BRIEF REPORTS

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Continuity equation in electron scattering from nuclei

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The role of the continuity equation in the calculations usually performed in the context of the electron scattering from nuclei is analyzed. Special attention is paid to the uncertainty introduced in the theoretical predictions after this equation is considered in the calculation of the transverse electric multipoles, when the models are not gauge invariant. Quantitative analysis of this point is done for a simple model based on a harmonic oscillator Hamiltonian with a constant spin-orbit potential.

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Electron scattering from nuclei has produced a large body of work about nuclear structure. Elastic scattering has provided the shape of the nuclear charge density. Inelastic scattering has provided information on the charge and current distributions involved in a given transition. The particular characteristics of the electron-nucleon interaction (i.e., it is rather well known and relatively weak) have ensured, to some extent, the reliability of the results obtained in the analysis of the experimental information. This is basically the cross section, which is given by [1]

$$\frac{d\sigma}{d\Omega}=\frac{4\pi\sigma_{\rm Mott}}{2J_i+1}\frac{1}{\eta}\left[\frac{Q_\mu^4}{q^4}\sum_{\lambda=0}^{\infty}|t_{C\lambda}(q)|^2+\left(-\frac{Q_\mu^2}{2q^2}+\tan^2\frac{\theta}{2}\right)\sum_{\lambda=1}^{\infty}\{|t_{E\lambda}(q)|^2+|t_{M\lambda}(q)|^2\}\right],$$

where $\sigma_{\rm Mott}$ is the well-known Mott cross section, J_i is the angular momentum of the initial state of the nucleus, $|i\rangle$, η is the nuclear recoil factor, and $Q^\mu=(\omega,\mathbf{q})$ is the four-momentum carried by the photon exchanged between the electron and the nucleus. This result is obtained in the framework of the plane wave Born approximation (PWBA), after assuming that the continuity equation (CE) is satisfied. Actually, electron waves are distorted by the nuclear field, the distorted wave Born approximation should be used, and the previous expression for the cross sections is no longer valid. Despite the fact that the effect of the distortion can be important, especially for heavy nuclei, we will continue within the PWBA in order to facilitate a clearer discussion.

The charge and current distributions mentioned above enter in the longitudinal (charge) $t_{C\lambda}$ and transverse (current) $t_{E\lambda}$ and $t_{M\lambda}$ multipoles. These are reduced matrix elements of multipole operators between the nuclear states involved in the transition considered. They are given as follows:

$$t_{C\lambda}(q)=\langle J_f\|M_\lambda^{\rm Coul}(q)\|J_i\rangle, \tag{1}$$

$$t_{E\lambda}(q)=\sqrt{\frac{\lambda+1}{2\lambda+1}}\langle J_f\|iT_{\lambda\lambda-1}(q)\|J_i\rangle-\sqrt{\frac{\lambda}{2\lambda+1}}\langle J_f\|iT_{\lambda\lambda+1}(q)\|J_i\rangle, \tag{2}$$

$$t_{M\lambda}(q)=\langle J_f\|iT_{\lambda\lambda}(q)\|J_i\rangle, \tag{3}$$

with

$$M_{\lambda\mu}^{\rm Coul}(q)=\int d\mathbf{r}j_\lambda(qr)Y_{\lambda\mu}(\hat{\mathbf{r}})\rho(\mathbf{r}),$$

$$T_{\lambda L}^{\mu}(q)=\int d\mathbf{r}j_\lambda(qr)\mathbf{Y}_{\lambda L}^{\mu}(\hat{\mathbf{r}})\cdot\mathbf{J}(\mathbf{r}).$$

Here j_λ is a spherical Bessel function, $\mathbf{Y}_{\lambda L}^{\mu}(\hat{\mathbf{r}})$ is a vector spherical harmonic, and $\rho(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ are the nuclear charge and current densities operators, respectively.

In this context the CE enters as a basic aspect. This equation is just the formulation of the charge-current conservation and it follows directly from the property of gauge invariance of the electromagnetic field and its coupling to the particle field [2]. As a consequence, the CE is expected to hold in quite general situations. It can be written as

$$[H,\rho(\mathbf{r})]=i\nabla\cdot\mathbf{J}(\mathbf{r}),$$

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where H is the nuclear Hamiltonian. If we take the λ multipole of the above equation and evaluate the reduced matrix element between the nuclear states $|J_i\rangle$ and $|J_f\rangle$, which are supposed to be eigenstates of H with eigenvalues E_i and E_f, respectively, it is straightforward to obtain [3,4]

$$\frac{\omega}{q}t_{C\lambda}(q)=-\sqrt{\frac{\lambda}{2\lambda+1}}t_{\lambda-}(q)-\sqrt{\frac{\lambda+1}{2\lambda+1}}t_{\lambda+}(q), \quad (4)$$

with $\omega=E_f-E_i$ the energy transfer and where we have defined the multipoles

$$t_{\lambda\pm}(q)=\langle J_f\|iT_{\lambda\lambda\pm1}(q)\|J_i\rangle.$$

One of the main implications of the CE is the constraint it introduces in the charge and current distributions. As we can see in Eq. (4), the longitudinal (charge) multipoles are related to the two transverse (current) multipoles of electric type. This means that, in general, the measurement of the cross section allows one to separate three independent entities: the charge multipole, the magnetic multipole, and one of the two electric ones, the other one being fixed by the CE.

When these multipoles are calculated, some problems appear, mainly because, in most cases, the wave functions used are nonrelativistic and non-nucleonic degrees of freedom are not considered. This problem is considerably more severe for the current than for the charge [4], and this is the reason why it is a common practice to eliminate one of the two terms in Eq. (2) by means of the CE (4). In this way, $t_{E\lambda}$ is not obtained directly from the current operator and part of the deficiency is taken out by including explicitly the charge multipole in the calculation.

Though, in principle, there is nothing to argue against this procedure, it is worth pointing out that usually the models taken into account to perform the calculations do not satisfy the CE and, as a consequence, a great limitation appears, because (i) the ansatz considered is not unique and (ii) the results one obtains by using the different possibilities could disagree to a very large extent.

In this paper we want to investigate in detail these two aspects, trying to evaluate the degree of incertitude introduced in the problem when such a procedure is followed.

First, we discuss the indefiniteness inherent to the method. It is evident from the previous comments that one can proceed in two different ways, each one corresponding to the elimination of $t_{\lambda-}$ or $t_{\lambda+}$ in Eq. (2). Then two new formulas to evaluate the multipole $t_{E\lambda}$ appear:

$$t_{E\lambda}^{(+)}(q)=-\sqrt{\frac{2\lambda+1}{\lambda}}t_{\lambda+}(q)-\sqrt{\frac{\lambda+1}{\lambda}}\frac{\omega}{q}t_{C\lambda}(q), \quad (5)$$

$$t_{E\lambda}^{(-)}(q)=\sqrt{\frac{2\lambda+1}{\lambda+1}}t_{\lambda-}(q)+\sqrt{\frac{\lambda}{\lambda+1}}\frac{\omega}{q}t_{C\lambda}(q). \quad (6)$$

Of course, these two equations produce the same result [and the same as Eq. (2)] if the CE is fulfilled, but it is not possible to ensure anything concerning which of them is better when it does not. This is very important because Eq. (5) is the one commonly used to rearrange the electric multipoles in calculations where the CE is not satisfied. From the experimental point of view, the choice selected depends on the type of transition which is being analyzed: $t_{\lambda-}$ dominates in transitions to collective states, while it is negligible in those excited levels of strong single particle-hole nature [4]. From a theoretical point of view, it is important to mention that the Siegert theorem [5] is satisfied if the prescription of Eq. (5) is adopted.

TABLE I. Parameters of the potential used in the toy model considered to discuss the violation of the CE (see text).

Nucleus	V_0[MeV]	b[fm]	V_{LS}[MeV]
^{16}O	-53.6	1.67	-4.20
^{40}Ca	-55.7	1.80	-1.90

In what follows we paid attention to the (rather usual) situations in which the CE is not verified. If this is the case, the use of Eq. (4) to eliminate one of the two multipoles in Eq. (2) can be understood also as a way to force the model to satisfy the CE. As we have previously seen, the determination of the current multipoles needs the knowledge of three reduced matrix elements $t_{\lambda-}$, $t_{\lambda+}$, and $t_{M\lambda}$. The use of $t_{E\lambda}^{(+)}$, given by Eq. (5), to evaluate the electric multipoles means that one modifies the first of these three matrix elements while the other two remain unchanged:

$$t_{\lambda-}(q)\rightarrow\tilde{t}_{\lambda-}(q)=-\sqrt{\frac{\lambda+1}{\lambda}}t_{\lambda+}(q)-\sqrt{\frac{2\lambda+1}{\lambda}}\frac{\omega}{q}t_{C\lambda}(q),$$

$$t_{\lambda+}(q)\rightarrow\tilde{t}_{\lambda+}(q)=t_{\lambda+}(q),$$

$$t_{M\lambda}(q)\rightarrow\tilde{t}_{M\lambda}(q)=t_{M\lambda}(q).$$

If Eq. (6) is chosen to evaluate the electric multipoles, the opposite occurs: $t_{\lambda+}$ is modified and $t_{\lambda-}$ and $t_{M\lambda}$ are not.

However, it is clear that these are not the only possibilities. One can find infinite pairs $(t_{\lambda-},t_{\lambda+})$ satisfying the CE at the level of Eq. (4), and each of them producing a different value for $t_{E\lambda}$, leaving $t_{M\lambda}$ unchanged. The two cases given in Eqs. (5) and (6) are only particular cases of this more general situation. The problem at this point is that any value of the electric multipole, ranging from $-\infty$ to $+\infty$, can be obtained in this way. Thus, the imposition of the CE following this method is completely out of control and, in some sense, has no physical meaning.

In order to quantify the different aspects pointed out in the previous discussion, we consider now a toy model to describe the nuclear structure by means of a single-particle Hamiltonian based in a harmonic oscillator potential and including a spin-orbit term:

$$H=-\frac{\hbar^2}{2m}\nabla^2+V_0+\frac{\hbar^2}{2m}\frac{r^2}{b^4}+V_{LS}\mathbf{l}\cdot\mathbf{s}.$$

The values of the parameters V_0, V_{LS}, and b used in the calculations have been adjusted to reproduce the energies of the single-particle states around the Fermi level in different double closed-shell nuclei and are given in Table I. The eigenfunctions corresponding to this Hamiltonian are harmonic oscillator wave functions R_{nl}, with oscillator parameter b, while the eigenvalues can be found to be

$$E_{nlj}=\frac{\hbar^2}{mb^2}\left(2n+l-\frac{1}{2}\right)+V_0+V_{\rm LS}c_{lj},$$

where

$$c_{lj}=\frac{1}{2}\left[j(j+1)-l(l+1)-\frac{3}{4}\right].$$

It is important to point out that, due to the fact that the spin-orbit piece of the potential does not depend on the spatial coordinates, the two single-particle states corresponding to a given value of l only differ in the energy values, the wave functions being exactly the same.

Within this model nuclear transitions in closed-shell nuclei can be described by considering wave functions for the excited states including a single one particle–one-hole (1p1h) configuration. In addition, transitions between single-particle (or single-hole) states describe nuclear transitions in nuclei with one nucleon above (or below) a closed shell. Obviously, the reliability of the results obtained in this way is related to the validity of the extreme shell model in each case.

The charge density operator has been taken to be the usual one:

$$\rho(\mathbf{r})=\sum_{k=1}^{A}\frac{1+\tau_3^k}{2}\delta(\mathbf{r}-\mathbf{r}_k),$$

while the nuclear current operator includes the convection and spin-magnetization one-body terms

$$\mathbf{J}^C(\mathbf{r})=\sum_{k=1}^{A}\frac{1}{2M_k}\frac{1}{i}\frac{1+\tau_3^k}{2}[\,\delta(\mathbf{r}-\mathbf{r}_k)\vec{\nabla}_k+\vec{\nabla}_k\delta(\mathbf{r}-\mathbf{r}_k)\,],$$

$$\mathbf{J}^M(\mathbf{r})=\sum_{k=1}^{A}\left(\mu_P\frac{1+\tau_3^k}{2}+\mu_N\frac{1-\tau_3^k}{2}\right)\vec{\nabla}_k\vec{\nabla}[\,\delta(\mathbf{r}-\mathbf{r}_k)\boldsymbol{\sigma}^k],$$

as well as the so-called spin-orbit current

$$\mathbf{J}^{\rm LS}(\mathbf{r})=\frac{1}{2}V_{\rm LS}\sum_{k=1}^{A}\frac{1+\tau_3^k}{2}\delta(\mathbf{r}-\mathbf{r}_k)\boldsymbol{\sigma}^k\times\mathbf{r}_k.$$

Here M_k is the mass of the k nucleon, $\mathbf{S}^k=\boldsymbol{\sigma}^k/2$ is its spin, and $\tau_3^k=1$ or -1 according this nucleon being proton or neutron, respectively. Finally, μ_P (μ_N) is the proton (neutron) magnetic moment.

It is a simple exercise to check that this model fulfills the CE and for the following discussion we take the multipoles calculated within it as ''pseudodata.'' The usual situation of a model not verifying the CE can be simulated by taking out the spin-orbit term in the current operator. It is to this model which we will apply the general procedure described above.

In order to visualize the point relative to the restoration of the CE we have plotted in Fig. 1 the $(t_{\lambda-},t_{\lambda+})$ plane for the transitions $0^+\rightarrow 1d_{5/2}1p_{1/2}^{-1}$ in ^{16}O (left) and $1d_{3/2}^{-1}\rightarrow 2s_{1/2}^{-1}$ in ^{39}K (right), and for two values of the momentum transfer, $q=1$ fm^{-1} (upper panel) and 2 fm^{-1} (lower panel). Therein the solid line represents the locus of the points satisfying the CE as given by Eq. (4). The point corresponding to the toy model in absence of the spin-orbit current term is P_0, while the point X is obtained for the full current operator.

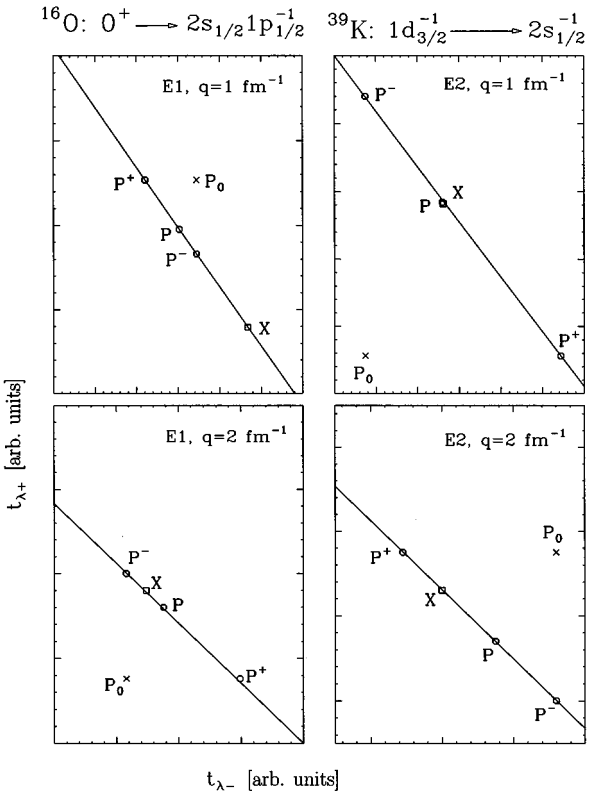

FIG. 1. $(t_{\lambda-},t_{\lambda+})$ plane for the transitions $0^+\rightarrow 1d_{5/2}1p_{1/2}^{-1}$ in ^{16}O (left) and $1d_{3/2}^{-1}\rightarrow 2s_{1/2}^{-1}$ in ^{39}K (right), and for $q=1$ fm^{-1} (upper panel) and 2 fm^{-1} (lower panel). The solid line represents the locus of the points satisfying the CE as given by Eq. (6). X and P_0 are the points corresponding to the model with and without spin-orbit current, respectively. P^+ is the point obtained by modifying $t_{\lambda-}$ only, leaving $t_{\lambda+}$ unchanged, while P^- corresponds to the opposite situation. Finally, P is the point giving the same electric multipole as P_0, but satisfying simultaneously the CE.

If the prescription in which the electric multipole is calculated by means of Eq. (5) is adopted, one reaches the point P^+. P^- corresponds to the situation provided by Eq. (6).

The cases shown in Fig. 1 are representative of the different situations that can be found. What is important to note is the lack of systematic results: It is not possible at all to ensure which of the two points P^+ or P^- is closer to X. In the usual situation, the recovering of the CE in the terms we are discussing, i.e., by going from P_0 to P^+ or P^-, does not permit one to establish any conclusion with respect to the feasibility of the electric multipole obtained.

To finish the discussion, in Fig. 1 we have plotted also the point P, which is the point obtained when both $t_{\lambda-}$ and $t_{\lambda+}$ are modified in order to satisfy the CE and give, simultaneously, the same value of $t_{E\lambda}$ as the one corresponding to P_0. As we can see, P is always between P^+ and P^- and closer to the last one [the reason for that can be easily deduced from Eqs. (5) and (6)].

It is interesting to analyze in detail the case of the ^{39}K at $q=1$ fm^{-1}. Therein (see upper right panel in Fig. 1) P coincides with X, which means that the effect of the inclusion of the spin-orbit term in the current is negligible, in

what refers to the calculation of the electric multipole. Despite the fact that P_0 corresponds to a situation in which the CE is not verified, it provides a better result than any of the two possibilities P^+ or P^-. This shows in a very clear way the fact that the restoration of the CE via the modification of the multipoles of the current cannot be considered as a mechanism which increases the goodness of the calculated electric multipoles.

Besides all these aspects, from the physical point of view, there is an additional important question following the procedure under discussion: Does there exist a current operator $\tilde{\mathbf{J}}(\mathbf{r})$ having the new multipoles $\tilde{t}_{\lambda-}$, $\tilde{t}_{\lambda+}$, and $\tilde{t}_{M\lambda}$? In principle, one cannot ensure the existence of such a current, which means that the modification of multipoles of the current to satisfy the CE at the level given by Eq. (6) has no physical basis.

This point has been investigated by Friar and Fallieros [6], who have developed new electric current operators in connection with the extension of the Siegert theorem for the retarded electric multipole field. As a result, new prescriptions to impose the CE constraints to the electric transverse multipoles are generated by these new currents, which, however, do not appear to be entirely satisfactory [7].

The main conclusion one can draw from the analysis presented here is that the standard procedures to impose the CE *by hand*, in calculations based on models which do not verify it, are misleading and do not ensure at all that a better or a more reasonable description of the data will be obtained.

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