Description of β decay to excited quadrupole phonon states within a boson-expansion formalism

A. A. Raduta* and J. Suhonen
Department of Physics, University of Jyväskylä, POB 35, SF-40351 Jyväskylä, Finland
(Received 5 July 1995)

A microscopic Hamiltonian including realistic two-body interaction, is used to describe the single β Gamow-Teller transitions to the ground state, the first excited quadrupole state (2_1^+) and the two-quadrupole-phonon states ($0_{2\text{-ph}}{}^+$,$2_{2\text{-ph}}{}^+$) of even-even isotopes ^{118,120}Sn from the two adjacent odd-odd nuclei ^{118,120}In and ^{118,120}Sb. The higher-RPA effects are evaluated within a boson-expansion formalism. The transition amplitudes are studied as functions of the particle-particle interaction strength. The corresponding $\log ft_{\pm}$ values are also calculated and compared with experimental data as well as with the predictions of some previous calculations. The perturbative components of the states involved give important contribution to the transitions feeding the two-phonon states. Adding these, the agreement with the experimental data is improved.

PACS number(s): 23.40.Hc, 13.10.+q, 21.60.Jz, 27.60.+j

I. INTRODUCTION

In the recent past a lot of effort has been put into explaining the data about single β and two-neutrino double β ($2\nu\beta\beta$) decays. The nuclear matrix elements which are used for these processes can be used for evaluating the rate for the $0\nu\beta\beta$ decay mode whose existence (or nonexistence) might provide an answer to the question whether neutrino is a Dirac or Majorana particle.

The first major progress in the field was achieved when it was realized that the particle-particle (pp) channel of the two-body proton-neutron interaction, which is usually neglected in the standard random-phase approximation (RPA), is very important for the β^+ decay strength [1]. This idea was extended to the calculation of $2\nu\beta\beta$ decay rates by several groups with the result that the Gamow-Teller transition amplitude is cancelled when the strength of the pp interaction, g_{pp}, reaches a value lying close to unity [2–6].

Since this cancellation point lies near the breaking-down value of the RPA, a natural question arose whether these results still hold when the first-order correction to the RPA is added. The answer was given by one of us (A.A.R.) in Refs. [7,8]. Indeed, by adding the higher-RPA effects through the boson-expansion method (BEM), only a moderate suppression of the Gamow-Teller amplitude is obtained near $g_{\mathrm{pp}}=1.0$. Moreover, there are transitions which are forbidden within the RPA but are allowed when the anharmonicities are switched on. An example of this type is the $2\nu\beta\beta$ transition to the excited 2^+ state of the daughter nucleus.

A short time after another formalism was emitted with a similar scope [9,10]. This was named as the multiple commutator method (MCM). A third method was formulated by Griffiths and Vogel in connection with the double β decay to the two-phonon 0^+ state [11].

Since some of the matrix elements which are involved in double β decay are describing virtual single β transitions to excited states it is natural to try to use a similar higher-RPA approach to describe single β transitions to excited states. Indeed, there are experimental data showing that excited states of some even-even nuclei, such as Sn and Cd isotopes, can be fed both by β^- and β^+ decay [12–18] and one of the authors (J.S.) has used the MCM to explain the $\log ft_{\pm}$ values corresponding to the transitions to one- and two-phonon states [19] in these nuclei.

The present paper is devoted to a similar study by using the BEM approach which was formulated earlier in Refs. [7,8]. Moreover, here we introduce anharmonic effects not only in the transition operator but also in the states involved in a given transition. When the higher-RPA corrections are resticted to the Gamow-Teller transition operator, a direct comparison of the two theoretical formalisms, BEM and MCM, is possible.

The above-mentioned scheme is discussed according to the following plan. In Secs. II and III we describe the quasiparticle representation of the model Hamiltonian and its expansion in terms of RPA bosons. The first-order boson expansion of the Gamow-Teller transition operator and analytical expressions for transition amplitudes are also derived in Sec. III. In Sec. IV, the initial and final states involved in the β^- and β^+ transitions are treated in the first order of perturbation and the corresponding expressions for the transition amplitudes are derived. Numerical results for the β^- transitions $^{118,120}\mathrm{In}\rightarrow{}^{118,120}$Sn and β^+ transitions ^{118,120}Sb $\rightarrow{}^{118,120}$Sn are presented in Sec. V. The final conclusions are drawn in Sec. VI.

II. THE MODEL HAMILTONIAN

We assume that the β^- and β^+ processes feeding the ground state and the low-lying excited states of the nucleus (N,Z), are described by the Hamiltonian which was used in Refs. [7,8] and which will be briefly presented here. It consists of three terms: (i) a one-body term describing independent motion of the nucleons in a Wood-Saxon (WS) potential including corrections due to the Coulomb interaction (here we use the same WS potential as in Ref. [20]), (ii) the proton-proton and neutron-neutron pairing and quadrupole interaction, and (iii) the proton-neutron dipole interaction.

*Permanent address: Institute of Atomic Physics, Bucharest, POB MG6, Romania.

The two-body interaction is taken as the Brueckner G matrix $\langle (ab)J|G|(cd)J\rangle$, $J=0,1,2$, calculated from the Bonn one-boson-exchange potential by solving the Bethe-Goldstone equation. We neglect the proton-neutron pairing and quadrupole-quadrupole interaction. The justification for this approximation is given in Ref. [8].

For the single-particle states we shall use here the abbreviation

$$|\tau,nljm\rangle=|\alpha\rangle=|a,m_\alpha\rangle,\ \ |-\alpha\rangle=|a,-m_\alpha\rangle,$$

where τ takes the values p for protons and n for neutrons. The corresponding creation operator is denoted by c_α^+. In the quasiparticle representation

$$\gamma_\alpha^+=U_a c_\alpha^+ - (-)^{j_a-m_\alpha}V_a c_{-\alpha}, \tag{2.1}$$

defined by the BCS approximation, the model Hamiltonian acquires the following form:

$$H=\sum_\alpha E_a\gamma_\alpha^+\gamma_\alpha+(H_{40}+H_{31}+H_{22}+\text{H.c.}), \tag{2.2}$$

where by H_{mn} we denote the terms consisting of m quasiparticle creation and n quasiparticle annihilation operators and E_α stands for the quasiparticle energy. The terms H_{mn} can be easily written in terms of the dipole and quadrupole operators,

$$A_{JM}^+(ab)=\sum_{m_\alpha,m_\beta} C_{m_\alpha m_\beta M}^{j_a j_b J}\gamma_\alpha^+\gamma_\beta^+,$$

$$B_{JM}^+(ab)=\sum_{m_\alpha,m_\beta} C_{m_\alpha -m_\beta M}^{j_a j_b J}\gamma_{m_\alpha}^+\gamma_{m_\beta}(-)^{j_b-m_\beta},\quad J=1,2, \tag{2.3}$$

and their Hermitian conjugates. The final expressions are

$$H_{40}=\sum_{a,b,c,d,J,M} h_{40}(abcd,J)A_{JM}^+(ab)A_{J-M}^+(cd)(-)^{J-M},$$

$$H_{31}=\sum_{a,b,c,d,J,M} h_{31}(abdc,J)A_{JM}^+(ab)B_{J-M}^+(dc)(-)^{J-M},$$

$$H_{22}=\sum_{a,b,c,d,J,M} h_{22}(abcd,J)A_{JM}^+(ab)A_{JM}(cd). \tag{2.4}$$

Here the following notations have been used:

$$h_{40}(abcd,J)=\tfrac{1}{2}G(abcdJ)U_aU_bV_cV_d,$$

$$h_{31}(abdc,J)=G(abdcJ)(U_aU_bU_cV_d-V_aV_bV_cU_d),$$

$$h_{22}(abcd,J)=-\tfrac{1}{4}[G(abcdJ)(U_aU_bU_cU_d+V_aV_bV_cV_d) + 4F(abcdJ)U_aV_bU_cV_d], \tag{2.5}$$

where G and F denote the G matrix for the Bonn potential in the Baranger notation [21]. All the G-matrix elements, including those of the pairing interaction, are multiplied by common factors (not depending on the states involved) accounting for the effects ignored by restricting the single-particle space. The factors are denoted by $g_{\text{pair}}^{(p)},g_{\text{pair}}^{(n)};g_{\text{ph}}^{(1)},g_{\text{pp}}^{(1)};g_{\text{ph}}^{(2)},g_{\text{pp}}^{(2)}$ for pairing, dipole-dipole, and quadrupole-quadrupole interaction, respectively.

In the present paper we treat only the Gamow-Teller decay which in medium-heavy nuclei dominates the Fermi decay. The transition amplitudes for β^+ and β^- processes are readily obtained if we know the single-particle matrix elements of the Gamow-Teller transition operators $\hat\beta^+$ and $\hat\beta^-$. These operators can be written as

$$(\hat\beta^+)_\mu=\sum_{n,p}\langle n|\sigma_\mu|p\rangle c_n^+c_p,\quad (\hat\beta^-)_\mu=-(\hat\beta^+)_{-\mu}^+(-)^{1-\mu},$$

$$\mu=\pm1,0. \tag{2.6}$$

Here $|p\rangle(|n\rangle)$ denotes a proton (neutron) single-particle state and σ_μ is the μth spherical component of the Pauli matrix. Obviously, $\hat\beta^\pm$ are dipole operators and therefore can be expressed in terms of the quasiparticle pair and density operators $A_{1\mu}^+,B_{1\mu}^+$:

$$(\hat\beta^+)_\mu=-\sum_k[\bar\sigma_kA_{1\mu}^+(k)+\sigma_kA_{1-\mu}(k)(-)^{1+\mu}+\bar\eta_kB_{1\mu}^+(k) + \eta_kB_{1-\mu}(k)(-)^{1+\mu}]. \tag{2.7}$$

Here the summation index k symbolizes a pair of proton and neutron states which is alternatively abbreviated by (j_p,j_n), where only the total angular momentum quantum numbers are specified. Also the following notations have been used:

$$\bar\sigma_k=\frac{\hat j_p}{\hat 1}V_{j_p}U_{j_n}\langle j_p\|\sigma\|j_n\rangle,\quad \sigma_k=\frac{\hat j_p}{\hat 1}U_{j_p}V_{j_n}\langle j_p\|\sigma\|j_n\rangle,$$

$$\bar\eta_k=-\frac{\hat j_p}{\hat 1}V_{j_p}V_{j_n}\langle j_p\|\sigma\|j_n\rangle,\quad \eta_k=\frac{\hat j_p}{\hat 1}U_{j_p}U_{j_n}\langle j_p\|\sigma\|j_n\rangle,$$

$$\hat j=\sqrt{2j+1}. \tag{2.8}$$

Throughout this paper, the reduced matrix elements are defined according to the convention of Rose [22]. The corresponding expressions for the $\hat\beta^-$ operator can be obtained from (2.7) by making the following replacements: $\bar\sigma_k\to-\sigma_k,\sigma_k\to-\bar\sigma_k,\bar\eta_k\to-\eta_k,\eta_k\to-\bar\eta_k$.

The model Hamiltonian is first treated by the RPA method. Within this approximation one defines the operators

$$\Gamma_{1\mu}^+(l)=\sum_{k=(j_p,j_n)}[X_l(k)A_{1\mu}^+(k)+Y_l(k)A_{1-\mu}(k)(-)^{1-\mu}], \tag{2.9}$$

$$\Gamma_{2\mu}^+(l)=\sum_{k=(j_p,j_p'),(j_n,j_n')}[R_l(k)A_{2\mu}^+(k) + S_l(k)A_{2-\mu}(k)(-)^\mu], \tag{2.10}$$

so that they have a boson character

$$[\Gamma_{\lambda\mu}(l),\Gamma^{+}_{\lambda'\mu'}(l')]=\delta_{\lambda\lambda'}\delta_{ll'}\delta_{\mu\mu'},\quad \lambda,\lambda'=1,2, \tag{2.11}$$

and they describe a harmonic approximation for H:

$$[H,\Gamma^{+}_{\lambda\mu}(l)]=\omega_{\lambda}(l)\Gamma^{+}_{\lambda\mu}(l). \tag{2.12}$$

The RPA excitation energies corresponding to the 2^{λ}-pole mode ($\lambda=1,2$) are denoted by $\omega_{\lambda}(l)$. The argument (l) is labeling the positive solutions of Eq. (2.12) which are ordered as follows:

$$\omega_{\lambda}(1)<\omega_{\lambda}(2)<\cdots<\omega_{\lambda}(N_{\lambda});\quad \lambda=1,2. \tag{2.13}$$

Equations (2.11) provide the following normalization for the boson amplitudes:

$$\sum_{k}[(X_l(k))^2-(Y_l(k))^2]=1, \tag{2.14}$$

$$2\sum_{k}[(R_l(k))^2-(S_l(k))^2]=1. \tag{2.15}$$

Once the RPA equations are solved, i.e., the excitation energies $\omega_{\lambda}(l)$ and the amplitudes X,Y,R,S are known, one can define the RPA boson space. This is generated by acting with the boson monomials $[\Gamma^{+}_{2\mu}(k)]^n[\Gamma^{+}_{1\mu}(k')]^m$ on the vacuum state $|0\rangle$, which satisfies the equations

$$\Gamma_{\lambda\mu}(l)|0\rangle=0;\quad \lambda=1,2;\quad l=1,2,\ldots,N_{\lambda}. \tag{2.16}$$

For example, the one-, two-, and three-phonon states which are needed in the present paper are defined by

$$\begin{aligned}
&|\lambda_m M\rangle=\Gamma^{+}_{\lambda M}(m)|0\rangle,\ \ \lambda=1,2;\ \ m=1,2,\ldots,N_{\lambda},\\
&|1_m 2_n;1M\rangle=[\Gamma^{+}_{1}(m)\Gamma^{+}_{2}(n)]_{1M}|0\rangle,\\
&\qquad m=1,2,..,N_1;\quad n=1,2,..,N_2,\\
&|2_m 2_n;JM\rangle=(1+\delta_{mn})^{-\frac{1}{2}}[\Gamma^{+}_{2}(m)\Gamma^{+}_{2}(n)]_{JM}|0\rangle,\\
&\qquad m,n=1,2,\ldots,N_2;\quad J=0,2,4,\\
&|1_{k_1}(2_{k_2}2_{k_3})_l;1M\rangle=N_{122}(k_1k_2k_3)[\Gamma^{+}_{1}(k_1)\\
&\qquad\times[\Gamma^{+}_{2}(k_2)\Gamma^{+}_{2}(k_3)]_l]_{1M}|0\rangle,\quad l=0,2,\\
&|2_{k_1}2_{k_2}2_{k_3};0\rangle=N_{30}(k_1k_2k_3)\\
&\qquad\times[\Gamma^{+}_{2}(k_1)\Gamma^{+}_{2}(k_2)\Gamma^{+}_{2}(k_3)]_0|0\rangle,\\
&|2_1 2_1 2_1;2M\rangle=N_{32}[\Gamma^{+}_{2}(1)[\Gamma^{+}_{2}(1)\Gamma^{+}_{2}(1)]_2]_{2M}|0\rangle,
\end{aligned} \tag{2.17}$$

where N_{122},N_{30}, and N_{32} denote the normalization factors. The expressions (2.9) and (2.10) can be easily inverted. In this way the quasiparticle 2^{λ}-pole operators $A^{+}_{\lambda\mu},A_{\lambda\mu}$ can be expressed in terms of the RPA bosons:

$$A^{+}_{1\mu}(j_p,j_n)=\sum_{k=1}^{N_1}[X_k(j_pj_n)\Gamma^{+}_{1\mu}(k)-Y_k(j_pj_n)\Gamma_{1-\mu}(k)(-)^{1-\mu}], \tag{2.18}$$

$$A^{+}_{2\mu}(j_\tau,j'_\tau)=2\sum_{k=1}^{N_2}[R_k(j_\tau j'_\tau)\Gamma^{+}_{2\mu}(k)-S_k(j_\tau j'_\tau)\Gamma_{2-\mu}(k)(-)^{\mu}]. \tag{2.19}$$

The expressions for $A_{1\mu}(j_p,j_n)$ and $A_{2\mu}(j_\tau,j_{\tau'})$ can be obtained from (2.18) and (2.19) by Hermitian conjugation operation, respectively. In order to satisfy the equations (2.11) one assumes quasiboson commutation relations for the operators $A_{\lambda\mu}$ and $A^{+}_{\lambda'\mu'}$. These yield the following properties:

$$\begin{aligned}
&\langle 0|[\Gamma_{1\mu}(k),A^{+}_{1\mu}(j_pj_n)]|0\rangle=X_k(j_pj_n),\\
&\langle 0|[\Gamma_{2\mu}(k),A^{+}_{2\mu}(j_\tau j'_\tau)]|0\rangle=2R_k(j_\tau j'_\tau),\\
&\langle 0|[\Gamma^{+}_{1-\mu}(k)(-)^{1-\mu},A^{+}_{1\mu}(j_pj_n)]|0\rangle=Y_k(j_pj_n),\\
&\langle 0|[\Gamma^{+}_{2-\mu}(k)(-)^{\mu},A^{+}_{2\mu}(j_\tau j'_\tau)]|0\rangle=2S_k(j_\tau j'_\tau),
\end{aligned} \tag{2.20}$$

$$\langle 0|B^{+}_{1\mu}(j_pj_n)|0\rangle=\langle 0|B^{+}_{2\mu}(j_\tau j'_\tau)|0\rangle=0;\quad \tau=p,n. \tag{2.21}$$

It is worth noting that due to Eqs. (2.18) and (2.19), one may say that the RPA approach provides a linear boson representation for the operators A^{+} and A. Consequently, the reduced matrix elements $\langle 1_k\|\hat{\beta}^{\pm}\|0\rangle$ can be expressed as linear combinations of amplitudes X and Y. Equations (2.18) and (2.19), with the amplitudes given by (2.20), will be generalized in the next section in order to include first-order corrections to the RPA operators.

III. FIRST-ORDER HIGHER-RPA CORRECTIONS FOR THE TRANSITION AMPLITUDES

The first improvement of the standard RPA approach was achieved by Cha in Ref. [1], where it was shown that the β^{+} transition rate is very sensitive to the variation of the strength of the two-body interaction in the particle-particle channel. A few years later this idea was used for the $2\nu\beta\beta$ decay [2–6] with the result that the Gamow-Teller amplitude for the transition $0^{+}_{i}\rightarrow 0^{+}_{f}$ is totally suppressed for a strength $g_{pp}\approx 1$. One of the authors (A.A.R.) showed [7,8] that this result does not hold any longer if one includes the higher-RPA corrections to the transition operator: the zero point has shifted towards higher values of g_{pp}. Indeed, due to the added corrections, the $\hat{\beta}^{+}$ operator has nonvanishing matrix elements not only between states like $|0\rangle_i,|1_k\rangle$ and $|1_k\rangle,|0\rangle_f$ but it may also connect either of the vacua $|0\rangle_i,|0\rangle_f$ to the higher boson states of the intermediate odd-odd nucleus. The final result was that for the $2\nu\beta\beta$ decay of ^{82}Se the Gamow-Teller amplitude is only moderately supressed for $g_{pp}=1$. Moreover, within the higher-RPA approach, the $2\nu\beta\beta$ mode $0^{+}_{i}\rightarrow 2^{+}_{f}$ is allowed although such a transition is forbidden on the RPA level.

It was the first time when it was shown that in order to account for the higher-RPA effects in the $2\nu\beta\beta$ process it is necessary to consider not only the proton-neutron bosons $\Gamma^{+}_{1\mu}$, but also the charge conserving bosons $\Gamma^{+}_{2\mu}(k)$. In this way one obtains the first-order boson expansion of the double Gamow-Teller transition operator. The results, concerning the transition to the excited 2^{+} state, were confirmed two years later in Refs. [9,10], where the higher-RPA approach is called the multiple commutator method (MCM). Here we shall show that the two methods, BEM and MCM, are identical at least when they are applied to the transitions $0^{+}_{i}\rightarrow 0^{+}_{f}, 0^{+}_{i}\rightarrow 2^{+}_{f}$. However, differences appear whenever the final nucleus is left in a two-phonon state.

It is worth noting that the double β decay may be viewed as a process taking place through two successive single β^{-} decays. Considering the amplitude of the second decay as the conjugate one of a β^{+} transition $|0^{+}_{f}\rangle\rightarrow|1_{k}\rangle$, the Gamow-Teller amplitude $|0^{+}_{i}\rangle\rightarrow|0^{+}_{f}\rangle$ contains a product of two amplitudes describing two virtual processes by which the intermediate odd-odd nucleus is fed by β^{-} and β^{+} decays of the mother and daughter nuclei, respectively. Therefore the results, provided by single β decay, can be used for improving the quality of the double β calculations.

In the present paper we shall use the BEM [7,8] to describe the $\beta^{\pm}$ transitions from the 1^{+} ground state of the odd-odd nucleus to the states $|0\rangle, |2_1 M\rangle, |2_1 2_1; JM\rangle$ $(J=0,2)$ of the even-even final nucleus. For the sake of completeness we shall briefly sketch the BEM, although it has been presented in detail in Ref. [8]. The dipole operators $A^{+}_{1\mu}$ and $B^{+}_{1\mu}$ are written as polynomials of the RPA bosons so that the mutual commutation relations are consistently preserved by the boson mapping. It is worth stressing that the boson representation for the dipole (pn) operators was possible only by considering the charge-conserving bosons together with the charge nonconserving ones. In Ref. [8] only the terms producing a nonvanishing contribution to double β transitions $0^{+}_{i}\rightarrow 0^{+}_{f}, 0^{+}_{i}\rightarrow 2^{+}_{f}$ were retained.

Since here we consider also the transition to the two-phonon states, some additional terms will appear. The first-order boson expansion of the operators $A^{+}_{1\mu}$ and $B^{+}_{1\mu}$ is

$$\begin{aligned}A^{+}_{1\mu}(j_p j_n)=&\sum_{k_1}[\mathscr{A}^{(10)}_{k_1}(j_p j_n)\Gamma^{+}_{1\mu}(k_1)+\mathscr{A}^{(01)}_{k_1}(j_p j_n)\Gamma_{1-\mu}(k_1)(-)^{1-\mu}]\\&+\sum_{k_1,k_2\leqslant k_3}\mathscr{A}^{(30)l}_{k_3k_2k_1}(j_p j_n)[[\Gamma^{+}_{2}(k_3)\Gamma^{+}_{2}(k_2)]_l\Gamma^{+}_{1}(k_1)]_{1\mu}+\sum_{k_1,k_2\leqslant k_3}\mathscr{A}^{(03)l}_{k_3k_2k_1}(j_p j_n)[[\Gamma_{2}(k_3)\Gamma_{2}(k_2)]_l\Gamma_{1}(k_1)]_{1\mu}\\&+\sum_{k_1,k_2\leqslant k_3}\mathscr{A}^{1(\bar{2}\bar{2})l}_{k_1k_3k_2}(j_p j_n)[\Gamma^{+}_{1}(k_1)[\Gamma_{2}(k_3)\Gamma_{2}(k_2)]_l]_{1\mu}+\sum_{k_1,k_2\leqslant k_3}\mathscr{A}^{(22)l\bar{1}}_{k_3k_2k_1}(j_p j_n)[[\Gamma^{+}_{2}(k_3)\Gamma^{+}_{2}(k_2)]_l\Gamma_{1}(k_1)]_{1\mu}\\&+\sum_{k_1,k_2,k_3}\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}(j_p j_n)[\Gamma^{+}_{1}(k_1)[\Gamma^{+}_{2}(k_2)\Gamma_{2}(k_3)]_l]_{1\mu}+\sum_{k_1,k_2,k_3}\mathscr{A}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}(j_p j_n)[[\Gamma^{+}_{2}(k_3)\Gamma_{2}(k_2)]_l\Gamma_{1}(k_1)]_{1\mu},\end{aligned}\tag{3.1}$$

$$\begin{aligned}B^{+}_{1\mu}(j_p j_n)=\sum_{k_1,k_2}\{&\mathbf{B}^{(20)}_{k_1k_2}(j_p j_n)[\Gamma^{+}_{1}(k_1)\Gamma^{+}_{2}(k_2)]_{1\mu}+\mathbf{B}^{(02)}_{k_1k_2}(j_p j_n)[\Gamma_{1}(k_1)\Gamma_{2}(k_2)]_{1\mu}\\&+\mathbf{B}^{(11;12)}_{k_1k_2}(j_p j_n)[\Gamma^{+}_{1}(k_1)\Gamma_{2}(k_2)]_{1\mu}+\mathbf{B}^{(11;21)}_{k_1k_2}(j_p j_n)[\Gamma^{+}_{2}(k_2)\Gamma_{1}(k_1)]_{1\mu}\}.\end{aligned}\tag{3.2}$$

The coefficients $\mathscr{A}$ and $\mathbf{B}$ are to be determined according to the algorithm which was already explained above. For illustration, let us consider the coefficient $\mathbf{B}^{(20)}_{k_1k_2}(j_p j_n)$. Commuting the equation (3.2) with $\Gamma_{2\mu_2}$ and then with $\Gamma_{1\mu_1}$ one obtains an equation containing in the right-hand side (rhs) the term $\mathbf{B}^{(20)}_{k_1k_2}(j_p j_n)$ plus some boson operators. We get rid of the latter terms by taking the vacuum expectation value. In this way one obtains

$$\mathbf{B}^{(20)}_{k_1k_2}(j_p j_n)=\sum_{\mu_1,\mu_2}C^{121}_{\mu_1\mu_2\mu}\langle 0|[\Gamma_{1\mu_1}(k_1),[\Gamma_{2\mu_2}(k_2),B^{+}_{1\mu}(j_p j_n)]]|0\rangle.\tag{3.3}$$

The double commutator involved in (3.3) is calculated without making any approximation. However, the vacuum expectation value of the exact result takes care of the RPA equality (2.21). Most of the coefficients $\mathscr{A}$ and $\mathbf{B}$ were analytically given in Ref. [8]. The new terms, which are needed here, are $\mathscr{A}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}, \mathscr{A}^{(22)l\bar{1}}_{k_3k_2k_1}, \mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}$, and $\mathscr{A}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}$. The expressions of these new coefficients are given in Appendix A.

Applying the Hermitian conjugation operation to the equations (3.1)and (3.2) one obtains the boson representation for the operators $A_{1-\mu}$ and $B_{1-\mu}$. Using the boson representation of these operators in connection with the operator $\hat{\beta}^{+}$, given by (2.7), one obtains

$$
\begin{aligned}
\hat{\beta}^{+}_{\mu}=&\sum_{k}\ [\mathscr{B}^{(10)}_{k}\Gamma^{+}_{1\mu}(k)+\mathscr{B}^{(01)}_{k}\Gamma_{1-\mu}(k)(-)^{1-\mu}]+\sum_{k_1,k_2}\ \{\mathscr{B}^{(20)}_{k_1k_2}[\Gamma^{+}_{1}(k_1)\Gamma^{+}_{2}(k_2)]_{1\mu}+\mathscr{B}^{(02)}_{k_1k_2}[\Gamma_{1}(k_1)\Gamma_{2}(k_2)]_{1\mu}\\
&+\mathscr{B}^{(12)}_{k_1k_2}[\Gamma^{+}_{1}(k_1)\Gamma_{2}(k_2)]_{1\mu}+\mathscr{B}^{(21)}_{k_2k_1}[\Gamma^{+}_{2}(k_2)\Gamma_{1}(k_1)]_{1\mu}\}+\sum_{l,k_1,k_2\leqslant k_3}\ \{\mathscr{B}^{1(22)l}_{k_1k_2k_3}[\Gamma^{+}_{1}(k_1)[\Gamma^{+}_{2}(k_2)\Gamma^{+}_{2}(k_3)]_{l}]_{1\mu}\\
&+\mathscr{B}^{\bar{1}(\bar{2}\bar{2})l}_{k_3k_2k_1}[[\Gamma_{2}(k_3)\Gamma_{2}(k_2)]_{l}\Gamma_{1}(k_1)]_{1\mu}\}+\sum_{l,k_1,k_2\leqslant k_3}\ \{\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}[\Gamma^{+}_{1}(k_1)[\Gamma_{2}(k_2)\Gamma_{2}(k_3)]_{l}]_{1\mu}\\
&+\mathscr{B}^{(22)l\bar{1}}_{k_3k_2k_1}[[\Gamma^{+}_{2}(k_3)\Gamma^{+}_{2}(k_2)]_{l}\Gamma_{1}(k_1)]_{1\mu}\}+\sum_{l,k_1,k_2,k_3}\ \{\mathscr{B}^{1(2\bar{2})l}_{k_1k_2k_3}[\Gamma^{+}_{1}(k_1)[\Gamma^{+}_{2}(k_2)\Gamma_{2}(k_3)]_{l}]_{1\mu}\\
&+\mathscr{B}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}[[\Gamma^{+}_{2}(k_3)\Gamma_{2}(k_2)]_{l}\Gamma_{1}(k_1)]_{1\mu}\}.
\end{aligned}
\tag{3.4}
$$

The coefficients $\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3},\mathscr{B}^{(22)l\bar{1}}_{k_3k_2k_1},\mathscr{B}^{1(2\bar{2})l}_{k_1k_2k_3},\mathscr{B}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}$ are given explicitly in Appendix B. The remaining coefficients have been calculated in Ref. [8].

The boson representation of the $\hat{\beta}^{-}$ operator is obtained from (3.4) by making use of Eq. (2.6). From Eq. (3.4) it is manifest that the matrix elements describing the β^{-} transitions are readily obtained. For example, the β^{-} transition $1^{+}_{1}\rightarrow 2^{+}_{1}$ is described by a reduced matrix element, which in Rose's convention [22] reads

$$
\langle 1^{+}_{1}\|\hat{\beta}^{+}\|2^{+}_{1}\rangle=\mathscr{B}^{(12)}_{k_1k_2}, \tag{3.5}
$$

where, according to our procedure, the coefficient $\mathscr{B}^{(12)}_{k_1k_2}$ is given by

$$
\begin{aligned}
\mathscr{B}^{(12)}_{k_1k_2}=&\sum_{\mu_1,\mu_2}C^{121}_{\mu_1\mu_2\mu}\langle 0|[[\Gamma_{1\mu_1}(k_1),\hat{\beta}^{+}_{\mu}],\\
&\Gamma^{+}_{2-\mu_2}(k_2)(-)^{\mu_2}]|0\rangle.
\end{aligned}
\tag{3.6}
$$

At this point we would like to mention that the multiple commutator method states that the reduced matrix element (m.e.) $\langle 1^{+}_{1}\|\hat{\beta}^{+}\|2^{+}_{1}\rangle$ is equal to the rhs of Eq. (3.6). It is clear now that the two procedures are identical, at least for the single β transitions to the ground state and excited one-phonon states.

One should mention that in the relation (3.6), the order in which the two commutators are performed is of no importance. Indeed, one can easily check that the same result is obtained when the commutation with $\Gamma^{+}_{2-\mu_2}(-)^{\mu_2}$ is made first and the result is then commuted with $\Gamma_{1\mu_1}$. Contrary to this, the commutation order is important when one calculates the coefficients of the monomials of third degree in bosons. In what follows we explain how different orderings of the commutators yield different boson representations for the operators under consideration. For example, within the BEM the factor $\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}$ has the expression

$$
\begin{aligned}
\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}=&\sum_{m_1,m_2,m'_2,M}C^{22l}_{m_2m'_2M}C^{l11}_{Mm_1\mu}\langle 0|\{[[\Gamma_{1m_1}(k_1),\beta^{+}_{\mu}],\\
&\times\Gamma^{+}_{2-m_2}(k_2)],\Gamma^{+}_{2-m'_2}(k_3)(-)^{M}\}|0\rangle.
\end{aligned}
\tag{3.7}
$$

One can easily check that this ordering corresponds to the Belyaev-Zelevinski (BZ) boson expansion [23] of the operators $A^{+}_{1\mu}(j_p,j_n)$ and $B^{+}_{1\mu}(j_p,j_n)$. Indeed, within the BZ expansion formalism the corrections to the operator $A^{+}_{1\mu}$ can be expressed in terms of the quasiboson operators $\mathring{A}^{+}_{2},\mathring{A}^{+}_{1},\mathring{A}_{2},\mathring{A}_{1}$ [these operators are defined by (2.3) but satisfy quasiboson commutation relations] as linear combinations of the operator products $\mathring{A}^{+}_{1}(j_p,j_n)\mathring{A}^{+}_{2}(j'_n,j''_n)\mathring{A}_{2}(j^{(1)}_{n},j^{(2)}_{n})$ and $\mathring{A}^{+}_{1}(j_p,j_n)\mathring{A}^{+}_{2}(j'_p,j''_p)\mathring{A}_{2}(j^{(1)}_{p},j^{(2)}_{p})$. The operators $B^{+}_{1\mu}(j_p,j_n)$ are linear combinations of terms like $\mathring{A}^{+}_{1}(j_p,j'_n)\mathring{A}_{2}(j'_n,j_n)$ and $\mathring{A}^{+}_{2}(j_p,j'_p)\mathring{A}_{1}(j'_p,j_n)$. Concluding, the boson representation we use here has the property that it preserves, to first order, the mutual commutation relations of the expanded operators.

Contrary to the BEM, the MCM approach commutes first $\hat{\beta}^{+}$ with Γ^{+}_{2}. The result is then commuted with Γ^{+}_{2} and the last commutation operation concerns the boson Γ_{1}. While in the BEM the result for $\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}$ is a superposition of the RPA amplitude products $X_{k_1}S_{k_2}R_{k_3}$, the MCM produces two types of term: $X_{k_1}R_{k_2}R_{k_3}$ and $Y_{k_1}S_{k_2}S_{k_3}$.

Obviously the two methods account for different anharmonic effects of the pair and density quasiparticle dipole operators $A^{+}_{1\mu}$ and $B^{+}_{1\mu}$. It is worth mentioning that the MCM yields the same result for $\mathscr{B}^{1(\bar{2}\bar{2})l}_{k_1k_2k_3}$ as that obtained in Ref. [11] by using a different method. For the sake of completeness we say few words about this method. To this purpose let us consider the matrix element

$$
\langle 1_1M|A_{1\mu}(j_pj_n)|2_12_1;JM\rangle. \tag{3.8}
$$

The authors of Ref. [11] claim that the leading term for this m.e. is of the type

$$
\begin{aligned}
&X_1(j'_pj'_n)R_1(j_{p_1}j_{p_2})R_1(j_{n_1}j_{n_2})\\
&\quad\times\langle 0|A_1(j'_pj'_n)A_1(j_pj_n)A^{+}_{2}(j_{p_1}j_{p_2})A^{+}_{2}(j_{n_1}j_{n_2})|0\rangle.
\end{aligned}
\tag{3.9}
$$

The first two operators involved in the above m.e. are recoupled into a product of charge conserving operators: $A_l(j'_p,j_p)A_l(j'_n,j_n)$. Furthermore, the quasiboson approximation is used for the proton- and neutron-type operators. Here two remarks are necessary. Since in the BEM one first commutes $A_{1\mu}$ with Γ_{1M} one misses the contribution coming from the operator A_{1M} since this commutes with $A_{1\mu}(j_p,j_n)$. The second remark concerns the fact that in Ref. [11] there is no place where the exact commutation relations are used. Due to this fact there is no doubt that important anharmonicities are lost.

In conclusion, concerning the decays to the two phonon states, the methods used by Griffiths and Vogel [11] and Suhonen and Civitarese [9,10] provide similar results although they are conceptually different from each other. Moreover, this common result is different from that obtained through the BEM which is the basic procedure adopted in the present paper. A deeper analysis of the advantages and drawbacks of each of the three methods will be presented in a subsequent publication.

Once the boson representation for the operators $\hat{\beta}^{\pm}$ is determined, the reduced m.e. describing the single β decays are readily obtained. The results are

$$\langle 1_1^+\|\hat{\beta}^+\|0^+\rangle_0=\mathscr{B}_1^{(10)},\quad \langle 1_1^+\|\hat{\beta}^+\|2_1^+\rangle_0=\mathscr{B}_{11}^{(12)},$$

$$\langle 1_1^+\|\hat{\beta}^+\|2_12_1;0^+\rangle_0=\sqrt{2}\mathscr{B}_{111}^{1(\overline{2}\overline{2})0},$$

$$\langle 1_1^+\|\hat{\beta}^+\|2_12_1;2^+\rangle_0=\sqrt{2}\mathscr{B}_{111}^{1(\overline{2}\overline{2})2}. \tag{3.10}$$

In the above equations we have specified also the parity of the initial and the final states. Also we introduce a lower index "0" in order to stress that here the higher-RPA corrections are considered only for the transition operator. Therefore, the states are those defined by the RPA approach. The reduced m.e. for $\hat{\beta}^-$ are obtainable from the corresponding m.e. of $\hat{\beta}^+$ by making the following interchanges: $\bar{\sigma}_k\rightleftharpoons-\sigma_k,\bar{\eta}_k\rightleftharpoons-\eta_k$, leading to

$$\langle 1_1^+\|\hat{\beta}^-\|0^+\rangle_0=-\mathscr{B}_1^{(01)},\quad \langle 1_1^+\|\hat{\beta}^-\|2_1^+\rangle_0=-\mathscr{B}_{11}^{(21)},$$

$$\langle 1_1^+\|\hat{\beta}^-\|2_12_1;0^+\rangle_0=-\sqrt{2}\mathscr{B}_{111}^{(22)0\overline{1}},$$

$$\langle 1_1^+\|\hat{\beta}^-\|2_12_1;2^+\rangle_0=-\sqrt{2}\mathscr{B}_{111}^{(22)2\overline{1}}. \tag{3.11}$$

The anharmonic effects in the states connected by the operators $\hat{\beta}^{\pm}$, will be treated perturbatively in the next section. Aiming at this goal, here we calculate the first-order boson expansion of the model Hamiltonian. The cubic terms in bosons are determined by the boson representations of the 2^{λ}-pole operators $B_{\lambda\mu}^+$ and $B_{\lambda\mu}$, with $\lambda=1,2$.

Following the procedure we outlined before, one obtains the following expressions for the quadrupole operators:

$$B_{2\mu}^+(j_1j_2)=\sum_{k_1,k_2}\{D_{k_1k_2}^{(20)}(j_1j_2)[\Gamma_2^+(k_1)\Gamma_2^+(k_2)]_{2\mu}+D_{k_1k_2}^{(11)}(j_1j_2)[\Gamma_2^+(k_1)\Gamma_2(k_2)]_{2\mu}+D_{k_1k_2}^{(02)}(j_1j_2)[\Gamma_2(k_2)\Gamma_2(k_1)]_{2\mu}\}, \tag{3.12}$$

where the coefficients $D^{(mn)}$ are those listed in Appendix C. Replacing the operators $A_{\lambda\mu}^+$ and $A_{\lambda\mu}$ by their RPA expansions (2.18) and (2.19) and the operators $B_{\lambda\mu}^+$ and $B_{\lambda\mu}$ by their first-order boson expansions (3.2) and (3.12) one obtains

$$H=\sum_{\lambda,k,\mu}\omega_\lambda(k)\Gamma_{\lambda\mu}^+(k)\Gamma_{\lambda\mu}(k)+\sum_{k_1,k_2,k_3}\mathscr{H}_{k_1k_2k_3}^{(30)}\{[\Gamma_2^+(k_1)\Gamma_2^+(k_2)\Gamma_2^+(k_3)]_0+\text{H.c.}\}+\sum_{k_1,k_2,k_3}\mathscr{H}_{k_1k_2k_3}^{(21)}\{[\Gamma_2^+(k_1)\Gamma_2(k_2)\Gamma_2(k_3)]_0+\text{H.c.}\}+\sum_{k_1,k'_1,k_2}\mathscr{H}_{k_1k'_1k_2}^{(pn)}\{[\Gamma_1^+(k_1)\Gamma_2^+(k_2)\Gamma_2(k'_1)]_0+\text{ H.c.}\}. \tag{3.13}$$

In the proton neutron (pn) Hamiltonian we did not consider terms of the type $\Gamma_1^+\Gamma_1^+\Gamma_2$ and $\Gamma_2^+\Gamma_1\Gamma_1$ since they perturb the states of the (N,Z) nucleus by connecting them with the states having the main components in the nuclei $(N-2,Z+2)$ and $(N+2,Z-2)$, respectively.

IV. PERTURBATION TREATMENT OF THE STATES INVOLVED IN THE β TRANSITIONS

In Refs. [7,8] the higher-RPA contribution to the double Gamow-Teller (GT) transition rate of the $2\nu\beta\beta$ process was evaluated by considering the first-order corrections to the RPA transition operator. The reason was that in case the states were perturbed as well, the completeness property for the states describing the odd-odd intermediate nucleus would have been lost. Due to this reason a full perturbation treatment requires special caution.

For single β transition we are, however, not confronted with such a problem and therefore the perturbation of the initial and final states is possible. Next we present the results for the perturbed states. Hereafter the first-order perturbed states will be denoted by $|\ \rangle'$. Also, to keep the notations simple, the "M" quantum number will not be written explicitly. The final result for the perturbed states reads

$$
\begin{aligned}
|1_1^+\rangle' &= \mathcal{N}_1\left(|1_1^+\rangle + \sum_{k_1,k_2} C_{12;1}(k_1,k_2)|(1_{k_1}2_{k_2});1^+\rangle + \sum_{k_1\leq k_2\leq k_3} C_{4,1}(k_1,k_2,k_3)|(2_{k_1}2_{k_2}2_{k_3})_0 1_1;1^+\rangle \right),\\
|2_1^+\rangle' &= \mathcal{N}_2\left(|2_1^+\rangle + \sum_{k_2\leq k_3} C_{2,1}(k_2,k_3)|(2_{k_2}2_{k_3});2^+\rangle \right),\\
|0^+\rangle' &= \mathcal{N}_0\left(|0^+\rangle + \sum_{k_1\leq k_2\leq k_3} C_{3,0}(k_1,k_2,k_3)|(2_{k_1}2_{k_2}2_{k_3});0^+\rangle \right),\\
|2_1 2_1;0^+\rangle' &= \mathcal{N}_{20}\left(|2_1 2_1;0^+\rangle + \sum_{k_2\leq k_3} C_{3,2}(k_2,k_3)|(2_1 2_{k_2}2_{k_3});0^+\rangle \right),\\
|2_1 2_1;2^+\rangle' &= C^{(2)}|(2_1 2_1);2^+\rangle + C^{(1)}|2_2^+\rangle.
\end{aligned}
\tag{4.1}
$$

The results for the perturbation coefficients C are given in Appendix D.

The two-phonon state $2^+_{2\text{-ph}}(\equiv|2_1 2_1;2^+\rangle)$ deserves special attention. Indeed, for the two isotopes of tin, ^{118,120}Sn, which are considered in our numerical application, this state lies close in energy to the second RPA state $|2_2^+\rangle$. This makes the perturbation treatment non-applicable. Due to this fact the influence of anharmonicities on the state $|2^+_{2-\mathrm{ph}}\rangle$ is obtained by diagonalizing the boson Hamiltonian in the space of the two quasidegenerate states. The coefficients $C^{(2)}$ and $C^{(1)}$, involved in (4.1), are therefore provided by a diagonalization procedure. Taking into account the first-order boson expansion for the $\hat{\beta}^{\pm}$ operators and the expressions (4.1) of the perturbed states, one easily calculates the corresponding reduced m.e. The final results are listed below:

$$
\begin{aligned}
'\langle 1_1^+\|\hat{\beta}^+\|2_1^+\rangle' &= \mathcal{N}_2\mathcal{N}_1\Bigg\{ \mathcal{B}_{11}^{(12)} + \sum_{k_1,k_2} C_{12,1}(k_1,k_2)\left[\delta_{k_2,1}\mathcal{B}_{k_1}^{(10)} + \mathcal{B}_{k_1k_21}^{1(2\bar{2})l}\left(\frac{1}{\sqrt{5}}\delta_{l0} + \delta_{l2}\sqrt{15}W(2211;21)\right)\right]\\
&\quad + \sum_{k_2\leq k_3}(1+\delta_{k_2k_3})^{\frac{1}{2}}\mathcal{B}_{1k_2k_3}^{1(\bar{2}\bar{2})2}C_{2,1}(k_2,k_3) + \frac{1}{\sqrt{5}}\sum_{k_2\leq k_3}\mathcal{B}_{1k_2k_3}^{1(22)2}C_{4,1}(1,k_2,k_3)N_{30}^{-1}(1,k_2,k_3)\Bigg\},\\
'\langle 1_1^+\|\hat{\beta}^+\|0^+\rangle' &= \mathcal{N}_1\mathcal{N}_0\left[\mathcal{B}_1^{(10)} + \sum_{k_1,k_2}\mathcal{B}_{k_1k_2}^{(20)}C_{12;1}(k_1,k_2) + \mathcal{B}_1^{(10)}\sum_{k_1\leq k_2\leq k_3} C_{4,1}(k_1,k_2,k_3)C_{3,0}(k_1,k_2,k_3)\right],\\
'\langle 1_1^+\|\hat{\beta}^+\|2_1 2_1;0^+\rangle' &= \mathcal{N}_1\mathcal{N}_{20}\left[\sqrt{2}\mathcal{B}_{111}^{1(\bar{2}\bar{2})0} + \sqrt{\frac{2}{5}}\sum_{k_1} C_{12;1}(k_1,1)\mathcal{B}_{k_1,1}^{(12)} + \sum_{k_2\leq k_3} C_{3,2}(k_2,k_3)C_{4,1}(1,k_2,k_3)\mathcal{B}_1^{(10)}\right],\\
'\langle 1_1^+\|\hat{\beta}^+\|2_1 2_1;2^+\rangle' &= \mathcal{N}_1\Bigg[C^{(1)}\left(\mathcal{B}_{12}^{(12)} + \sum_k \mathcal{B}_k^{(10)}C_{12;1}(k,2)\right) + C^{(2)}\Bigg(\sqrt{2}\mathcal{B}_{111}^{1(\bar{2}\bar{2})2} + \sqrt{30}W(1212;12)\sum_k \mathcal{B}_{k1}^{(12)}C_{12;1}(k,1)\\
&\quad + \frac{1}{\sqrt{10}}\sum_k \mathcal{B}_{1k}^{(20)}N_{30}^{-1}(1,1,k)C_{4,1}(1,1,k)\Bigg)\Bigg].
\end{aligned}
\tag{4.2}
$$

The corresponding m.e. for the β^- operator are obtained by the same replacements as in the unperturbed case. The normalization factors $N_3(k_1,k_2,k_3)$ are defined in Appendix D, in Eq. (D4). In evaluating these reduced m.e. we ignored terms which are of third degree in the perturbation of the states and expansion coefficients for the $\hat{\beta}^{\pm}$ operator, Indeed, inclusion of them would require second-order boson expansion for the transition operator and second-order perturbation for the nuclear states. The amplitude for the Gamow-Teller single $\beta^{\pm}$ transition is equal to the reduced m.e. of the $\hat{\beta}^{\mp}$ operator.

$$
M^{\mp}_{\mathrm{GT}}(J_i\to J_f) = \langle J_i^+\|\hat{\beta}^{\pm}\|J_f^+\rangle. \tag{4.3}
$$

Comparison with experimental data is made in terms of log $ft_{\pm}$ where $ft_{\pm}$ is defined by

$$
ft_{\pm} = \frac{6050}{g_A^2(M^{\pm}_{\mathrm{GT}})^2}, \tag{4.4}
$$

with $g_A=1$. Once again it is to be stressed that the notation of Rose [22] for the reduced matrix element has been used.

V. NUMERICAL RESULTS

The formalism we developed in the previous sections is applied to ^{118} Sn and ^{120}Sn. The single-particle basis, both for protons and neutrons, consists of the major shells $3\hbar\omega$

TABLE I. The $\log ft_-$ values for the β^- decay ^{118}In$(1^+)\rightarrow^{118}$Sn(J_f^π). First column: predictions using unperturbed states and expansion for the $\hat{\beta}^+$ operator. Second column: results produced by the MCM. Third column: both the transition operator and the states are perturbed. Fourth column: experimental data (taken from Refs. [12–19]).

J_f^π	Unperturbed states	MCM	Perturbed states	Expt. data
$0^+_{\text{g.s.}}$	4.8	4.8	4.8	4.7
2^+_1	5.2	5.2	5.2	5.5
$0^+_{\text{2-ph}}$	6.9	6.1	6.6	
$2^+_{\text{2-ph}}$	7.4	6.8	6.5	6.2

TABLE III. The $\log ft_+$ values for the β^+ decay ^{118}Sb$(1^+)\rightarrow^{118}$Sn(J_f^π). First column: predictions using unperturbed states and expansion for the $\hat{\beta}^-$ operator. Second column: results produced by the MCM. Third column: both the transition operator and the states are perturbed. Fourth column: experimental data (taken from Refs. [12–19]).

J_f^π	Unperturbed states	MCM	Perturbed states	Expt. data
$0^+_{\text{g.s.}}$	5.2	5.2	5.2	4.5
2^+_1	7.0	7.0	6.9	5.8
$0^+_{\text{2-ph}}$	8.2	7.7	7.4	5.2
$2^+_{\text{2-ph}}$	8.0	9.9	6.9	6.3

and $4\hbar\omega$ plus the intruder $h_{11/2}$. The strength parameters for the pairing and 2^λ-pole interaction ($\lambda=1,2$) were taken from Ref. [19].

The $\log ft$ values were calculated by using for the reduced m.e. alternatively the relations (3.10),(3.11), and (4.2). The former case corresponds to the situation when only the transition operators are affected by anharmonicities. Results for β^- transitions are collected in the first columns of Tables I and II for ^{118}Sn and ^{120}Sn, respectively. They correspond to a strength of $g_{\text{pp}}\approx 0.9$ of the particle-particle proton-neutron interaction. These predictions are to be compared with those of the second columns which are obtained by using the MCM approach for evaluating the higher-RPA effects.

Results for the β^+ transitions feeding ^{118}Sn and ^{120}Sn are presented in Tables III and IV, respectively. Comparing the first two columns of Tables I–IV, one confirms our previous statement that the two procedures, the BEM and the MCM, are identical when the transition leads to the ground state or to a one-phonon state, but they are different when a multiphonon state is fed. On the third columns of Tables I–IV the results, obtained by using Eq. (4.2) for the reduced m.e., are presented. It is remarkable that perturbing the states connected by the $\hat{\beta}^\pm$ operator, the transition amplitudes to the ground state as well as to the first one-phonon state, 2^+_1, are practically unmodified. This is a nice numerical confirmation of the procedure used in Refs. [7,8], where the states, involved in the double-β-decay process, are not perturbed.

Concerning the transition to the two-phonon states, $0^+_{\text{2-ph}}$ and $2^+_{\text{2-ph}}$, the modifications, determined by the perturbative components of states, are comparable with those produced by anharmonic boson corrections to the transition operator. The effect of perturbing the states on the $\log ft$ values is to improve the agreement with the experimental data.

From the tables one can see that the quality of the unperturbed description and the MCM description is roughly the same for the two-phonon states. It means that on the basis of the present experimental data one cannot decide which one of the two higher-RPA descriptions is the more realistic one. Futhermore, from Tables I and II it can be noted that for the β^- feeding the MCM and the BEM, with perturbation of the states included, yield results of similar quality, close to the experimental data. The description of the experimental β^+ feeding seems to be more difficult for both models, as seen from Tables III and IV. Here the BEM, including the state perturbation, yields consistently better results.

Concerning the $2\nu\beta\beta$ decay, the nearest double-β-decay transition is ^{116}Cd$(0^+_{gs})\rightarrow^{116}In(1^+)\rightarrow^{116}Sn(J_f^\pi)$, where $J_f^\pi=0^+_{\text{g.s.}},2^+_1,0^+_{\text{2-ph}},2^+_{\text{2-ph}}$. These decays consist of the β^- decay transitions ^{116}In$(1^+)\rightarrow^{116}$Sn(0^+) and the conjugate of the β^+ decay transition ^{116}In$(1^+)\rightarrow^{116}$Cd(0^+). Extrapolating slightly the discussion of the previous paragraph one can draw the following conclusions concerning this $2\nu\beta\beta$ decay: (i) the β^- transitions seem to be well described, both for the ground state and the excited states;

(ii) the inverse β^+ branch for the ground-state transition is described reasonably well, as seen from Tables III and IV.

The above observations suggest that the $2\nu\beta\beta$ decay to the ground state, as well as to the excited states of ^{116}Sn, should be reasonably well described by the BEM and the MCM. Of course this statement concerns only the decay via the lowest virtual 1^+ state. However , this 1^+ state is usually the one giving the dominating contribution in the $2\nu\beta\beta$ decay to the 0^+ states. This dominance is even more pronounced to the 2^+ final states since there the energy denominator of the perturbation expression of the $2\nu\beta\beta$ amplitude is a cubic one [8].

In summary, the above analysis suggests that the $2\nu\beta\beta$ decay of ^{116}Cd is rather well described by both the BEM and

TABLE II. The same as in Table I but for the decay ^{120}In $\rightarrow^{120}$Sn.

J_f^π	Unperturbed states	MCM	Perturbed states	Expt. data
$0^+_{\text{g.s.}}$	4.8	4.8	4.8	5
2^+_1	5.3	5.3	5.2	5.2
$0^+_{\text{2-ph}}$	6.6	6.0	6.1	5.9–6.8
$2^+_{\text{2-ph}}$	7.0	6.6	7.5	5.8–6.3

TABLE IV. The same as in Table III but for the β^+ decay ^{120}Sb$\rightarrow^{120}$Sn.

J_f^π	Unperturbed states	MCM	Perturbed states	Expt. data
$0^+_{\text{g.s.}}$	5.0	5.0	5.0	4.5
2^+_1	6.5	6.5	6.4	5.6
$0^+_{\text{2-ph}}$	7.8	7.4	6.7	6.0-6.1
$2^+_{\text{2-ph}}$	7.5	8.9	7.4	

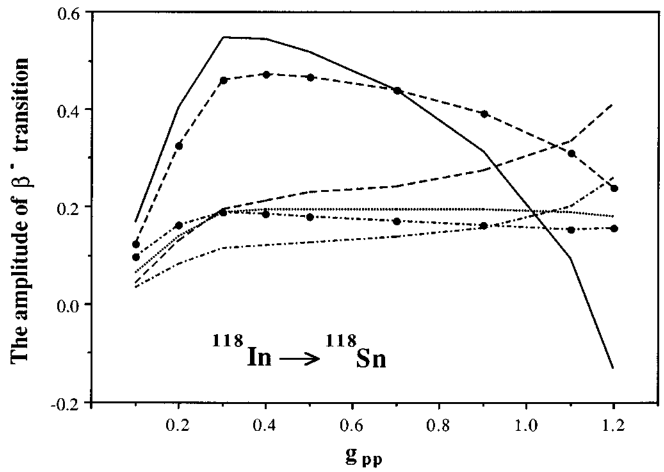

FIG. 1. Transition amplitudes describing the β^- transitions ^{118}In$(1^+)\rightarrow{}^{118}$Sn(J_f^+) with $|J_f^+\rangle$ taking the values $|0^+\rangle$ (solid line), $|2_1^+\rangle$ (short dash), $|2_12_1;0^+\rangle$ (long dash), $|2_12_1;0^+\rangle'$ (long dash with point), $|2_12_1;2^+\rangle$ (dash-dot), and $|2_12_1;2^+\rangle'$ (dash-dot with point). These transition amplitudes are plotted as functions of $g_{\rm pp}$. For a better presentation, the transition amplitudes having as final state either $0^+_{\text{2-ph}}$ or $2^+_{\text{2-ph}}$, are first multiplied by 10 and then plotted.

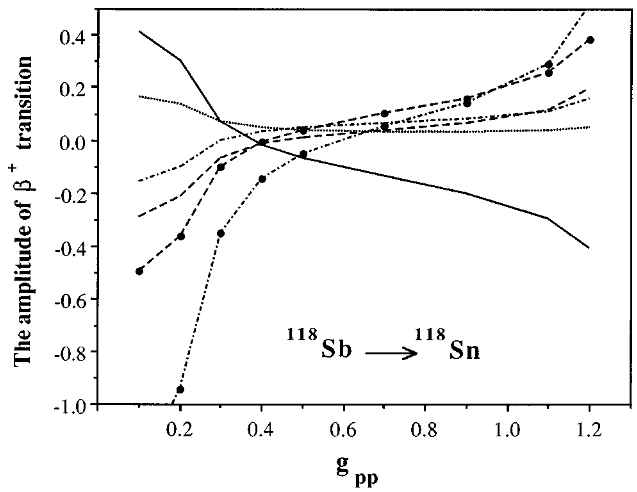

FIG. 2. Transition amplitudes describing the β^+ transitions ^{118}Sb$(1^+)\rightarrow{}^{118}$Sn(J_f^+) with $|J_f^+\rangle$ taking the values $|0^+\rangle$ (solid line), $|2_1^+\rangle$ (short dash), $|2_12_1;0^+\rangle$ (long dash), $|2_12_1;0^+\rangle'$ (long dash with point), $|2_12_1;2^+\rangle$ (dash-dot), and $|2_12_1;2^+\rangle'$ (dash-dot with point). These transition amplitudes are plotted as functions of $g_{\rm pp}$. For a better presentation, the transition amplitudes having as final state either $0^+_{\text{2-ph}}$ or $2^+_{\text{2-ph}}$, are first multiplied by 10 and then plotted.

the MCM. However, it is dangerous to extrapolate the present result further away from the Cd and Sn region and only a careful study of the nuclei involved in the $2\nu\beta\beta$-decay processes can test the quality of these models in description of $2\nu\beta\beta$-decay transitions.

The reduced m.e. for β^- and β^+ decays are plotted in Figs. 1 and 2, respectively, as functions of $g_{\rm pp}$ for ^{118}Sn. From Fig. 1 one sees that except for the transition $1^+\rightarrow 0^+$, all the other β^- transitions are characterized by reduced m.e. which do not change their sign in the range $g_{\rm pp}=0.0-1.2$. The amplitudes $\langle 1^+\|\hat{\beta}^{\pm}\|2^+\rangle$ are almost constant within a large interval of $g_{\rm pp}$ values. The β^- transition to the state $0^+_{\text{2-ph}}$ is more affected by perturbation than the transition to the other two-phonon state $2^+_{\text{2-ph}}$. From Fig. 2 one observes that increasing $g_{\rm pp}$ from 0 to 1.2 the matrix elements of the β^+ decay change their sign. An exception is the m.e. $\langle 1^+\|\hat{\beta}^-\|2_1^+\rangle$. For $g_{\rm pp}\leqslant 0.63$ and $g_{\rm pp}\geqslant 1.0$ the state $2^+_{\text{2-ph}}$ is affected by perturbation to a larger extent than the state $0^+_{\text{2-ph}}$.

The energy corrections due to the first-order perturbation are given in Table V. As we already mentioned, the corrected energy of $2^+_{\text{2-ph}}$ state was obtained through the diagonalization procedure. From Table V one sees that the perturbation does not affect significantly the excitation energy of the states 2_1^+ and 1_1^+. The notable effect in the present scheme is the energy splitting characterizing the two members of the two-phonon triplet. The magnitudes of these splittings are close to the experimental data. Also our prediction concerning the energy ordering of the perturbed states (i.e., $2^+_{\text{2-ph}},0^+_{\text{2-ph}}$) agrees with the experimental data.

VI. CONCLUSIONS

In the previous sections we developed a formalism to calculate the Gamow-Teller single β^- and β^+ transition amplitudes from a state 1^+, describing the odd-odd nuclei $(N+1,Z-1)$ and $(N-1,Z+1)$, respectively, to the even-even nucleus (N,Z). We considered the cases where the even-even nucleus is fed into one of the states $0^+,2_1^+,0^+_{\text{2-ph}}$ or $2^+_{\text{2-ph}}$.

Two distinct situations are separately analyzed. In the first case the states describing the initial and the final nuclei are RPA states but the transititon operator is expanded to first order in terms of bosons. It is to be noted that considering the transition operator in zeroth order of boson expansion (i.e., within the RPA approach), only the ground state can be fed. In the second step both the transition operator and the states are modified due to anharmonicities.

Numerical application is made for ^{118}Sn and ^{120}Sn. In this case the β^- transitions emerge from ^{118}In and ^{120}In while

TABLE V. First order perturbative corrections to energies of the states 0^+ (ground state), 2_1^+, 1_1^+, $0^+_{\text{2- ph}}$, and $2^+_{\text{2- ph}}$ in units of keV. Also the energy splitting for the two phonon states ($\Delta E_{0,2}$) is listed.

	ΔE_{0^+}	$\Delta E_{2_1^+}$	$\Delta E_{1_1^+}$	$\Delta E_{0^+_{\text{2- ph}}}$	$\Delta E_{2^+_{\text{2- ph}}}$	$\Delta E_{0,2}$
^{118}Sn	38	54	43	115	-132	247
^{120}Sn	72	88	77	198	-111	309

the β^+ transitions from ^{118}Sb and ^{120}Sb, respectively. The predicted amplitudes for the transitions $^{118}\mathrm{In}\rightarrow{}^{118}\mathrm{Sn}$ and $^{118}\mathrm{Sb}\rightarrow{}^{118}\mathrm{Sn}$ are represented in Figs. 1 and 2, respectively, as functions of g_{pp}. For $g_{\mathrm{pp}}=0.9$, the $\log ft_{\pm}$ values were calculated for the two schemes of perturbation mentioned above. In the first case the predictions of the present work are compared with those of the MCM approach. One concludes that the two formalisms (BEM and MCM) are identical for the transitions to 2_1^+ but they differ in predictions when the transitions to either the state $0^+_{2\text{-ph}}$ or $2^+_{2\text{-ph}}$ are considered.

The corrections to the transition amplitudes coming from the perturbation of the initial and final states are negligible when the final state is either 0^+ or 2_1^+ but they are comparable to those generated by anharmonicities in the transition operator when the daughter nucleus is left in a two-phonon state.

Based on the present study, it is reasonable to expect that the $2\nu\beta\beta$ decay of ^{116}Cd to the ground state and excited states of ^{116}Sn, should be rather well described by both the BEM and MCM. The verification of this statement, both in the ^{116}Cd decay and the other $2\nu\beta\beta$ decays, will be the subject of future investigations.

Perturbation does not affect significantly the excitation energies (see Table V) for the states 2_1^+ and 1_1^+, but produces a splitting in energy for the two-phonon states $0^+_{2\text{-ph}}$ and $2^+_{2\text{-ph}}$, which is equal to 247 keV for ^{118}Sn and 309 keV for ^{120}Sn.

Finally, we would like to mention that there are two sources generating divergences for the perturbative series. One was already mentioned and arises from the fact that there are RPA one-phonon states lying close in energy to $2^+_{2\text{-ph}}$. In this case the interaction between the states $|2_1 2_1;2^+\rangle$ and $|2_2^+\rangle$ cannot be treated perturbatively. Another source for divergences appears whenever one of the two RPA approaches (the charge-conserving or the charge nonconserving) breaks down. These cases are associated with large values of the backward-going amplitudes (Y or S) which yield a slowly converging (or even divergent) boson-expansion series. Concluding, for these two limiting cases the perturbative treatment cannot be applied. The first case of divergence could be avoided by diagonalizing the boson Hamiltonian in the space of the quasidegenerate states, but if the strength of the two-body interaction is close to its critical value (where the RPA breaks down) the method is not applicable at all and we have to look for some other procedures.

APPENDIX A

Here we give explicit expressions for the coefficients $\mathscr{A}$ which define the boson representation for the dipole operators $A^+_{1\mu}(j_p j_n)$ and which were not calculated in Ref. [8]:

$$
\begin{aligned}
\mathscr{A}^{1(\overline{22})l}_{k_1k_2k_3}(j_pj_n)=\frac{-4(2l+1)\sqrt{15}}{1+\delta_{k_2k_3}}\sum_{j'_n,j''_n}\Big\{ & Z^{11l}_{n'np}Z^{22l}_{n'nn''}X_{k_1}(j_pj'_n)[S_{k_2}(j''_nj_n)R_{k_3}(j''_nj'_n)+S_{k_3}(j''_nj_n)R_{k_2}(j''_nj'_n)] \\
&+\sum_{j'_p,j''_p}(-)^{j_p-j'_p}Z^{11l}_{p'pn}Z^{22l}_{p'pp''}X_{k_1}(j'_pj_n)[S_{k_2}(j''_pj_p)R_{k_3}(j''_pj'_p)+S_{k_3}(j''_pj_p)R_{k_2}(j''_pj'_p)]\Big\},\\
\mathscr{A}^{(22)l\bar{1}}_{k_3k_2k_1}(j_pj_n)=\frac{-4(2l+1)\sqrt{15}}{1+\delta_{k_2k_3}}\sum_{j'_n,j''_n}\Big\{ & Z^{11l}_{n'np}Z^{22l}_{n'nn''}Y_{k_1}(j_pj'_n)[R_{k_2}(j''_nj_n)S_{k_3}(j''_nj'_n)+R_{k_3}(j''_nj_n)S_{k_2}(j''_nj'_n)] \\
&+\sum_{j'_p,j''_p}(-)^{j_p-j'_p}Z^{11l}_{p'pn}Z^{22l}_{p'pp''}Y_{k_1}(j'_pj_n)[R_{k_2}(j''_pj_p)S_{k_3}(j''_pj'_p)+R_{k_3}(j''_pj_p)S_{k_2}(j''_pj'_p)]\Big\}.
\end{aligned}
\tag{A1}
$$

The coefficients $\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}$ and $\mathscr{A}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}$ are obtainable from $\mathscr{A}^{1(\overline{22})l}_{k_1k_2k_3}$ and $\mathscr{A}^{(22)l1}_{k_3k_2k_1}$, respectively, by interchanging the forward-going and backward-going amplitudes

$$
S_{k_2}(j''_nj_n)\rightarrow R_{k_2}(j''_nj_n);R_{k_2}(j''_nj'_n)\rightarrow S_{k_2}(j''_nj'_n),
$$

$$
S_{k_2}(j''_pj_p)\rightarrow R_{k_2}(j''_pj_p);R_{k_2}(j''_pj'_p)\rightarrow S_{k_2}(j''_pj'_p). \tag{A2}
$$

In Eqs. (A1), the following notations have been used:

$$
Z^{l_1l_2l}_{pp'n}=(-)^{j_p-j_n}W(l_1j_pl_2j'_p;j_nl),
$$

$$
Z^{l_1l_2l}_{nn'p}=(-)^{j_n-j_p}W(l_1j_nl_2j'_n;j_pl),
$$

$$
Z^{22l}_{\tau\tau'\tau''}=(-)^{j_\tau-j''_\tau}W(2j_\tau 2j'_\tau;j''_\tau l). \tag{A3}
$$

APPENDIX B

Here we list the coefficients $\mathscr{B}$ involved in the equations (3.4):

$$\mathscr{B}^{1(\overline{22})l}_{k_1k_2k_3}=-\sum_k\,[\bar{\sigma}_k.\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}(k)+\sigma_k.\mathscr{A}^{(22)l\bar{1}}_{k_3k_2k_1}(k)],$$

$$\mathscr{B}^{(22)l\bar{1}}_{k_3k_2k_1}=-\sum_k\,[\bar{\sigma}_k.\mathscr{A}^{(22)l\bar{1}}_{k_3k_2k_1}(k)+\sigma_k.\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}(k)],$$

$$\mathscr{B}^{1(2\bar{2})l}_{k_1k_2k_3}=-\sum_k\,[\bar{\sigma}_k.\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}(k)+\sigma_k.\mathscr{A}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}(k)],$$

$$\mathscr{B}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}=-\sum_k\,[\bar{\sigma}_k.\mathscr{A}^{(2\bar{2})l\bar{1}}_{k_3k_2k_1}(k)+\sigma_k.\mathscr{A}^{1(2\bar{2})l}_{k_1k_2k_3}(k)]. \tag{B1}$$

APPENDIX C

The expansion coefficients of Eq. (3.12) for the quadrupole operators $B^{+}_{2\mu}$ read

$$D^{(20)}_{k_1k_2}(j_1j_2)=-\frac{20}{1+\delta_{k_1k_2}}\sum_{j_1'}Z^{222}_{j_1j_2j_1'}[R_{k_1}(j_1'j_1)S_{k_2}(j_1'j_2)$$
$$+S_{k_1}(j_1'j_2)R_{k_2}(j_1'j_1)],$$

$$D^{(11)}_{k_1k_2}(j_1j_2)=-20\sum_{j_1'}Z^{222}_{j_1j_2j_1'}[R_{k_1}(j_1'j_1)R_{k_2}(j_1'j_2)$$
$$+R_{k_1}(j_1'j_2)R_{k_2}(j_1'j_1)],$$

$$D^{(02)}_{k_1k_2}(j_1j_2)=-\frac{20}{1+\delta_{k_1k_2}}\sum_{j_1'}Z^{222}_{j_1j_2j_1'}[S_{k_1}(j_1'j_1)R_{k_2}(j_1'j_2)$$
$$+R_{k_1}(j_1'j_2)S_{k_2}(j_1'j_1)]. \tag{C1}$$

The factors Z^{222}_{abc} were defined in Appendix B. The coefficients defining the first-order expanded Hamiltonian have the expressions:

$$\mathscr{H}^{(30)}_{k_1k_2k_3}=\sum_{a,b,c,d}f(abcd)[R_{k_1}(ab)D^{(20)}_{k_2k_3}(dc)$$
$$-S_{k_1}(ab)D^{(02)}_{k_2k_3}(dc)],$$

$$\mathscr{H}^{(21)}_{k_1k_2k_3}=\mathscr{H}^{1;02}_{k_1k_2k_3}+\mathscr{H}^{1;11}_{k_3k_2k_1},$$

$$\mathscr{H}^{1;02}_{k_1k_2k_3}=\sum_{a,b,c,d}f(abcd)[R_{k_1}(ab)D^{(02)}_{k_2k_3}(dc)$$
$$-S_{k_1}(ab)D^{(20)}_{k_2k_3}(dc)],$$

$$\mathscr{H}^{1;11}_{k_1k_2k_3}=\sum_{a,b,c,d}f(abcd)[R_{k_1}(ab)D^{(20)}_{k_2k_3}(dc)$$
$$-S_{k_1}(ab)D^{(02)}_{k_2k_3}(dc)],$$

$$\mathscr{H}^{(pn)}_{k_1k_1'k_2}=\sum_{a,b,c,d}f(abcd)[X_{k_1'}(ab)\mathbf{B}^{(02)}_{k_1k_2}(dc)$$
$$-Y_{k_1'}(ab)\mathbf{B}^{(20)}_{k_1k_2}(dc)+X_{k_1'}(ab)\mathbf{B}^{(11,21)}_{k_1'k_2}(dc)$$
$$-Y_{k_1'}(ab)\mathbf{B}^{(11,12)}_{k_1'k_2}(dc)]. \tag{C2}$$

Here we abbreviated by $f(a,b,c,d)$ the following expression:

$$f(abcd)=2\sqrt{5}G(abcd2)(U_aU_bU_cV_d-V_aV_bV_cU_d).$$

APPENDIX D

The perturbation coefficients involved in (4.1) and (4.2) have the expressions:

$$C_{12;1}(k_1,k_2)=\frac{\mathscr{H}^{(pn)}_{k_11k_2}}{\sqrt{3}[\omega_1(k_1)+\omega_2(k_2)-\omega_1(1)]},$$

$$C_{4,1}(k_1,k_2,k_3)=\frac{\bar{\mathscr{H}}^{(30)}_{k_1,k_2,k_3}}{N_{30}(k_1,k_2,k_3)[\omega_2(k_1)+\omega_2(k_2)+\omega_2(k_3)]}. \tag{D1}$$

Here we have used the notation:

$$\bar{\mathscr{H}}^{(30)}_{k_1k_2k_3}=2(\mathscr{H}^{(30)}_{k_1k_2k_3}+\mathscr{H}^{(30)}_{k_2k_1k_3}+\mathscr{H}^{(30)}_{k_3k_2k_1})\theta(k_2-k_1)\theta(k_3-k_2)+(2.\mathscr{H}^{(30)}_{k_1k_1k_3}+\mathscr{H}^{(30)}_{k_3k_1k_1})\delta_{k_1,k_2}\theta(k_3-k_2)$$
$$+(\mathscr{H}^{(30)}_{k_1k_2k_2}+2.\mathscr{H}^{(30)}_{k_2k_1k_2})\delta_{k_2,k_3}\theta(k_2-k_1)+\mathscr{H}^{(30)}_{k_1k_1k_1}\delta_{k_1,k_2}\delta_{k_2,k_3}, \tag{D2}$$

where $\theta(x)$ denotes the step function having the value 1 for $x>0$ and 0 for $x\leqslant 0$, respectively. Furthermore

$$N_{30}(k_1,k_2,k_3)=\theta(k_2-k_1)\theta(k_3-k_2)+\frac{1}{\sqrt{2}}[\delta_{k_1,k_2}(1-\delta_{k_2,k_3})+\delta_{k_2,k_3}(1-\delta_{k_1,k_2})]+\frac{1}{\sqrt{6}}\delta_{k_1,k_2}\delta_{k_2,k_3}, \tag{D3}$$

$$C_{3,0}=C_{4,1},$$

$$C_{2,1}(k_2,k_3)=\sqrt{\frac{2}{5}}\frac{2\mathscr{H}^{(1;02)}_{1k_2k_3}+\mathscr{H}^{(1;11)}_{k_3k_21}+\mathscr{H}^{(1;11)}_{k_2k_31}}{(1+\delta_{k_2k_3})^{\frac{1}{2}}[\omega_2(k_2)+\omega_2(k_3)-\omega_2(1)]},$$

$$C_{3,2}(k_2,k_3)=\frac{2\mathscr{H}^{(1;02)}_{1,k_2,k_3}+\mathscr{H}^{(1;11)}_{k_3k_21}+\mathscr{H}^{(1;11)}_{k_2k_31}}{(1+\delta_{k_2k_3})[\omega_2(k_2)+\omega_2(k_3)-\omega_2(1)]N_{30}(1,k_2,k_3)}. \tag{D4}$$

The norms are given by

$$\mathscr{N}_1=\left[1+\sum_{k_1,k_2}C_{12;1}(k_1,k_2)^2+\sum_{k_1\leqslant k_2\leqslant k_3}C_{4,1}(k_1,k_2,k_3)^2\right]^{-\frac{1}{2}},\quad \mathscr{N}_2=\left[1+\sum_{k_2,k_3}C_{2,1}(k_2,k_3)^2\right]^{-\frac{1}{2}},$$

$$\mathscr{N}_0=\left[1+\sum_{k_1\leqslant k_2\leqslant k_3}C_{30}(k_1,k_2,k_3)^2\right]^{-\frac{1}{2}},\quad \mathscr{N}_{20}=\left[1+\sum_{k_2,k_3}C_{3,2}(k_2,k_3)^2\right]^{-\frac{1}{2}}. \tag{D5}$$

[1] D. Cha, Phys. Rev. C **27**, 2269 (1983).
[2] P. Vogel and P. Fischer, Phys. Rev. C **32**, 1362 (1985).
[3] P. Vogel and M. R. Zirnbauer, Phys. Rev. Lett. **57**, 3148 (1986).
[4] O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. B **194**, 11 (1986).
[5] K. Grotz, H. V. Klapdor, and J. Metzinger, Phys. Lett. B **132**, 22 (1983); H. V. Klapdor and K. Grotz, *ibid.* **142**, 323 (1984).
[6] K. Muto, E. Bender, and H. V. Klapdor, Z. Phys. A **334**, 177 (1989).
[7] A. A. Raduta, A. Faessler, S. Stoica, and W. Kaminski, Phys. Lett. B **254**, 7 (1991).
[8] A. A. Raduta, A. Faessler, and S. Stoica, Nucl. Phys. **A534**, 149 (1991).
[9] J. Suhonen, Nucl. Phys. **A563**, 205 (1993).
[10] J. Suhonen and O. Civitarese, Phys. Lett. B **308**, 212 (1993); O. Civitarese and J. Suhonen, Nucl. Phys. **A578**, 62 (1994).
[11] A. Griffiths and P. Vogel, Phys. Rev. C **46**, 181 (1992).
[12] P. De Gelder, E. Jacobs, and D. De Frenne, Nucl. Data Sheets **38**, 545 (1983).
[13] *Table of Isotopes*, edited by C. M. Lederer and V. S. Shirley (Wiley, New York, 1978).
[14] J. Blachot and G. Marguier, Nucl. Data Sheets **59**, 333 (1990).
[15] T. Tamura, K. Miyamo, and S. Ohya, Nucl. Data Sheets **51**, 329 (1987).
[16] A. Hashisume, Y. Tendow, and M. Oshima, Nucl. Data Sheets **52**, 641 (1987).
[17] K. Kitao, M. Kanbe, Z. Matsumoto, and T. Seo, Nucl. Data Sheets **49**, 315 (1986).
[18] M. Bertschy, S. Drissi, P. E. Garrett, J. Jolie, J. Kern, S. J. Mannanal, J. P. Vorlet, N. Warr, and J. Suhonen, Phys. Rev. C **51**, 103 (1995).
[19] J. Suhonen and O. Civitarese, Nucl. Phys. **A584**, 449 (1995).
[20] A. Bohr and B. R. Mottelson, *Nuclear Structure, Vol. I* (Benjamin, New York 1963).
[21] M. Baranger, Phys. Rev. **120**, 957 (1960).
[22] M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley, New York, 1957).
[23] S. T. Belyaev and G. Zelevinski, Nucl Phys. **39**, 582 (1962).