right|\mathbf{k}-\mathbf{Q}/2\right\rangle. \tag{5}$$

However, in order that Eq. (5) becomes amenable for calculations, still another simplification is made, by assuming that at nucleon energies E above 100 MeV it is reasonable to consider average energies $\langle\epsilon_n\rangle$ characteristic for protons and neutrons.

B. Full treatment

We shall now show, however, that our expression [cf. Eq. (3)] for the optical model potential can be further simplified (in an almost exact way) without recourse to any such approximations as those introduced above for the standard full-folding optical potential or introducing any specific model to describe target states as in previous three-body approaches [2]. For this purpose, let us at first expand the operator $t_{pj}(\mathscr{E})$ in Eq. (3) around some optimum value $\overline{\mathscr{E}}$ of the effective energy $\mathscr{E}$, i.e.,

$$t_{pj}(\mathscr{E})=t_{pj}(\overline{\mathscr{E}})+(\mathscr{E}-\overline{\mathscr{E}})dt_{pj}(\mathscr{E})/d\mathscr{E}|_{\mathscr{E}=\overline{\mathscr{E}}}+\cdots \tag{6a}$$

we shall then choose $\overline{\mathscr{E}}$ such that

$$\sum_n \int d\mathbf{Q}\; d\mathbf{Q}'\,\phi_j^+(\mathbf{Q}'-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n(\mathbf{Q}')\,\phi_n^+(\mathbf{Q}) \times\langle \mathbf{k}'-a\mathbf{Q}'|(\mathscr{E}-\overline{\mathscr{E}})dt_{pj}(\mathscr{E})/d\mathscr{E}|_{\mathscr{E}=\overline{\mathscr{E}}}|\mathbf{k}-a\mathbf{Q}'\rangle=0. \tag{6b}$$

Further, for the sake of clarity, let us introduce the following notations:

$$\mathscr{T}(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q}')=\langle \mathbf{k}'-a\mathbf{Q}'|dt_{pj}(\mathscr{E})/d\mathscr{E}|_{\mathscr{E}=\overline{\mathscr{E}}}|\mathbf{k}-a\mathbf{Q}'\rangle \tag{7a}$$

and

$$\mathscr{T}_n(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q}')=\phi_j^+(\mathbf{Q}'-c\mathbf{k}')\,\phi_n(\mathbf{Q}')\mathscr{T}(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q}'). \tag{7b}$$

In that way, the condition given by Eq. (6b) can be cast in the form (observing completeness)

$$\int d\mathbf{Q}\left(E+\frac{ac\hbar^2Q^2}{2\overline{\mu}}\right)\phi_j^+(\mathbf{Q}-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\mathscr{T}(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q})+\sum_n\int d\mathbf{Q}\; d\mathbf{Q}'(\epsilon_j-\epsilon_n)\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{Q})\mathscr{T}_n(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q}')$$
$$\equiv\int d\mathbf{Q}\;\overline{\mathscr{E}}\phi_j^+(\mathbf{Q}-c\mathbf{k}')\,\phi_j(\mathbf{Q}-c\mathbf{k})\mathscr{T}(\overline{\mathscr{E}},\mathbf{k},\mathbf{k}',\mathbf{Q}). \tag{8}$$

Next, making use of the Schrödinger equation in momentum space for each of ϕ_j and ϕ_n^+, one directly obtains

$$(\epsilon_j-\epsilon_n)\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{Q})=\frac{\hbar^2}{2\overline{\mu}}\{(\mathbf{Q}-c\mathbf{k})^2-Q^2\}\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{Q})+\frac{1}{(2\pi)^3}\int d\mathbf{y}\{U_{jc}(\mathbf{Q}-\mathbf{y})\,\phi_j(\mathbf{y}-c\mathbf{k})\,\phi_n^+(\mathbf{Q})$$
$$-U_{jc}(\mathbf{y}-\mathbf{Q})\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{y})\}, \tag{9}$$

where U_{jc} is the interaction between the active nucleon j and the core c in the target nucleus. Inserting Eq. (9) into Eq. (8) and noting that the second term on the RHS of Eq. (9) will vanish identically while integrating over $\mathbf{Q}$ [since $U_{jc}(\mathbf{x})=U_{jc}(-\mathbf{x})$], one arrives at the following expression for the optimal effective energy $\overline{\mathscr{E}}$:

$$\overline{\mathscr{E}}=E+\frac{ac\hbar^2}{2\overline{\mu}}Q^2+\frac{ac\hbar^2}{2\overline{\mu}}\{c^2k^2-2c\mathbf{k}\mathbf{Q}\}=\frac{\hbar^2}{2\mu}(\mathbf{k}-a\mathbf{Q})^2, \tag{10}$$

where $\mu=m_p\cdot m_j/(m_p+m_j)$ is the reduced mass of the incident particle-active target nucleon system. Moreover, it is shown in the Appendix that higher order terms in the expansion given in Eq. (6a) will give negligible contribution to the above optical model potential [cf. Eq. (3)] and can be safely neglected. Summing up, then by inserting Eqs. (6) and (10) into Eq. (3), one finally arrives at the following approximate expression for the optical model potential:

$$\langle \mathbf{k}'|V_{\rm opt}(E)|\mathbf{k}\rangle=\frac{A-1}{A}\sum_{j=1}^{A}\int d\mathbf{Q}\;\phi_j^+(\mathbf{Q}-c\mathbf{k})\,\phi_j(\mathbf{Q}-c\mathbf{k}) \times\langle \mathbf{k}'-a\mathbf{Q}|t_{pj}(\overline{\mathscr{E}})|\mathbf{k}-a\mathbf{Q}\rangle. \tag{11}$$

This expression will thus constitute our main result for modified full-folding-type optical model potential. It is not only more exact than those expressions given in previous approaches, but also simpler and more amenable for calculations as the two-body t matrix appearing in the integrand in Eq. (11) comes half-on-shell.

To that end, it seems that the above treatment may be limited to the case when the t matrix $t_{pj}(\mathscr{E})$ is a slowly varying function of the energy $\mathscr{E}$ [where the series (6a) can be trusted]. It is very interesting, however, to note that Eq. (11) can be obtained in another (more obvious) way free from the above (seeming) limitation. Indeed, one at first observes that the denominator of the second term in the square brackets in Eq. (2a) can be expanded in the form

$$\frac{1}{E^++\epsilon_j-\epsilon_n-E_\lambda+(ac\hbar^2/2\overline{\mu})Q'^2}=\sum_{l=0}\frac{(-)^{\ell}(\epsilon_j-\epsilon_n)^{\ell}}{\{E^+-E_\lambda+(ac\hbar^2/2\overline{\mu})Q'^2\}^{\ell+1}}. \tag{12a}$$

Further, by applying Eq. (9) as many times as necessary and adopting the mathematical observations given in the Appendix, one arrives at (after some manipulations):

$$\int d\mathbf{Q}(\epsilon_j-\epsilon_n)^{\ell}\,\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{Q}) \approx\left(\frac{\hbar^2}{2\overline{\mu}}\{(\mathbf{Q}-c\mathbf{k})^2-Q^2\}\right)^{\ell}\phi_j(\mathbf{Q}-c\mathbf{k})\,\phi_n^+(\mathbf{Q}). \tag{12b}$$

Then, inserting Eq. (12a) into Eq. (2a) while observing Eq. (12b), one directly obtains Eq. (11).

C. Optimal approximation

Equation (11) can be, however, further simplified. To show this, let us rewrite it in the following form:

$$\langle \mathbf{k}'|V_{\mathrm{opt}}(E)|\mathbf{k}\rangle = \frac{A-1}{A}\sum_{j=1}^{A}\int d\mathbf{Q}\ \phi_j^{+}(\mathbf{Q}+c\mathbf{q})\phi_j(\mathbf{Q}) \times \langle \mathbf{k}'-ac\mathbf{q}-a\mathbf{Q}|t_{pj}(\overline{E})|\mathbf{k}-a\mathbf{Q}\rangle, \tag{13a}$$

where $\overline{E}=(\hbar^2/2\mu)(b\mathbf{k}-a\mathbf{Q})^2$ and $\mathbf{q}=\mathbf{k}-\mathbf{k}'$ is the momentum transfer, while $b=1-ac$. It has been shown [5], that by expanding the t-matrix element in the above equation around some preferable value $\mathbf{Q}_0$ and choosing that value such that

$$\int d\mathbf{Q}(\mathbf{Q}-\mathbf{Q}_0)\phi_j^{+}(\mathbf{Q}+c\mathbf{q})\phi_j(\mathbf{Q})=0, \tag{13b}$$

then it may be reasonable to take out the t-matrix element from under the integral sign in Eq. (13a) at that value $\mathbf{Q}_0$. Such a process will lead to the following optimal approximation for the optical model potential [with $\mathbf{Q}_0=-(c/2)\mathbf{q}$]:

$$\langle \mathbf{k}'|V_{\mathrm{opt}}(E)|\mathbf{k}\rangle=(A-1)\rho(c\mathbf{q})\langle \mathbf{p}'|t_{pj}(E_p)|\mathbf{p}\rangle, \tag{14}$$

where $\mathbf{p}=b\mathbf{k}+(ac/2)\mathbf{q}$; $\mathbf{p}'=b\mathbf{k}'-(ac/2)\mathbf{q}$; $E_p=(\hbar^2/2\mu)p^2$, and the single-particle density ρ is given by:

$$\rho(c\mathbf{q})=\int d\mathbf{Q}\phi_j^{+}(\mathbf{Q}+c\mathbf{q})\phi_j(\mathbf{Q})$$

(here for simplicity we assumed the same single-particle density for both protons and neutrons in the target nucleus). Equation (14) will thus constitute a modified $t\rho$ approximation for the optical model potential [note that the t matrix in Eq. (14) is fully on shell]. It has been also argued [5] that such an optimal approximation is very reliable in the sense that higher order corrections in the above expansion of the t matrix can be neglected except perhaps at very large scattering angles. Further, it is also interesting at this point to note that if one assumes the core to be infinitely heavy (i.e., $c=1$), Eq. (14) will reduce exactly to the optimal approximation derived (in the spirit of a three-body problem) by Gurvitz *et al.* [6] in that special case (cf. Eq. (21) of [6]).

III. CONCLUSIONS

In summary, we have shown, based on an approximate solution to the three-body equations for the optical model potential, that a new modified (full-folding-type) expression for such a potential is obtained. Also, a new factorized $t\rho$ form was derived for that potential. This modified optical model potential is more exact than previous versions in that recoil effects are treated properly and the full active target nucleon-core spectrum is considered. It is in this way that the effective energy in the projectile-active target nucleon t matrix is approximately given by Eq. (10b), and consequently t_{pj} in the integrand in Eq. (11) is half-on-shell. Had one truncated the full active target nucleon-core spectrum, one would have obtained an off-shell t_{pj} amplitude (as in the usual full-folding models). Therefore, care must be exercised in interpreting the results based on previous approaches, where the whole spectrum of the bound target nucleon states is not fully accounted for.

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APPENDIX A

One can easily see that the second-order term in the expansion given in Eq. (6a), will lead to the following correction term $\Delta V_{\mathrm{opt}}(\mathbf{k}',\mathbf{k})$ for the optical potential:

$$\Delta V_{\mathrm{opt}}(E)=\frac{A-1}{2A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}\ d\mathbf{Q}'\,\phi_j^{+}(\mathbf{Q}'-c\mathbf{k}')\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})\phi_n(\mathbf{Q}')(\mathscr{E}-\overline{\mathscr{E}})^2 \times\langle \mathbf{k}'-a\mathbf{Q}'|d^2t_{pj}(\mathscr{E})/d\mathscr{E}^2|_{\mathscr{E}=\overline{\mathscr{E}}}|\mathbf{k}-a\mathbf{Q}'\rangle. \tag{A1}$$

Let us for the sake of clarity introduce the following notations:

$$T(\mathbf{Q},\mathbf{Q}')=\langle \mathbf{k}'-a\mathbf{Q}'|d^2t_{pj}(\mathscr{E})/d\mathscr{E}^2|_{\mathscr{E}=\overline{\mathscr{E}}}|\mathbf{k}-a\mathbf{Q}'\rangle,$$

$$G(Q,Q')=\frac{\hbar^2}{2\mu}\{Q^2-Q'^2\},$$

and

$$F(\mathbf{Q},\mathbf{x})=\frac{\hbar^2}{2\mu}[(\mathbf{Q}-c\mathbf{k})^2-x^2],$$

so that $\mathscr{E}-\overline{\mathscr{E}}=\epsilon_j-\epsilon_n-\{G(Q,Q')+F(\mathbf{Q},\mathbf{Q})\}$. Consequently, $\Delta V_{\mathrm{opt}}(E)$ can be cast in the form

$$\Delta V_{\mathrm{opt}}(E)=I_1+I_2+I_3,$$

where

$$I_1=\frac{A-1}{2A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}\,d\mathbf{Q}'\,\phi_j^{+}(\mathbf{Q}'-c\mathbf{k}')\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})\phi_n(\mathbf{Q}')T(\mathbf{Q},\mathbf{Q}')[G(Q,Q')+F(\mathbf{Q},\mathbf{Q}')]^2 \tag{A2a}$$

and

$$I_2=-\frac{A-1}{A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}\,d\mathbf{Q}'(\epsilon_j-\epsilon_n)\phi_j^{+}(\mathbf{Q}'-c\mathbf{k}')\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})\phi_n(\mathbf{Q}')T(\mathbf{Q},\mathbf{Q}')[G(Q,Q')+F(\mathbf{Q},\mathbf{Q}')], \tag{A2b}$$

while

$$I_3=\frac{A-1}{2A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}\,d\mathbf{Q}'(\epsilon_j-\epsilon_n)^2\phi_j^{+}(\mathbf{Q}'-c\mathbf{k}')\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})\phi_n(\mathbf{Q}')T(\mathbf{Q},\mathbf{Q}'). \tag{A2c}$$

It may be now apparent that Eq. (A2a) can be recast in the form (using completeness)

$$I_1=\frac{A-1}{2A}\sum_{j=1}^{A}\sum_{n}\int d\mathbf{Q}\;\phi_j^{+}(\mathbf{Q}-c\mathbf{k}')\phi_j(\mathbf{Q}-c\mathbf{k})T(\mathbf{Q},\mathbf{Q})F^2(\mathbf{Q},\mathbf{Q}). \tag{A3}$$

Further, observing that Eq. (9) will take the following form in the above notations:

$$\begin{aligned}(\epsilon_j-\epsilon_n)\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})&=F(\mathbf{Q},\mathbf{Q})\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{Q}-c\mathbf{k})+\frac{1}{(2\pi)^3}\int d\mathbf{y}\{U_{jc}(\mathbf{Q}-\mathbf{y})\phi_n^{+}(\mathbf{Q})\phi_j(\mathbf{y}-c\mathbf{k})\\&\quad-U_{jc}(\mathbf{y}-\mathbf{Q})\phi_n^{+}(\mathbf{y})\phi_j(\mathbf{Q}-c\mathbf{k})\},\end{aligned} \tag{A4}$$

one can then write (after some manipulations and using completeness)

$$I_2=-\frac{A-1}{A}\sum_{j=1}^{A}\int d\mathbf{Q}[T(\mathbf{Q},\mathbf{Q})F^2(\mathbf{Q},\mathbf{Q})\phi_j^{+}(\mathbf{Q}-c\mathbf{k}')\phi_j(\mathbf{Q}-c\mathbf{k})]-\frac{A-1}{(2\pi)^3A}B(\mathbf{k},\mathbf{k}'), \tag{A5}$$

where

$$\begin{aligned}B(\mathbf{k},\mathbf{k}')=\int d\mathbf{Q}d\mathbf{y}U_{jc}(\mathbf{Q}-\mathbf{y})[&T(\mathbf{Q},\mathbf{Q})\{G(Q,Q)+F(\mathbf{Q},\mathbf{Q})\}\phi_j^{+}(\mathbf{Q}-c\mathbf{k}')\phi_j(\mathbf{y}-c\mathbf{k})\\&-T(\mathbf{Q},\mathbf{y})\{G(Q,y)+F(\mathbf{Q},\mathbf{Q})\}\phi_j^{+}(\mathbf{y}-c\mathbf{k}')\phi_j(\mathbf{Q}-c\mathbf{k})].\end{aligned}$$

In the same way, one finds that

$$I_3=\frac{A-1}{2A}\sum_{j=1}^{A}\int d\mathbf{Q}T(\mathbf{Q},\mathbf{Q})F^2(\mathbf{Q},\mathbf{Q})\phi_j^{+}(\mathbf{Q}-c\mathbf{k}')\phi_j(\mathbf{Q}-c\mathbf{k})+\frac{A-1}{2(2\pi)^3A}C(\mathbf{k},\mathbf{k}'), \tag{A6}$$

where

$$C(\mathbf{k},\mathbf{k}')=\int d\mathbf{Q}\;d\mathbf{y}U_{jc}(\mathbf{Q}-\mathbf{y})F(\mathbf{Q},\mathbf{Q})[T(\mathbf{Q},\mathbf{Q})\phi_j^{+}(\mathbf{Q}-c\mathbf{k}')\phi_j(\mathbf{y}-c\mathbf{k})-T(\mathbf{Q},\mathbf{y})\phi_j^{+}(\mathbf{y}-c\mathbf{k}')\phi_j(\mathbf{Q}-c\mathbf{k})].$$

Summing up, then

$$\Delta V_{\rm opt}(E)=\frac{A-1}{2(2\pi)^3A}[C(\mathbf{k},\mathbf{k}')-2B(\mathbf{k},\mathbf{k}')]. \tag{A7}$$

Now as for realistic potentials (e.g., Woods-Saxon) $U_{jc}(\mathbf{q})$ is sharply peaked at $\mathbf{q}\approx 0$ with an exponential decrease times an oscillating function. So that, $\Delta V_{\rm opt}$ to a very good approximation will vanish. Moreover, $\Delta V_{\rm opt}$ will display $1/E^2$ fall off [due to $d^2t_{pj}(\mathscr{E})/d\mathscr{E}^2$] with respect to $V_{\rm opt}$ in the case of intermediate and/or high energies.

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