# Unified picture of the BCS nuclear model by use of the 1/*N* expansion

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Several BCS-type approximations encountered in the literarature were correlated using a 1/*N* expansion over a set of dispersion type energy equations, which are the exact solution of a system of fermions interacting through a pure pairing force. One of the interesting results obtained is that the BCS+QRPA approximation coincides with the first order term of the 1/*N* expansion.

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## I. INTRODUCTION

For nearly three decades the nuclear pairing model has played an important role in the description of nuclear spectra [1]. For many reasons discussed elsewhere [2] the mean field Bardeen-Cooper-Schrieffer (BCS) method [3] is the most useful approximation to solve this model. Since it is a mean field theory, it violates a symmetry of the system, namely the particle number conservation. Throughout the literature [4–15] there are many attempts to improve the BCS approximation. Reversing this procedure and starting with an exact solution of the pairing Hamiltonian, we would be able to derive and correlate the existing refinements to the BCS approximation. As a consequence it is possible to present a unified picture of the various solutions of the BCS pairing model. A straightforward diagonalization of the pairing Hamiltonian, either in the usual shell-model basis or in quasispin basis, cannot serve our purpose since both methods are essentially numerical. Alternatively, Richardson and Sherman [16] have derived a set of dispersion-type equations as an exact solution of the pairing Hamiltonian. Physically this set of dispersion-type equations plays a similar role to the gap equation in the BCS theory and contains all the relevant physics. Mathematically, these equations represent a two-dimensional problem describing an equilibrium distribution of a collection of parallel lines of charge. As the number of particles ($2N$) tends to infinity, such a collection of parallel lines of charge coalesce to form a sheet of charge [17].

On the other hand, the 1/*N* expansion has been a popular tool in different branches of physics. In this paper this concept will be used to present a unified view of several improvements of the BCS nuclear model. Using complex variable analysis, Richardson [17] was able to perform an expansion in powers of 1/*N* of the total energy of the system, obtained through a set of dispersion equations, where BCS is the zero-order expansion. This expansion allows a comparison among the different BCS refinements in the literature. To carry out this comparison, we have derived analytical expressions in a 1/*N* expansion up to third order in a symmetric and solvable two-level model. We have found that the expression to the first-order correction in this 1/*N* expansion corresponds to the BCS+quasiparticle random phase approximation (QRPA). The present paper is organized as follows. In Sec. II we review the 1/*N* expansion of the set of dispersion-type equations. In Sec. III, the various existing refinements to the BCS theory are presented. The comparison of the 1/*N* expansion and the BCS-like theories are presented in Sec. IV along with some conclusions and general remarks.

## II. 1/*N* EXPANSION OVER THE DISPERSION TYPE OF ENERGY EQUATION

A system of $N$ pairs of particles interacting with a standard pairing force is described by the Hamiltonian

$$H=\sum_{jm}\epsilon_j c^{\dagger}_{jm}c_{jm}-\frac{g}{4}\sum_{jm,j'm'}(-)^{j-m}(-)^{j'-m'}\times c^{\dagger}_{j'm},c^{\dagger}_{j'-m},c_{j-m}c_{jm}, \tag{2.1}$$

where $c^{\dagger}_{jm}(c_{jm})$ is the creation (annihilation) operator in the single particle shell ($\epsilon_j$) orbit. $g$ is the pairing strength. An exact solution for this Hamiltonian was obtained long ago by Richardson and Sherman [16] starting with the following wave function:

$$|\Psi\rangle=\prod_{i=1}^{N}\left(\sum_j \frac{\sqrt{\Omega_j}}{2\epsilon_j-E_i}A^{\dagger}_j\right)|O\rangle. \tag{2.2}$$

Here, the pair creation operator $A^{\dagger}_j=(1/\sqrt{\Omega_j})\Sigma_m(-)^{j-m}c^{\dagger}_{jm}c^{\dagger}_{j-m}$ is introduced with $\Omega_j=(j+1/2)$ as the half degeneracy of the level $j$. With these wave functions and performing some algebraic manipulations, the ground-state energy and the excited seniority zero states are given by

$$E=2\sum_{i=1}^{N}e_i. \tag{2.3}$$

 

The $e_i$ are the roots, in general complex, of the following coupled system of equations:

$$-\frac{N}{G}+\sum_{i\neq l}^{N}\frac{1}{e_l-e_i}-\frac{1}{2}\sum_j\frac{\Omega_j}{e_l-\epsilon_j}=0, \tag{2.4}$$

where the sum over $j$ extends over the $M$ single particle states and $l=1,2,\ldots,N$. In the above expression, the pairing strength is taken to be $g=G/N$. It is also assumed that $\Omega_j$ are extensive quantities and therefore of order $N$. It is interesting to note that all terms in Eq. (2.4) are proportional to $N$.

Equation (2.4) represents a problem in two-dimensional electrostatics, i.e., the equilibrium distribution of a collection of parallel lines of charge. Therefore, the complex numbers $e_i$ may be thought of as being the locations of $N$ free lines of charge of unit strength in the complex plane. These free lines of charge lay in a uniform external field, $-N/G$, and in the field generated by a number of fixed lines of charge, with strength $-\Omega_j/2$, located at $\epsilon_j$ on the real axis.

The electrostatic field generated by such a distribution at any position $z$ in the complex plane is given by

$$F(z)=\sum_i\frac{1}{z-e_i}-\frac{1}{2}\sum_j\frac{\Omega_j}{z-\epsilon_j}-\frac{N}{G}. \tag{2.5}$$

The solutions of Eq. (2.4) can be sought through an expansion in powers of $1/N$. In order to perform such an expansion it is necessary to choose some limiting form for the field when $N\to\infty$. Following Richardson [17], we assume that all the results already obtained by BCS theory come from the above function. The leading term $F_0(z)$ of a $1/N$ expansion represents the ground state, where the number of lines of charge increase and coalesce to form a sheet. As a result, the poles of $F(z)$ arising from the first term of (2.5) merge and form a branch cut. Further, it may be assumed that this branch cut extends from the point $a$ to the point $a^*$, since the roots of Eq. (2.4) occur always in complex conjugate pairs. Therefore it is suggested that [17]

$$F_0(z)=-\frac{1}{2}\sqrt{(z-a)(z+a^*)}\sum_j\frac{\Omega_j}{E_j(z-\epsilon_j)}. \tag{2.6}$$

Here $a=\lambda+i\Delta$ is yet to be determined and $E_j=\sqrt{(\epsilon_j-a)(\epsilon_j+a^*)}$. From the above function it is possible to obtain all BCS-theory equations, namely the gap equation, the number equation, and the ground-state energy. It is also possible to identify the real and imaginary parts of $a$ with the well-known parameters of the BCS theory $\lambda$ (chemical potential) and $\Delta$ (energy gap), respectively. Up to here, only the basic equations of BCS were obtained. From the dispersion equations one can also get $1/N$ corrections to the BCS approximation. This was obtained in Ref. [17] providing the following expression for the energy:

$$E=(2\lambda-G)N+G-2\sum_{m=1}^{M-1}h(m). \tag{2.7}$$

In Eq. (2.7) each $h(m)$ is a term of an expansion of the field $H(j)$ produced by the free charges of unity strength located at the positions of the fixed charges

$$H(j)=\sum_{i=1}^{N}\frac{1}{\epsilon_j-e_i}=\frac{1}{2\pi i}\oint\frac{F(z)}{\epsilon_j-z}dz, \tag{2.8}$$

namely

$$H(j)=\frac{h(0)}{E_j}+\sum_{m=1}^{M-1}\frac{h(m)}{E_j(x_m-\epsilon_j)}, \tag{2.9}$$

where the $x_m$ are the real zeros of $F_0(z)$.

Expanding $h(m)$ in powers of $1/N$ gives the BCS ground-state energy and the different orders of improvement. Richardson [17] has presented explicit formulas for the first- and second-order energy corrections, $h_1(m)$ and $h_2(m)$, respectively. In this paper the third-order correction is also calculated in a solvable symmetric two-level model ($M=2$) and the results will be given in Sec. IV. In this model two levels with the same $j$ containing total number of particles $n=\Omega=j+1/2$ are separated by energy $\epsilon$. In order to give a flavor of the way this method works, we present below the first-order correction to BCS, obtained with the above described method. $h_1(m)$ is given by

$$h_1(m)=\frac{1}{2}\left\{\frac{\Sigma_j\dfrac{\Omega_j}{(x_m-\epsilon_j)^2}}{\Sigma_j\dfrac{\Omega_j}{E_j(x_m-\epsilon_j)^2}}-[(x_m-\lambda)^2+\Delta^2]^{1/2}\right\}. \tag{2.10}$$

In the particular case of the solvable symmetric two-level model we have

$$h_1(m=1)=\frac{1}{2}(g\Omega-\Delta)=g\Omega\left[1-\sqrt{1-\left(\frac{\epsilon}{2g\Omega}\right)^2}\right]. \tag{2.11}$$

Further results will be presented in Table I.

## III. THE EXISTING REFINEMENTS TO THE BCS SOLUTION OF THE NUCLEAR PAIRING MODEL

The well known BCS approximation for the schematic nuclear pairing Hamiltonian is presented in several places [2]. Here, only the essential points of this approximation will be discussed. Using the quasiparticle operators $a_{jm}$ and $a_{jm}^{\dagger}$ the Hamiltonian (2.1) can be split up in the form

$$H=H_{00}+H_{11}+H_{20}+H_{02}+H_{22}+H_{31}+H_{13}+H_{40}+H_{04}, \tag{3.1}$$

where $H_{mn}$ is written in terms of $(a^{\dagger})^n(a)^m$ or $(a)^n(a^{\dagger})^m$. Assuming that the residual terms $H_{31}+H_{13}+H_{40}+H_{04}$ can be neglected and making $H_{20}+H_{02}=0$, the well-known gap equation is obtained. Physically, it means that the original system of interacting quasiparticles and thus the complicated many-body problem is replaced by the simple one-body problem. However, one has to pay for this simplification. The new quasiparticle vacuum, which is the ground state of

TABLE I. Corrections to the BCS approximation in the symmetric two-level model in terms of a dimensionless parameter, $\kappa=(\epsilon/2g\Omega)^2$.

| | First order | Order $1/N$ | Order $1/N^2$ |
|---|---|---|---|
| This work | $-g\Omega[2-\sqrt{1-\kappa}]$ | $\frac{g}{4}\left[\frac{4\sqrt{1-\kappa}}{1-\kappa}-\frac{4-\kappa}{1-\kappa}\right]$ | $-\frac{g}{8\Omega}\left[\frac{(12-24\kappa)}{(1-\kappa)^2}-\frac{(36+39\kappa-43\kappa^2)}{3(1-\kappa)^{5/2}}\right]$ |
| QRPA [5] | $-g\Omega[2-\sqrt{1-\kappa}]$ | – | – |
| PBCS [8] | $-g\Omega[1+\frac{1}{2}\kappa]$ | – | – |
| Nogami [9] | $-g\Omega[1+\frac{1}{2}\kappa]$ | $-\frac{g}{4}\kappa$ | $-\frac{g}{8\Omega}\kappa$ |
| Nilsson [9] | $-g\Omega[1+\frac{1}{2}\kappa]$ | $-\frac{g}{4}[1-\kappa]$ | – |

the approximate Hamiltonian, is no longer an eigenvector of the number operator ($\hat{\mathbf{n}}$); thus one has to introduce some Lagrangian multiplier ($\lambda_1$) in order to keep at least the mean value of the number of nucleons in the system fixed. Lipkin and Nogami [9] introduced a second factor $\lambda_2\hat{\mathbf{n}}^2$ in the Hamiltonian (3.1) to obtain a smaller deviation in the number of particles. The following result is reported by Pradham *et al.* [10]:

$$\frac{4\lambda_2}{g}=\frac{(\Sigma_j\Omega_j u_j^3 v_j)(\Sigma_j\Omega_j u_j v_j^3)-\Sigma_j\Omega_j(u_j v_j)^4}{[\Sigma_j\Omega_j(u_j v_j)^2]^2-\Sigma_j\Omega_j(u_j v_j)^4}. \quad (3.2)$$

The $u_j$ and $v_j$ in the above formula are the well-known Bogoliubov-Valatin coefficients slightly modified to

$$v_j^2=\frac{1}{2}[1-\gamma_j^{-1}(\epsilon_j^0-\lambda)] \quad u_j^2=1-v_j^2, \quad (3.3)$$

where

$$\epsilon_j^0=\epsilon_j+(4\lambda_2-g)v_j^2, \quad (3.4)$$

$$\lambda=\lambda_1+2\lambda_2(n+1), \quad (3.5)$$

$$\gamma_j=[(\epsilon_j^0-\lambda)^2+\Delta^2]^{1/2}. \quad (3.6)$$

In the above expressions, $n$ is the average number of particles in the BCS ground state. Hence the correction to the BCS ground-state energy due to the term $-\lambda_2(\langle\hat{\mathbf{n}}^2\rangle-n^2)$ is given by $-4\lambda_2\Sigma\Omega_j(u_j v_j)^2$.

Using the same philosophy mentioned above, Nilsson [11] considered the nuclear wave function written in terms of wave functions with good particle number $p$:

$$\Psi(n)=\sum_p C_p(n)\varphi_p(n). \quad (3.7)$$

Let $\mathscr{E}(n,p)$ be defined as the energy associated with the $p$ component. Up to second order, the mean value of $\mathscr{E}(n,p)$ is given by

$$\langle\mathscr{E}\rangle=\mathscr{E}(n,p)+\frac{1}{2}\langle(\hat{p}-n)^2\rangle\left[\frac{\partial^2\mathscr{E}(n,p)}{\partial p^2}\right]_{p=n}. \quad (3.8)$$

Bang *et al.* [12] obtained the following expression for $[\partial^2\mathscr{E}(n,p)/\partial p^2]_{p=n}$:

$$\frac{1}{2}\langle(\hat{p}-n)^2\rangle[\partial^2\mathscr{E}(n,p)/\partial p^2]_{p=n} = g\frac{\Sigma_{j,j'}\Omega_j(\Omega_{j'}-\delta_{jj'})u_j v_j^3 u_{j'}^3 v_{j'}}{\Sigma_j\Omega_j u_j^2 v_j^2}, \quad (3.9)$$

which is very similar to the previous expression. We will use it to obtain one of the refinements to the BCS model; the result is shown in Sec. IV.

Another way to restore the number conservation, as suggested by Bayman [8], is to introduce a gauge transformation $|\mathrm{BCS}(\theta)\rangle$ into the BCS wave function in such a way that the ground state energy is given by

$$E=\frac{\int_0^{2\pi}e^{-iN\theta}\langle\mathrm{BCS}|\hat{H}|\mathrm{BCS}(\theta)\rangle d\theta}{\int_0^{2\pi}e^{-iN\theta}\langle\mathrm{BCS}|\mathrm{BCS}(\theta)d\theta}\equiv\frac{I_E}{I_o}, \quad (3.10)$$

where $\theta$ is the gauge angle. Transforming both overlaps into polynomial forms it was possible to perform the above integrations analytically [13], resulting in

$$I_E=\sum_{l=N}\left\{\prod_j B_j\left[\sum_j 2\left(\epsilon_j-\frac{g}{2}\right)l_j-g\sum_{ij}\frac{u_i v_j}{u_j v_i}(\Omega_j-l_j)l_i\right]\right\} \quad (3.11)$$

and

$$I_o=\sum_{l=N}\left(\prod_j B_j\right), \quad (3.12)$$

with

$$B_j=(u_j^2)^{\Omega_j-l_j}(v_j^2)^{l_j}. \quad (3.13)$$

Here $l=\Sigma_j l_j$ and $l_j$ are integers. The last method considered here to restore the number conservation is an explicit treatment of the neglected residual interaction in the BCS approximation using the QRPA [4]. In this case the residual term $H_{\rm res}=H_{22}+H_{03}+H_{30}+H_{40}+H_{04}$ is rewritten as

$$H_{\rm res}=H_c+H_a\,, \tag{3.14}$$

where

$$H_c=-g\sum_j\ \sqrt{\Omega_i}\sqrt{\Omega_j}(u_i^2\mathscr{A}_i^\dagger-v_i^2\mathscr{A}_i)(u_j^2\mathscr{A}_j-v_j^2\mathscr{A}_j^\dagger) \tag{3.15}$$

and

$$H_a=-g\sum_j\ \sqrt{\Omega_j}(-u_iv_i)[\mathscr{N}_i(u_j^2\mathscr{A}_j-v_j^2\mathscr{A}_j^\dagger)$$

$$+(u_j^2\mathscr{A}_j^\dagger-v_j^2\mathscr{A}_j)\mathscr{N}_i]+g\sum\ u_jv_ju_iv_i\mathscr{N}_i\mathscr{N}_j\,. \tag{3.16}$$

As before, quasiparticle pair operators can be defined by $\mathscr{A}_j^\dagger=1/\sqrt{\Omega_j}\,\Sigma_m(-)^{j-m}a_{jm}^\dagger a_{j-m}^\dagger$ and $\mathscr{N}_j=\Sigma_m a_{jm}^\dagger a_{jm}$. It is worthwhile to note that the self-energy term, $-g\Sigma\Omega_j v_j^4$, is included in $H_c$. From ( 3.15) and (3.16) it is possible to see that $H_c$ is of the order of $g\Omega$, while $H_a$ contains terms of the order of $g\sqrt{\Omega}$ and $g$. Within the quasiparticle scheme it is reasonable to consider that the lowest state has few quasiparticles. If $\Omega_j$ is large, then a consistent approximation is to consider only $H_c$ and neglect the Pauli principle for quasiparticle pairs which therefore satisfy boson commutation relations $[\mathscr{A}_i,\mathscr{A}_j^\dagger]=\delta_{ij}$. In the present work we are looking for ground-state corrections due to this quasiboson approximation, resulting in the well-known correlated ground state, given by

$$E_{\rm QRPA}-E_{\rm BCS}=\frac{1}{2}\left[\sum_j\ 2E_j-\sum_n\ W_n\right], \tag{3.17}$$

where $E_j$ is the single quasiparticle energy and $W_n$ is the RPA eigenvalue given in Ref. [4].

All the results presented in this section can be connected with those obtained in a $1/N$ expansion discussed in Sec. II. In the next section, these approximations will be discussed in the framework of the symmetric two-level model.

## IV. DISCUSSION AND CONCLUSIONS

In this section we are going to present a unified picture of the various existing refinements to the BCS theory using the $1/N$ expansion of the dispersion-type energy equations. This picture can be made with the help of analytical expressions in the symmetric two-level model. Here, both number and gap equations can be solved analytically, resulting in closed expressions written in terms of $\Omega$, $g$, and $\epsilon$ for the BCS ground state and its refinements.

At first, in the soluble symmetric two-level model, $W_n$ is obtained analytically and the correlated ground-state energy is then given by

$$E_{\rm QRPA}=E_{\rm BCS}-2g\Omega+\sqrt{(g\Omega)^2-(\epsilon/2)^2}. \tag{4.1}$$

Other expressions mentioned in Secs. II and III can be calculated and are presented in Table I in terms of a dimensionless parameter $\kappa=(\epsilon/2g\Omega)^2$, which just gauges the validity of the BCS approach. For $\kappa\geqslant 1$, which would correspond to nuclei near the closed shell, no superconducting solution exists, and therefore BCS is not valid anymore. Table I can be read in two different, although not completely independent, ways. A given column contains information on the behavior of different approximation methods at a fixed order in powers of $1/N$, being the strength of the pairing interaction, measured by $\kappa$, the relevant parameter for comparison among the different models. On the other hand, by comparing different columns, we are able to assess the quality of each model regarding the amount of the correction, it exhausts at a given power of $1/N$. By just glancing over this table, we can notice some interesting connections. The first order of the $1/N$ expansion can be readily identified with the BCS + QRPA approximation. Since the $1/N$ expansion provides exact expressions for each order of the expansion, we can conclude that the QRPA correction has exhausted the first term of the $1/N$ expansion. Thus, since the BCS+QRPA represents the lowest order of a quasiboson expansion, we can expect that our $1/N$ expansion contains the various orders of the quasiboson expansion. We can also test the other refinements listed in Sec. III by simply noting whether or not it has exhausted the equivalent term of the $1/N$ expansion. Looking again at Table I we note that the projection procedure (PBCS) does not exhaust the term proportional to $g\Omega$, because it is only part of the Taylor expansion of the QRPA correction given by

$$-2g\Omega+g\Omega\sqrt{1-\kappa}\simeq-g\Omega\left[1+\frac{1}{2}\kappa\right]. \tag{4.2}$$

As a consequence it takes into account only part of the $g\Omega$ contribution.

It is worthwhile to discuss here the analytically obtained number projected ground-state energy in the symmetric two-level model given by

$$E=\frac{\displaystyle\sum_{l=0}^{l=\Omega}B_l^2\left(2\epsilon[\Omega-l]-g\Omega\left\{1+\frac{u^2}{v^2}l+\left[\frac{v^2}{u^2}-\frac{l}{\Omega}\left(\frac{u^2-v^2}{uv}\right)^2\right](\Omega-l)\right\}\right)}{\displaystyle\sum_{l=0}^{l=\Omega}B_l^2}, \tag{4.3}$$

where $B_l$ is given in Eq. (3.13). As in Ref. [15] we can transform this factor into the so-called Moivre-Laplace limit for the binomial coefficient. Introducing the new variable $l=\Omega v^2+\delta$ into this limit we get, after some calculations, the simple Gaussian form

$$B(\delta)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\delta^2}{2\sigma^2}\right), \tag{4.4}$$

where $\sigma^2=\Omega uv$. Replacing this new variable into the expression (4.4) and transforming the summation into integrals through

$$\sum_{l=0}^{l=\Omega} \xrightarrow{l=\Omega v^2+\delta} \int_{-\Omega v^2}^{\Omega u^2} d\delta \xrightarrow{\Omega\to\infty} \int_{-\infty}^{\infty} d\delta, \tag{4.5}$$

we obtain the ground-state energy presented in Table I [18].

The last two improvements shown in Table I are the Lipkin-Nogami and Nilsson prescriptions. In the last case we can notice the exclusive presence of terms of the order of 1 and $1/N$, while the Nogami correction contains terms up to the order of $1/N^2$. A Nilsson correction of the order of $1/N$ does not give the correct behavior at the order of $1/N$ since it does not go to zero in the limit of very strong pairing $(\kappa\to 0)$. On the other hand, Nogami's correction of the order of $1/N$ is identical to the exact one up to first order in powers of $\kappa$. This behavior clearly shows the advantages of the Lipkin-Nogami method in restoring particle number conservation. As a matter of fact, it has been used very much in the literature (see for example Ref. [19], where a thorough analysis over different regions on the periodic table has been undertaken).

As a final remark we would like to stress again that the several scattered refinements to the BCS nuclear model found in the literature can be presented in a unified way through the use of $1/N$ expansion. Moreover, we can see in Table I how the different improvements to the BCS model concatenate in a coherent way.

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