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Exact solution of the bound-state Faddeev-Yakubovsky equations for one-dimensional systems with a δ-function interaction

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The Faddeev-Yakubovsky equations for four identical spinless particles in one-dimensional space interacting via δ-function potentials are solved analytically for a bound state.

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During the last decade the Faddeev-Yakubovsky (FY) equations [1] were utilized to obtain approximate wave functions of various four-body systems [2–10]. However, due to the complexity of these equations, to check the validity of the approximations used is not an easy task. Thus it is important to investigate the exact solutions of the FY equations with model two-body forces. In spite of the fact that FY equations are necessary to solve uniquely quantum four-body scattering and reaction problems, the Faddeev method has become increasingly popular also in bound-state calculations.

In our recent paper [11] we derived the FY wave function for any excitation and arbitrary value of the total angular momentum, for four identical particles in three-dimensional space interacting via harmonic oscillator two-body forces. Up to now in the literature there are no solvable examples of the FY equations for four particles interacting through a short range pair potential. That is why it is interesting to find an exact solution of the bound-state FY equations for four identical spinless particles in one-dimensional space interacting via δ-function potentials. The Schrödinger equation in this case is exactly soluble. The bound and scattering states for these systems have been found by McGuire [12] and by Yang [13]. In [14] it was shown that the three-body Faddeev equations, in this case, are exactly solvable.

In this paper we show that the FY equations are exactly solvable for the bound state of a one-dimensional system of four identical particles connected via δ-function two-body forces. We do it using the general approach of Ref. [11].

In the case of four identical particles we have only two independent FY components $\Psi_{(a_3,a_2)}$ and $\Psi_{(b_3,b_2)}$, corresponding to partitions $a_2 = (1,2,3)(4)$, $b_2 = (1,2)(3,4)$, and $a_3 = b_3 = (1,2)(3)(4)$, which can be represented as follows [11]:

$$\Psi_{(a_3,a_2)} = \frac{1}{E-H_0-V_{12}} V_{12} \frac{1}{E-H_0}(V_{13}+V_{23})\Psi \ , \quad (1)$$

$$\Psi_{(b_3,b_2)} = \frac{1}{E-H_0-V_{12}} V_{12} \frac{1}{E-H_0} V_{34}\Psi \ , \quad (2)$$

where H_0 is the kinetic energy operator, Ψ is the solution of the Schrödinger equation, and V_{ij} is the potential between particles i and j

$$V_{ij} = -g\delta(r_{ij}) \ . \quad (3)$$

Let us introduce the Jacobi coordinates

$$r_{ij} = x_i - x_j, \quad z_{ij} = x_k - x_l,$$
$$y_{ij} = \tfrac{1}{2}(x_i + x_j - x_k - x_l) \ ; \quad (4)$$

here $\{i,j,k,l\}$ are four numbers forming a permutation of $\{1,2,3,4\}$ and the masses of the four identical particles are set to unity.

The only bound-state solution of the Schrödinger equation

$$\left(-\frac{\partial^2}{\partial r_{ij}^2} - \frac{\partial^2}{\partial z_{ij}^2} - \frac{1}{2}\frac{\partial^2}{\partial y_{ij}^2}\right)\Psi + V\Psi = E\Psi \ ,$$
$$V = -g\sum_{i<j}\delta(r_{ij}) \quad (5)$$

corresponds to the energy $E = -\frac{5}{2}g^2$ and has the form [12,13]

$$\langle r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}|\Psi\rangle = -e^{(g/2)(|r_{12}|+|r_{13}|+|r_{14}|+|r_{23}|+|r_{24}|+|r_{34}|)} \ . \quad (6)$$

In order to calculate the Green function $1/(E-H_0-V_{12})$ let us consider the Green function equation

$$\frac{1}{E-H_0-V_{12}} = \frac{1}{E-H_0} + \frac{1}{E-H_0}V_{12}\frac{1}{E-H_0-V_{12}} \ , \quad (7)$$

which in the momentum representation has the form

$$\langle k_{12}, q_{12}, p_{12}|\frac{1}{E-H_0-V_{12}}|k'_{12}, q'_{12}, p'_{12}\rangle$$
$$= \delta(k_{12}-k'_{12})\delta(q_{12}-q'_{12})\delta(p_{12}-p'_{12})\frac{1}{E-k_{12}^2-q_{12}^2-\frac{1}{2}p_{12}^2}$$
$$-\frac{g}{2\pi}\frac{1}{E-k_{12}^2-q_{12}^2-\frac{1}{2}p_{12}^2}\int_{-\infty}^{+\infty} dk''_{12}\langle k''_{12}, q_{12}, p_{12}|\frac{1}{E-H_0-V_{12}}|k'_{12}, q'_{12}, p'_{12}\rangle \ . \quad (8)$$

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Here k_{12}, q_{12}, and p_{12} are the Jacobi momenta connected with the Jacobi coordinates r_{12}, z_{12}, and y_{12}.
Using the notation

$$\langle k_{12},q_{12},p_{12}|\frac{1}{E-H_0-V_{12}}|k'_{12},q'_{12},p'_{12}\rangle = \delta(q_{12}-q'_{12})\delta(p_{12}-p'_{12})\mathcal{I}(q_{12},p_{12},k_{12},k'_{12}) , \tag{9}$$

we can get from (7) and (8)

$$\begin{aligned}\mathcal{I}(q_{12},p_{12},k_{12},k'_{12}) &= -\frac{\delta(k_{12}-k'_{12})}{\kappa^2+k_{12}^2+q_{12}^2+\frac{1}{2}p_{12}^2}\\ &\quad+\frac{g\sqrt{\kappa^2+q_{12}^2+\frac{1}{2}p_{12}^2}}{\pi(g-2\sqrt{\kappa^2+q_{12}^2+\frac{1}{2}p_{12}^2})}\frac{1}{(\kappa^2+k_{12}^2+q_{12}^2+\frac{1}{2}p_{12}^2)}\frac{1}{\kappa^2+k_{12}'^2+q_{12}^2+\frac{1}{2}p_{12}^2},\\ E &= -\kappa^2<0,\quad \kappa^2=\tfrac{5}{2}g^2 .\end{aligned} \tag{10}$$

Substitution of Eqs. (8), (9), and (10) into (1) and (2) gives the following expressions for the FY components:

$$\langle k_{12},q_{12},p_{12}|\Psi_{(a_3,a_2)}\rangle$$

$$= \left(\frac{g}{2\pi}\right)\int_{-\infty}^{+\infty}\mathcal{I}(q_{12},p_{12},k_{12},k'_{12})\frac{1}{(\kappa^2+k_{12}'^2+q_{12}^2+\frac{1}{2}p_{12}^2)}\langle k'_{12},q_{12},p_{12}|(V_{13}+V_{23})\Psi\rangle dk'_{12}, \tag{11}$$

$$\langle k_{12},q_{12},p_{12}|\Psi_{(b_3,b_2)}\rangle$$

$$= \left(\frac{g}{2\pi}\right)\int_{-\infty}^{+\infty}\mathcal{I}(q_{12},p_{12},k_{12},k'_{12})\frac{1}{(\kappa^2+k_{12}'^2+q_{12}^2+\frac{1}{2}p_{12}^2)}\langle k'_{12},q_{12},p_{12}|V_{34}\Psi\rangle dk'_{12} .$$

In order to calculate $\langle k'_{12},q_{12},p_{12}|(V_{13}+V_{23})\Psi\rangle$ and $\langle k'_{12},q_{12},p_{12}|V_{34}\Psi\rangle$ let us introduce

$$\begin{aligned}\chi(q_{ij},p_{ij}) &= \langle k_{ij},q_{ij},p_{ij}|V_{ij}\Psi\rangle\\ &= \frac{g}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}dz_{ij}\int_{-\infty}^{+\infty}dy_{ij}e^{-i(q_{ij}z_{ij}+p_{ij}y_{ij})}e^{-g/2[|z_{ij}|+2|(\frac{1}{2}z_{ij}+y_{ij})|+2|(\frac{1}{2}z_{ij}-y_{ij})|]}\\ &= \frac{2g}{(2\pi)^{3/2}}\left[\frac{3g^2-(q_{ij}+p_{ij}/2)p_{ij}}{[\frac{9}{4}g^2+(q_{ij}+p_{ij}/2)^2](4g^2+p_{ij}^2)}+\frac{3g^2+(q_{ij}-p_{ij}/2)p_{ij}}{[\frac{9}{4}g^2+(q_{ij}-p_{ij}/2)^2](4g^2+p_{ij}^2)}\right.\\ &\quad\left.+\frac{\frac{9}{4}g^2+(p_{ij}^2/4-q_{ij}^2)}{[\frac{9}{4}g^2+(q_{ij}+p_{ij}/2)^2][\frac{9}{4}g^2+(q_{ij}-p_{ij}/2)^2]}\right] .\end{aligned} \tag{12}$$

Using the relations

$$\begin{aligned}&k_{13}=\tfrac{1}{2}k_{12}-\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},\quad k_{23}=-\tfrac{1}{2}k_{12}-\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},\\ &q_{13}=-\tfrac{1}{2}k_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},\quad q_{23}=\tfrac{1}{2}k_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},\\ &p_{13}=k_{12}+q_{12},\quad p_{23}=-k_{12}+q_{12},\quad k_{34}=q_{12},\quad p_{34}=-p_{12},\quad q_{34}=k_{12}\end{aligned} \tag{13}$$

and (12) we can get from (11) the following expressions for the FY components:

$$\langle k_{12},q_{12},p_{12}|\Psi_{(a_3,a_2)}\rangle$$

$$\begin{aligned}&= \frac{g}{(2\pi)}\int_{-\infty}^{+\infty}\mathcal{I}(q_{12},p_{12},k_{12},k'_{12})\frac{1}{\kappa^2+k_{12}'^2+q_{12}^2+\frac{1}{2}p_{12}^2}[\chi(-\tfrac{1}{2}k'_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},k'_{12}+q_{12})\\ &\quad+\chi(\tfrac{1}{2}k'_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},-k'_{12}+q_{12})]dk'_{12} ,\end{aligned} \tag{14}$$

$$\langle k_{12},q_{12},p_{12}|\Psi_{(b_3,b_2)}\rangle = \frac{g}{(2\pi)}\int_{-\infty}^{+\infty}\mathcal{I}(q_{12},p_{12},k_{12},k'_{12})\frac{1}{\kappa^2+k'^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12}}\chi(k'_{12},p_{12})dk'_{12}\ . \tag{15}$$

It is very easy to see that integrals (14) and (15) can be calculated analytically. Indeed, substitution of (10) and (12) into (14) and (15) gives

$$\begin{aligned}\langle k_{12},q_{12},p_{12}|\Psi_{(a_3,a_2)}\rangle = &-\frac{g}{(2\pi)}\frac{1}{(\kappa^2+k^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})^2}[\chi(-\tfrac{1}{2}k_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},k_{12}+q_{12})\\ &+\chi(\tfrac{1}{2}k_{12}+\tfrac{1}{2}q_{12}+\tfrac{1}{2}p_{12},-k_{12}+q_{12})]\\ &+\frac{g^2}{2\pi^2}\frac{\sqrt{\kappa^2+q^2_{12}+\frac{1}{2}p^2_{12}}}{(g-2\sqrt{\kappa^2+q^2_{12}+\frac{1}{2}p^2_{12}})}\frac{\eta_1(q_{12},p_{12})}{(\kappa^2+k^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})}\ ,\end{aligned} \tag{16}$$

$$\begin{aligned}&\langle k_{12},q_{12},p_{12}|\Psi_{(b_3,b_2)}\rangle\\ &=-\frac{g}{(2\pi)}\frac{\chi(k_{12},p_{12})}{(\kappa^2+k^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})^2}+\frac{g^2}{2\pi^2}\frac{\sqrt{\kappa^2+q^2_{12}+\frac{1}{2}p^2_{12}}}{(g-2\sqrt{\kappa^2+q^2_{12}+\frac{1}{2}p^2_{12}})}\frac{\eta_2(q_{12},p_{12})}{(\kappa^2+k^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})}\ .\end{aligned} \tag{17}$$

Here

$$\eta_1(q_{12},p_{12}) = \int_{-\infty}^{+\infty}\frac{\chi(\frac{1}{2}k'_{12}+\frac{1}{2}q_{12}+\frac{1}{2}p_{12},-k'_{12}+q_{12})+\chi(-\frac{1}{2}k'_{12}+\frac{1}{2}q_{12}+\frac{1}{2}p_{12},k'_{12}+q_{12})}{(\kappa^2+k'^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})^2}dk'_{12} \tag{18}$$

and

$$\eta_2(q_{12},p_{12}) = \int_{-\infty}^{+\infty}\frac{\chi(k'_{12},p_{12})}{(\kappa^2+k'^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})^2}dk'_{12}\ . \tag{19}$$

Integrals (18) and (19) have the structure

$$\int_{-\infty}^{+\infty}\frac{g_n(x)}{h_n(x)h_n(-x)}dx\ , \tag{20}$$

where

$$\begin{aligned}g_n(x) &= b_0x^{2n-2}+b_1x^{2n-4}+\cdots+b_{n-1},\\ h_n(x) &= a_0x^{n-1}+a_1x^{n-2}+\cdots+a_n\ .\end{aligned} \tag{21}$$

One can use therefore the result of [15]

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{g_n(x)dx}{h_n(x)h_n(-x)} = \frac{\pi i M_n}{a_0\Delta_n},\\ &\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0\\ a_0 & a_2 & a_4 & \cdots & 0\\ 0 & a_1 & a_3 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & a_n\end{vmatrix},\quad M_n = \begin{vmatrix} b_0 & b_1 & \cdots & b_{n-1}\\ a_0 & a_2 & \cdots & 0\\ 0 & a_1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & a_n\end{vmatrix}\ .\end{aligned} \tag{22}$$

In order to illustrate it let us now evaluate, for example, the FY component $\Psi_{(b_3,b_2)}$ given by Eqs. (17) and (19). The evaluation of the FY component $\Psi_{(a_3,a_2)}$ given by Eqs. (16) and (18) could be done in exactly the same way.

We can rewrite (19) as

$$\begin{aligned}\eta_2(q_{12},p_{12}) = &\frac{g^3}{(2\pi)^{3/2}(4g^2+p^2_{12})}\\ &\times\int_{-\infty}^{+\infty}\frac{(45g^2+4k'^2_{12}+5p^2_{12})}{[\frac{9}{4}g^2+(k'_{12}+p_{12}/2)^2][\frac{9}{4}g^2+(k'_{12}-p_{12}/2)^2](\kappa^2+k'^2_{12}+q^2_{12}+\frac{1}{2}p^2_{12})^2}dk'_{12}\ .\end{aligned} \tag{23}$$

Comparison of (23) and (20) gives

$$g_4(k'_{12}) = 4k'^2_{12} + (5p^2_{12} + 45g^2),$$
$$h_4(k'_{12}) = \left(k'_{12} - i\sqrt{\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12}}\right)^{2}\left(k'_{12} - \frac{p_{12}}{2} - \tfrac{3}{2}ig\right)\left(k'_{12} + \frac{p_{12}}{2} - \tfrac{3}{2}ig\right) . \tag{24}$$

Using (24), (22), and (17) we obtain

$$\begin{aligned}\langle k_{12}, q_{12}, p_{12}|\Psi_{(b_3,b_2)}\rangle = &-\frac{g}{2\pi}\frac{\chi(k_{12},p_{12})}{(\kappa^2 + k^2_{12} + q^2_{12} + \frac{1}{2}p^2_{12})^2}\\ &+\left(\frac{g^2}{2\pi}\right)^{5/2}\frac{\sqrt{\kappa^2 + q^2_{12} + \frac{1}{2}p^2_{12}}}{(g - 2\sqrt{\kappa^2 + q^2_{12} + \frac{1}{2}p^2_{12}})(\kappa^2 + k^2_{12} + q^2_{12} + \frac{1}{2}p^2_{12})}\\ &\times\frac{(45g^2 + 5p^2_{12})[(\tilde{a}_3 - \tilde{a}_1\tilde{a}_2)/\tilde{a}_4] - 4\tilde{a}_1}{(\tilde{a}^2_3 + \tilde{a}^2_1\tilde{a}_4 - \tilde{a}_1\tilde{a}_2\tilde{a}_3)(4g^2 + p^2_{12})} ,\end{aligned} \tag{25}$$

where

$$\begin{aligned}\tilde{a}_1 &= 3g + 2\sqrt{\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12}},\\ \tilde{a}_2 &= \left(\frac{p^2}{4} + \frac{9}{4}g^2\right) + 6\sqrt{\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12}}\\ &\quad +\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12},\\ \tilde{a}_3 &= 2\sqrt{\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12}}\left(\frac{9}{4}g^2 + \frac{p^2_{12}}{4}\right)\\ &\quad +3g(\kappa^2 q^2_{12} + \tfrac{1}{2}p^2_{12}),\\ \tilde{a}_4 &= (\kappa^2 + q^2_{12} + \tfrac{1}{2}p^2_{12})\left(\frac{9}{4}g^2 + \frac{p^2_{12}}{4}\right) .\end{aligned} \tag{26}$$

In conclusion, we have found analytically solutions of the four-body Faddeev-Yakubovsky equations in the case of four identical spinless particles in one-dimensional space interacting via a δ-function two-body potential. We have shown that the FY components in the momentum space have simple momentum dependence and are elementary algebraic functions. The FY components in the coordinate space could be found by the three-dimensional Fourier transformation, which, however, though feasible, is rather tedious and is expected to lead to a complicated answer. It therefore will not be performed here. As we expected [11] the FY components in this case have no oscillations and no arbitrary constants, in contrast to the FY components corresponding to four particles interacting via harmonic-oscillator two-body potentials.

The δ-function potential example gives a unique possibility to check the validity of the different approximations of the FY equation for four particles interacting via short-range potentials. Note that in the same manner, in principle, it is possible to evaluate the five-body FY components.

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