Baryon mapping of quark systems

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We discuss a mapping procedure from a space of colorless three-quark clusters into a space of elementary baryons and illustrate it in the context of a three-color extension of the Lipkin model recently developed. Special attention is addressed to the problem of the formation of unphysical states in the mapped space. A correspondence is established between quark and baryon spaces and the baryon image of a generic quark operator is defined both in its Hermitian and non-Hermitian forms. Its spectrum (identical in the two cases) is found to consist of a physical part containing the same eigenvalues of the quark operator in the cluster space and an unphysical part consisting only of zero eigenvalues. A physical subspace of the baryon space is also defined where the latter eigenvalues are suppressed. The procedure discussed is quite general and applications of it can be thought also in the correspondence between systems of $2n$ fermions and n bosons.

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I. INTRODUCTION

Among the QCD-inspired quark models of baryons, nonrelativistic constituent models have attracted considerable attention in recent years [1]. Here, baryons are assumed to be clusters of three quarks, each of them carrying color, spin, and isospin and interacting via a potential whose main terms are a confining and a hyperfine term. These models have provided interesting results in the description of single baryon properties and, based on that, attempts have been made to extend their application to the study of the baryon-baryon interaction [2] as well as of few-baryon bound systems [3].

It is in this framework that mapping techniques traditionally developed in nuclear physics for the description of collective excitations and establishing a correspondence between systems of $2n$ fermions and n bosons [4] have been recently extended to the correspondence between systems of $3n$ fermions and n fermions. More precisely, it is the mapping of three-quark clusters onto "elementary baryons," namely, fermions carrying the same quantum numbers as the clusters, which has become the object of investigation.

Recent works on this subject have been those of Nadjakov in 1990 [5], Pittel, Engel, Dukelsky and Ring in 1990 [7], and Meyer in 1991 [8]. Although different among themselves, these procedures all have a common point: they follow the Belyaev-Zelevinsky method which is that based on the mapping of the operators in such a way that their commutation relations are preserved [9]. More particularly, it is the Dyson mapping [11] or generalizations of it which they employ. Applications of these procedures can be found within the so-called quark nuclear-plasma model of Nadjakov [5] as well as within the so-called Bonn quark shell model of Petry *et al.* [12], in Ref. [7].

A quite interesting scenario appeared in the more recent paper (1994) of Pittel, Arias, Dukelsky, and Frank [6] (hereafter referred to as PADF). Here, the authors have developed a three-color extension of the so-called Lipkin-Meshkov-Glick model [13] which has been widely used in the past as a testing bench for nuclear many-body approximations. The quark Hamiltonian of the model includes one-body, two-body, and three-body interactions and, as for the model in its original form, group theoretical techniques have been developed for an exact solution of its eigenvalue problem.

By following also in this case the Belyaev-Zelevinsky method, PADF have developed a new mapping procedure which has been found able to overcome some limitations evidenced in the previous approaches [6]. Among these limitations, for instance, is the "preference" of these approaches toward special forms of Hamiltonians. The image of the Lipkin Hamiltonian has been constructed by PADF in both a Hermitian and a non-Hermitian form and, in both cases, all the original quark eigenvalues have been exactly reproduced in the baryon space. However, besides these eigenstates, all with a corresponding one in the quark cluster space, several other states have appeared which are a pure artifact of the mapping procedure. It is the mixing in the spectrum of these "physical" and "unphysical" states which has been analyzed by PADF.

In 1991, Catara and Sambataro [14] (hereafter referred to as CS) have proposed a mapping procedure which is different from those discussed so far in that it does not follow the Belyaev-Zelevinsky method. The starting point of the procedure has been a "simple" (as will also be discussed later) correspondence between a space of quark clusters and a space of elementary baryons. Therefore the baryon image of a generic quark operator has been constructed such that all the eigenvalues of the quark operator in the cluster space were also eigenvalues of its image. This does not imply that corresponding matrix elements of the quark operator in the cluster space and of its image must be equal. However, a further correspondence has also been established between quark and baryon spaces such that matrix elements were indeed preserved as within the so-called Marumori approach [10].

This procedure has been first applied to a realistic Hamiltonian of Oka and Yazaki [15] and the derived nucleon-nucleon Hamiltonian analyzed [14]. As a second application [16], the authors have derived the nucleon image of the one-body quark density operator and expectation values of this operator have been calculated in the ground state of doubly magic nuclei like ^{4}He, ^{16}O, and ^{40}Ca described within the nuclear shell model. This has allowed an analysis of quark exchange effects on the quark densities of these nuclei.

As also evidenced by CS, the realizations just discussed have referred to cases in which corresponding states were forming a set of linearly independent states on one side, the *composite* space, and a set of orthonormal states on the other side, the *elementary* space. This has created the conditions for the nonappearance of unphysical states in the mapped space. In circumstances different from these, the appearance of these states would have made the mapping considerably more complicated and a study of this problem was left to future developments of the theory.

The three-color extension of the Lipkin model proposed by PADF has offered the opportunity of investigating this problem thoroughly. The mapping procedure of CS has now been reviewed with reference to the new model. After establishing a "simple" correspondence (as in CS) between the spaces of three-quark clusters and of elementary baryons, the baryon image of a quark operator has been defined, in both its Hermitian and non-Hermitian forms, and its spectrum analyzed. As a general result, this spectrum (identical in the two cases) has been found to consist of (a) a physical part whose eigenvalues are identical to those of the quark operator in the cluster space and (b) an unphysical part whose eigenvalues are all zero. Moreover, a further correspondence between quark and baryon spaces has been established such as to guarantee the equality of corresponding matrix elements and so a physical baryon subspace has been defined.

As an important point, we remark that the procedure which is discussed in this paper has been developed in a quite general form so that applications of it can be considered for very different cases like, for instance, the correspondence between systems of $2n$ fermions and n bosons.

The paper is organized as follows. In Sec. II, we briefly review the three-color extension of the Lipkin model developed by PADF. In Sec. III, we discuss the mapping procedure, and, precisely, in Sec. III A, we establish the correspondence between the quark and baryon spaces; in Sec. III B, we derive the baryon image of a quark operator in its non-Hermitian form; in Sec. III C, we derive the image in its Hermitian form. In Sec. IV, we discuss the n-body structure of the image operator and consider, as an example, the Hamiltonian of the Lipkin model. Finally, in Sec. V, we summarize the results and give some closing remarks.

II. THE THREE-COLOR LIPKIN MODEL

As anticipated in the Introduction, the three-color Lipkin model has been presented and discussed thoroughly by PADF. Here, we will briefly review its main points.

The model is a natural extension of the standard Lipkin model [13] to fermions characterized by three colors. Therefore, there are two levels, each one 3Ω-fold degenerate, separated by an energy Δ. Each single-particle state in these levels is characterized by three quantum numbers: c, the color, σ, which individuates whether the state belongs to the level "up" ($\sigma=+$) or "down" ($\sigma=-$), and, finally, p, which runs from 1 up to Ω. In the unperturbed ground state, it is assumed that $N=3\Omega$ particles occupy all the single-particle states in the lower level.

The model is discussed in second quantized form and so creation and annihilation operators $q^\dagger_{c\sigma p}$ and $q_{c\sigma p}$ are introduced. These satisfy the fermion commutation relations

$$\{q^\dagger_{c\sigma p},q^\dagger_{c'\sigma' p'}\}=\{q_{c\sigma p},q_{c'\sigma' p'}\}=0, \tag{1}$$

$$\{q_{c\sigma p},q^\dagger_{c'\sigma' p'}\}=\delta_{c,c'}\delta_{\sigma,\sigma'}\delta_{p,p'} \tag{2}$$

where $\{A,B\}=AB+BA$. The model Hamiltonian includes one-body, two-body, and three-body interactions and scatters particles among the levels without changing the p values and maintaining all states "colorless" (as will be pointed out in the next section). Its form is

$$\hat{H}_C=\hat{H}_C^{(1)}+\hat{H}_C^{(2)}+\hat{H}_C^{(3)}, \tag{3}$$

with

$$\hat{H}_C^{(1)}=\frac{\Delta}{2}\sum_{cp}(q^\dagger_{c+p}q_{c+p}-q^\dagger_{c-p}q_{c-p}), \tag{4}$$

$$\begin{aligned}\hat{H}_C^{(2)}=-\frac{\chi_2}{\Omega}\sum_{c_1c_2c_3c_4c_5p_1p_2}&\epsilon_{c_1c_2c_3}\epsilon_{c_1c_4c_5}\\ &\times(q^\dagger_{c_2+p_1}q^\dagger_{c_3+p_2}q_{c_5-p_2}q_{c_4-p_1}\\ &+q^\dagger_{c_4-p_1}q^\dagger_{c_5-p_2}q_{c_3+p_2}q_{c_2+p_1}),\end{aligned} \tag{5}$$

and

$$\begin{aligned}\hat{H}_C^{(3)}=-\frac{\chi_3}{\Omega^2}\sum_{c_1c_2c_3c_4c_5c_6p_1p_2p_3}&\epsilon_{c_1c_2c_3}\epsilon_{c_4c_5c_6}\\ &\times(q^\dagger_{c_1+p_1}q^\dagger_{c_2+p_2}q^\dagger_{c_3+p_3}q_{c_6-p_3}q_{c_5-p_2}q_{c_4-p_1}\\ &+q^\dagger_{c_4-p_1}q^\dagger_{c_5-p_2}q^\dagger_{c_6-p_3}q_{c_3+p_3}q_{c_2+p_2}q_{c_1+p_1}),\end{aligned} \tag{6}$$

where $\epsilon_{c_1c_2c_3}$ is the totally antisymmetric tensor of rank 3.

III. BARYON MAPPING OF QUARK OPERATORS

A. Quark and baryon spaces: The correspondence

Let us define

$$F^\dagger_{\mu_1\mu_2\mu_3}=\frac{1}{6}\sum_{c_1c_2c_3}\epsilon_{c_1c_2c_3}q^\dagger_{c_1\mu_1}q^\dagger_{c_2\mu_2}q^\dagger_{c_3\mu_3} \tag{7}$$

(where $q^\dagger_{c\mu}\equiv q^\dagger_{c\sigma p}$) the operator which creates a colorless cluster of three particles (the "quarks") characterized by the quantum numbers $\sigma_1p_1,\sigma_2p_2,\sigma_3p_3$. This operator is symmetric with respect to the indices 1,2,3. We define $C^{(\Omega)}$ the vector space spanned by the states which are obtained by acting with Ω cluster creation operators on a vacuum state $|0\rangle$,

$$F^\dagger_{\mu_1^{(1)}\mu_2^{(1)}\mu_3^{(1)}}F^\dagger_{\mu_1^{(2)}\mu_2^{(2)}\mu_3^{(2)}}\cdots F^\dagger_{\mu_1^{(\Omega)}\mu_2^{(\Omega)}\mu_3^{(\Omega)}}|0\rangle, \tag{8}$$

the vacuum being defined by the condition

$$q_{c\mu}|0\rangle=0. \tag{9}$$

In the following, we will discuss in some detail the cases $\Omega=1$ and 2. The second one is particularly interesting since the associated mapping is representative of all cases with larger Ω.

States of $C^{(1)}$ are

$$|\mu_1\mu_2\mu_3\rangle \equiv F^\dagger_{\mu_1\mu_2\mu_3}|0\rangle \tag{10}$$

and their overlap is

$$\langle\mu_1\mu_2\mu_3|\mu_1'\mu_2'\mu_3'\rangle = \tfrac{1}{6}S(\mu_1\mu_2\mu_3,\mu_1'\mu_2'\mu_3') \tag{11}$$

where

$$S(\mu_1\mu_2\mu_3,\mu_1'\mu_2'\mu_3') = \sum_{i,j,k=1}^{3} |\epsilon_{ijk}|\delta_{\mu_1,\mu_i'}\delta_{\mu_2,\mu_j'}\delta_{\mu_3,\mu_k'}. \tag{12}$$

States of $C^{(2)}$ are

$$|i\rangle = F^\dagger_{\mu_1^{(i)}\mu_2^{(i)}\mu_3^{(i)}}F^\dagger_{\mu_{1'}^{(i)}\mu_{2'}^{(i)}\mu_{3'}^{(i)}}|0\rangle \tag{13}$$

and, differently from the previous case, they are neither orthogonal nor linearly independent. By constructing the overlap matrix and diagonalizing it, one finds indeed $\bar{N}=20$ orthonormal states to be compared with the total number of $N=52$ states.

In correspondence with the quark cluster operator $F^\dagger_{\mu_1\mu_2\mu_3}$ (7), let us define the "baryon" operator $f^\dagger_{\mu_1\mu_2\mu_3}$, symmetric with respect to the three indices and satisfying the commutation relations

$$\{f^\dagger_{\mu_1\mu_2\mu_3}, f^\dagger_{\mu_1'\mu_2'\mu_3'}\} = \{f_{\mu_1\mu_2\mu_3}, f_{\mu_1'\mu_2'\mu_3'}\} = 0, \tag{14}$$

$$\{f_{\mu_1\mu_2\mu_3}, f^\dagger_{\mu_1'\mu_2'\mu_3'}\} = \tfrac{1}{6}S(\mu_1\mu_2\mu_3,\mu_1'\mu_2'\mu_3'), \tag{15}$$

where the function $S(\mu_1\mu_2\mu_3,\mu_1'\mu_2'\mu_3')$ is defined in Eq. (12). We call $E^{(\Omega)}$ the vector space spanned by the states which are generated by the action of Ω of these operators on a vacuum $|0)$

$$f^\dagger_{\mu_1^{(1)}\mu_2^{(1)}\mu_3^{(1)}}f^\dagger_{\mu_1^{(2)}\mu_2^{(2)}\mu_3^{(2)}}\cdots f^\dagger_{\mu_1^{(\Omega)}\mu_2^{(\Omega)}\mu_3^{(\Omega)}}|0). \tag{16}$$

A "simple" (see also [14]) correspondence can be established between the states of $C^{(\Omega)}$ and $E^{(\Omega)}$. States (16) can be, in fact, formally obtained from states (8) by simply replacing cluster creation operators $F^\dagger_{\mu_1^{(i)}\mu_2^{(i)}\mu_3^{(i)}}$ with baryon creation operators $f^\dagger_{\mu_1^{(i)}\mu_2^{(i)}\mu_3^{(i)}}$ and the quark vacuum $|0\rangle$ with the baryon vacuum $|0)$. In correspondence with the state (10) of $C^{(1)}$, for instance, we have for $E^{(1)}$

$$|\mu_1\mu_2\mu_3) \equiv f^\dagger_{\mu_1\mu_2\mu_3}|0) \tag{17}$$

and, similarly, in correspondence with the state (13) of $C^{(2)}$, we have for $E^{(2)}$

$$|i) = f^\dagger_{\mu_1^{(i)}\mu_2^{(i)}\mu_3^{(i)}}f^\dagger_{\mu_{1'}^{(i)}\mu_{2'}^{(i)}\mu_{3'}^{(i)}}|0). \tag{18}$$

An important feature of these states is their orthogonality. This orthogonality, on one side, and the linear dependence of the corresponding states of $C^{(\Omega)}$ (at least for $\Omega>1$), on the other side, clearly reflect the "elementary" or "composite" nature of the baryons entering in the definitions of the spaces $E^{(\Omega)}$ and $C^{(\Omega)}$, respectively, and are associated with the different commutator algebras of the operators $f^\dagger_{\mu_1\mu_2\mu_3}$ and $F^\dagger_{\mu_1\mu_2\mu_3}$.

Results similar to those just discussed in the correspondence between spaces $C^{(\Omega)}$ and $E^{(\Omega)}$ for $\Omega=2$ also hold for larger spaces ($\Omega=3,4,\ldots$). Also in these cases, in fact, corresponding states while forming an orthogonal set in $E^{(\Omega)}$ are a set of linearly dependent states in $C^{(\Omega)}$.

In the next section, we will describe a procedure aiming to derive the image in $E^{(\Omega)}$ of a generic operator acting within $C^{(\Omega)}$. For simplicity, we will refer to the case $\Omega=2$ and, therefore, to the correspondence between the states (13) and (18). However, the value of Ω having no relevance (only $\Omega>1$), if not for the difficulty in actual calculations, in the following we will suppress the indication of Ω.

B. The image operator and its physical and unphysical eigenstates: Its non-Hermitian form

On the basis of what has been said at the end of the last subsection, let C be the vector space spanned by the N states $\{|1\rangle,|2\rangle,\ldots,|N\rangle\}$ defined by Eq. (13). Similarly, let E be the vector space spanned by the N states $\{|1),|2),\ldots,|N)\}$ defined by Eq. (18). With respect to this last equation, we are only supposing that these states have been normalized so that they now satisfy the condition

$$(i|j) = \delta_{i,j}, \quad \forall i,j=1,2,\ldots,N. \tag{19}$$

In this section, in correspondence with a Hermitian operator $\hat{O}_C$ acting within C, we will search for an operator $\hat{O}_E$ acting within E such that all the eigenvalues of $\hat{O}_C$ in C be also eigenvalues of $\hat{O}_E$ in E. We will refer to $\hat{O}_E$ as the image operator of $\hat{O}_C$ in E. This operator will be first derived in its non-Hermitian form $\hat{O}_E^{(\mathrm{nH})}$. The study of the Hermitian image $\hat{O}_E^{(H)}$ will then be reserved to the next subsection.

Let $\bar{N}$ be the number of orthonormal states which can be constructed in terms of the N states $|i\rangle$: therefore the space C is $\bar{N}$-dimensional. In Sec. III A we have already seen that it is $N=52$ and $\bar{N}=20$ in the case $\Omega=2$. In all coming equations of this subsection as well as of the next one, indices written in terms of the latin letters $k,i,j,\ldots$ will be meant to vary in the interval $(1,N)$ while those written in terms of the greek letters $\alpha,\beta,\gamma,\ldots$ will be meant to vary in the interval $(1,\bar{N})$.

Each orthonormal state within C, $\overline{|\alpha\rangle}$, is a linear combination of states $|i\rangle$ which we write as

$$\overline{|\alpha\rangle} = \frac{1}{\sqrt{\mathcal{N}_\alpha}}\sum_i f_{i\alpha}|i\rangle \equiv \sum_i \bar{f}_{i\alpha}|i\rangle, \tag{20}$$

where $f_{i\alpha}$ and $\mathcal{N}_\alpha$ are real quantities satisfying the equations

$$\sum_l \langle i|l\rangle f_{lj} = \mathcal{N}_j f_{ij}, \tag{21}$$

$$\sum_i f_{ij} f_{ij'} = \delta_{jj'}, \tag{22}$$

and

$$\sum_j f_{ij} f_{i'j} = \delta_{ii'}. \tag{23}$$

The identity operator within C is

$$\hat{I}_C = \sum_\alpha \overline{|\alpha\rangle}\, \overline{\langle\alpha|} \tag{24}$$

and, by making use of Eq. (20), it can be written as

$$\hat{I}_C = \sum_{ij} |i\rangle B(i,j) \langle j| \tag{25}$$

with

$$B(i,j) = \sum_\alpha \bar{f}_{i\alpha} \bar{f}_{j\alpha}. \tag{26}$$

Only in the special case of a set of linearly independent states $(|1\rangle,|2\rangle,\ldots,|N\rangle)$ (i.e., $\bar{N}=N$), would this matrix coincide with the inverse of the overlap matrix $\langle i|j\rangle$.

We observe that, in general,

$$\hat{O}_C|i\rangle \notin C. \tag{27}$$

By defining the operator $\hat{\hat{O}}_C \equiv \hat{I}_C \hat{O}_C$, we notice that

$$\text{(i)}\ \ \hat{\hat{O}}_C|l\rangle = \sum_i |i\rangle \left\{ \sum_j B(i,j) \langle j|\hat{O}_C|l\rangle \right\}, \tag{28}$$

$$\text{(ii)}\ \ \langle i|\hat{\hat{O}}_C|l\rangle = \langle i|\hat{O}_C|l\rangle; \tag{29}$$

namely, (i) $\hat{\hat{O}}_C$ still gives rise to a state of C when acting on a state of this space and (ii) $\hat{\hat{O}}_C$ is equivalent to $\hat{O}_C$ within C.

To understand the role played by the operator $\hat{\hat{O}}_C$, let us notice that the eigenvalues λ_γ of an operator $\hat{O}_C$, within the space C, can be found by solving the system of $\bar{N}$ equations

$$\sum_\beta \overline{\langle\alpha|}\hat{O}_C\overline{|\beta\rangle} c_{\beta\gamma} = \lambda_\gamma c_{\alpha\gamma}. \tag{30}$$

By multiplying this expression on both sides by $\overline{|\alpha\rangle}$ and summing over all these states, one gets

$$\hat{\hat{O}}_C|\Psi_\gamma\rangle = \lambda_\gamma |\Psi_\gamma\rangle, \tag{31}$$

where $|\Psi_\gamma\rangle \equiv \Sigma_\alpha c_{\alpha\gamma}\overline{|\alpha\rangle}$. Therefore the states $|\Psi_\gamma\rangle$ associated with the eigenvalues λ_γ are eigenstates of the operator $\hat{\hat{O}}_C$.

Let us now turn to the space E. Here, due to the orthonormality of the states $|i)$, the identity operator is simply

$$\hat{I}_E = \sum_i |i)(i|. \tag{32}$$

Let $\hat{O}_E^{(\mathrm{nH})}$ be an operator acting within this space. Its action on a state of E is given by

$$\hat{O}_E^{(\mathrm{nH})}|l) = \sum_i |i)(i|\hat{O}_E^{(\mathrm{nH})}|l). \tag{33}$$

By comparing Eqs. (28) and (33), one sees that if, $\hat{O}_E^{(\mathrm{nH})}$ is defined such that

$$(i|\hat{O}_E^{(\mathrm{nH})}|l) = \sum_j B(i,j)\langle j|\hat{O}_C|l\rangle, \tag{34}$$

its action on states of E is formally identical to that of $\hat{\hat{O}}_C$ on the corresponding states of C. As a result of that, if the state

$$|\Psi_\gamma\rangle = \sum_\alpha c_{\alpha\gamma}\overline{|\alpha\rangle} = \sum_i \left(\sum_\alpha \bar{f}_{i\alpha} c_{\alpha\gamma} \right) |i\rangle \equiv \sum_i a_{i\gamma}|i\rangle \tag{35}$$

is an eigenstate of $\hat{\hat{O}}_C$ corresponding to the eigenvalue λ_γ, then also the state

$$|\Psi_\gamma) = \sum_i a_{i\gamma}|i) \tag{36}$$

is an eigenstate of $\hat{O}_E^{(\mathrm{nH})}$ with the same eigenvalue. Therefore $\bar{N}$ of the N eigenvalues of $\hat{O}_E^{(\mathrm{nH})}$ in E are the same as the eigenvalues of $\hat{O}_C$ in C and the associated eigenkets of $\hat{O}_E^{(\mathrm{nH})}$ are states "simply" corresponding to the eigenkets of $\hat{\hat{O}}_C$ in C. Equation (34) defines the image operator, non-Hermitian, of $\hat{O}_C$ in E.

Equation (34) recalls Eq. (23) of CS where, however, the matrix $B(i,j)$ is replaced by the inverse of the overlap matrix. As we have already seen, this can happen only in the case that $\bar{N}=N$. In this case, characterized by the fact that both $\hat{O}_C$ in C and $\hat{O}_E^{(\mathrm{nH})}$ in E would have the same number of eigenvalues, each eigenstate $|\Psi_\gamma)$ of $\hat{O}_E^{(\mathrm{nH})}$ would be in a one-to-one correspondence with an eigenstate $|\Psi_\gamma\rangle$ of $\hat{\hat{O}}_C$. Therefore, no ambiguities would exist about the "physicality" of all the eigenstates of $\hat{O}_E^{(\mathrm{nH})}$.

The case $\bar{N}<N$, instead, namely, the case under investigation, appears more complicated. In this case, in fact, only $\bar{N}$ eigenstates of $\hat{O}_E^{(\mathrm{nH})}$ in E can be in a one-to-one correspondence with eigenstates of $\hat{\hat{O}}_C$ in C and so have a physical meaning, while the remaining $N-\bar{N}$ are unphysical and only a result of the mapping procedure. In the following, we want to study these eigenstates on the basis of the definition (34) of the image operator.

Let us first define, corresponding to each state (20), the state

$$\overline{|\alpha)} = \sum_i \bar{f}_{i\alpha}|i) \tag{37}$$

and let us call $\bar{E}$ the subspace of E spanned by these states. By noticing that

$$\overline{(\alpha|\alpha')} = \frac{1}{\mathcal{N}_\alpha}\delta_{\alpha\alpha'}, \tag{38}$$

we conclude that this space is $\bar{N}$ dimensional. By multiplying both sides of Eq. (34) by the state $|i)$ and summing over all these states, one obtains

$$\sum_i |i)(i|\hat{O}_E^{(\mathrm{nH})}|l) = \sum_{ij} |i)B(i,j)\langle j|\hat{O}_C|l\rangle. \tag{39}$$

Moreover, by making use of Eqs. (20), (26), and (37), one obtains that

$$\hat{O}_E^{(\mathrm{nH})}|l) = \sum_\alpha \overline{|\alpha)}\overline{\langle\alpha}|\hat{O}_C|l\rangle \tag{40}$$

and also

$$\hat{O}_E^{(\mathrm{nH})}\overline{|\beta)} = \sum_\alpha \overline{|\alpha)}\overline{\langle\alpha}|\hat{O}_C|\overline{\beta}\rangle. \tag{41}$$

Now let $|\Psi_j)=\Sigma_l x_{lj}|l)$ be an eigenstate of $\hat{O}_E^{(\mathrm{nH})}$ with eigenvalue λ_j and $|\Psi_j\rangle=\Sigma_l x_{lj}|l\rangle$ its (nonzero) image state. Then

$$\hat{O}_E^{(\mathrm{nH})}|\Psi_j) = \sum_l x_{lj}\hat{O}_E^{(nh)}|l) = \sum_l x_{lj}\sum_\alpha \overline{|\alpha)}\overline{\langle\alpha}|\hat{O}_C|l\rangle$$
$$= \lambda_j\sum_\alpha \overline{|\alpha)}\overline{\langle\alpha}|\Psi_j\rangle \tag{42}$$

and since

$$\hat{O}_E^{(\mathrm{nH})}|\Psi_j) = \lambda_j|\Psi_j) \tag{43}$$

one can conclude that, if $\lambda_j \neq 0$,

$$|\Psi_j) = \sum_\alpha |\overline{\alpha})\ \langle\overline{\alpha}|\Psi_j\rangle. \tag{44}$$

That is, in correspondence with an eigenvalue $\lambda_j \neq 0$, an eigenket of the image operator $\hat{O}_E^{(\mathrm{nH})}$ must belong to $\bar{E}$. But it has already been seen that $\hat{O}_E^{(\mathrm{nH})}$ has the $\bar{N}$ eigenkets (36) and these can be rewritten as

$$|\Psi_\gamma) = \sum_l a_{l\gamma}|l) = \sum_\alpha c_{\alpha\gamma}\overline{|\alpha)}. \tag{45}$$

These states belong to $\bar{E}$, are linearly independent [similarly to the states (35)], and, this space being $\bar{N}$ dimensional, no extra eigenket linearly independent from these can be accepted within this space. As will be seen in Sec. III C, even in the presence of degeneracies, all the eigenstates of the non-Hermitian image $\hat{O}_E^{(\mathrm{nH})}$ in the space E must be linearly independent. Since, on the basis of Eq. (44), a nonzero eigenvalue would force its eigenstate to belong to $\bar{E}$ but this state could not be linearly independent from the previous $\bar{N}$ eigenstates (45), one can only conclude that a λ_j beyond those of these $\bar{N}$ eigenstates cannot be different from zero. One finally notices that the case of an image state $|\Psi_j\rangle$ with zero norm (we considered so far a nonzero state) corresponds to an eigenvalue $\lambda_j=0$, as can be seen from Eq. (42).

Equation (34) clearly shows that corresponding matrix elements of $\hat{O}_C$ and of its image operator $\hat{O}_E^{(\mathrm{nH})}$ are not equal. In the following, however, we will show that a correspondence can be established such that matrix elements in the quark and baryon spaces can be preserved. Due to the non-Hermiticity of $\hat{O}_E^{(\mathrm{nH})}$, eigenbras and eigenkets need not be dual vectors of one another. In order to individuate the eigenbra corresponding to each eigenket (45), let us define the bras

$$\overline{\overline{(\alpha|}} = \sum_i \bar{\bar{f}}_{i\alpha}(i|, \tag{46}$$

where

$$\bar{\bar{f}}_{i\alpha} = \sqrt{\mathcal{N}_\alpha}f_{i\alpha}. \tag{47}$$

The space spanned by these states, $\bar{\bar{E}}$, is a subspace of E^*, the dual space of E. Moreover, since

$$(\bar{\bar{\alpha}}|\bar{\bar{\alpha}}') = \mathcal{N}_\alpha\delta_{\alpha\alpha'}, \tag{48}$$

the dimension of $\bar{\bar{E}}$ is the same as that of $\bar{E}$, that is, $\bar{N}$.

An important property of bras (46) and kets (37) is that

$$\overline{\overline{(\alpha|}}\overline{\alpha')} = \delta_{\alpha\alpha'}. \tag{49}$$

It follows from this and from Eq. (40) that

$$\overline{\overline{(\alpha|}}\hat{O}_E^{(\mathrm{nH})} = \sum_l \overline{\langle\alpha}|\hat{O}_C|l\rangle(l|. \tag{50}$$

By making use of this expression one can verify that the bra

$$(\overline{\overline{\Psi}}_\gamma| = \sum_\alpha c_{\alpha\gamma}\overline{\overline{(\alpha|}} \tag{51}$$

is an eigenbra of $\hat{O}_E^{(\mathrm{nH})}$ corresponding to the eigenvalue λ_γ.

It also follows from (41) and (49) that

$$\overline{\overline{(\alpha|}}\hat{O}_E^{(\mathrm{nH})}\overline{|\beta)} = \overline{\langle\alpha}|\hat{O}_C\overline{|\beta\rangle}. \tag{52}$$

Therefore, to any basis state $\overline{|\alpha\rangle}$ (20) of C one can associate a ket $\overline{|\alpha)}$ (37) of $\bar{E}$ and a bra $\overline{\overline{(\alpha|}}$ (46) of $\bar{\bar{E}}$ such that matrix elements of $\hat{O}_C$ between basis states of C are equal to matrix elements of the image $\hat{O}_E^{(\mathrm{nH})}$ between the corresponding bra and ket of $\bar{\bar{E}}$ and $\bar{E}$. Restricting the action of $\hat{O}_E^{(\mathrm{nH})}$ within $\bar{E}$ and $\bar{\bar{E}}$, then, eliminates the unphysical zero eigenvalues which emerge from the diagonalization of $\hat{O}_E^{(\mathrm{nH})}$ in the full E.

C. The image operator: Its Hermitian form

In order to derive the Hermitian form $\hat{O}_E^{(H)}$ of the image operator defined in Eq. (34), let us first introduce the operators $\hat{B}_E^{1/2}$ and $\hat{B}_E^{-1/2}$ such that

$$(i|\hat{B}_E^{1/2}|j) \equiv B^{1/2}(i,j) \tag{53}$$

and

$$(i|\hat{B}_E^{-1/2}|j)\equiv B^{-1/2}(i,j), \tag{54}$$

where $B^{1/2}(i,j)$ and $B^{-1/2}(i,j)$ are matrices real, symmetric, and such that

$$\sum_{i'} B^{1/2}(i,i')B^{1/2}(i',j)=B(i,j) \tag{55}$$

and

$$\sum_{i'} B^{1/2}(i,i')B^{-1/2}(i',j)=\delta_{ij}. \tag{56}$$

$\hat{B}_E^{1/2}$ and $\hat{B}_E^{-1/2}$ are, therefore, Hermitian and such that

$$\hat{B}_E^{1/2}\hat{B}_E^{-1/2}=\hat{B}_E^{-1/2}\hat{B}_E^{1/2}=\hat{I}_E. \tag{57}$$

If $|\Psi_j)$ is an eigenket of $\hat{O}_E^{(\mathrm{nH})}$ associated with the eigenvalue λ_j, one deduces that

$$\hat{B}_E^{-1/2}\hat{O}_E^{(\mathrm{nH})}\hat{B}_E^{1/2}\hat{B}_E^{-1/2}|\Psi_j)=\lambda_j\hat{B}_E^{-1/2}|\Psi_j). \tag{58}$$

By defining

$$\hat{O}_E^{(H)}\equiv\hat{B}_E^{-1/2}\hat{O}_E^{(\mathrm{nH})}\hat{B}_E^{1/2} \tag{59}$$

and

$$|\tilde{\Psi}_j)\equiv\hat{B}_E^{-1/2}|\Psi_j), \tag{60}$$

Eq. (58) can be rewritten as

$$\hat{O}_E^{(H)}|\tilde{\Psi}_j)=\lambda_j|\tilde{\Psi}_j), \tag{61}$$

namely, $|\tilde{\Psi}_j)$ is an eigenket of $\hat{O}_E^{(H)}$ associated with the same eigenvalue λ_j. It can be derived that

$$(i|\hat{O}_E^{(H)}|m)=\sum_{l,k} B^{1/2}(i,l)\langle l|\hat{O}_C|k\rangle B^{1/2}(k,m) \tag{62}$$

from which one deduces that $\hat{O}_E^{(H)}$ is indeed Hermitian. This equation defines the image operator of $\hat{O}_C$ in E in its Hermitian form.

Equation (60) establishes a relation between eigenkets of $\hat{O}_E^{(\mathrm{nH})}$ and $\hat{O}_E^{(H)}$ corresponding to the same eigenvalue. The linear independence of the first ones, stated in the previous subsection and not guaranteed, in the presence of degeneracies, for a non-Hermitian operator, is forced, as an effect of this relation, by the linear independence of the eigenstates of $\hat{O}_E^{(H)}$ (obligatory for a Hermitian operator).

It has been seen in the previous subsection, Eq. (45), that eigenkets corresponding to physical eigenvalues of $\hat{O}_E^{(\mathrm{nH})}$ are combinations of states of $\bar{E}$. The state (60) becomes in this case

$$|\tilde{\Psi}_\gamma)=\sum_\alpha c_{\alpha\gamma}\hat{B}_E^{-1/2}\overline{|\alpha)}=\sum_\alpha c_{\alpha\gamma}\widetilde{|\alpha)} \tag{63}$$

where

$$\widetilde{|\alpha)}\equiv\hat{B}_E^{-1/2}\overline{|\alpha)}=\sum_j\left\{\sum_i \tilde{f}_{i\alpha}B^{-1/2}(j,i)\right\}|j). \tag{64}$$

One can verify that

$$\widetilde{(\alpha}\,\widetilde{|\alpha')}=\delta_{\alpha\alpha'} \tag{65}$$

and that

$$\widetilde{(\alpha|}\;\hat{O}_E^{(H)}\widetilde{|\alpha')}=\overline{\langle\alpha|}\hat{O}_C\overline{|\alpha'\rangle}. \tag{66}$$

Therefore, if $\tilde{E}$ is the $\bar{N}$-dimensional subspace of E spanned by the states $\{\widetilde{|1)},\widetilde{|2)},\ldots,\widetilde{|\bar{N})}\}$ defined in Eq. (64), one can say that to any basis state $\overline{|\alpha\rangle}$ (20) of C it is possible to associate a state $\widetilde{|\alpha)}$ (64) of $\tilde{E}$ such that a matrix element of $\hat{O}_C$ between basis states of C is equal to the matrix element of $\hat{O}_E^{(H)}$ between the corresponding states of $\tilde{E}$. The spectrum of $\hat{O}_E^{(H)}$ in $\tilde{E}$ contains, then, only the physical part of the spectrum of $\hat{O}_E^{(H)}$ in E. The space $\tilde{E}$ so defined individuates the physical subspace of E.

It is interesting to discuss also the case of the eigenbra $(\overline{\overline{\Psi}}_j|$ of $\hat{O}_E^{(\mathrm{nH})}$. Similarly to Eq. (58), one can write

$$(\overline{\overline{\Psi}}_j|\hat{B}_E^{1/2}\hat{B}_E^{-1/2}\hat{O}_E^{(\mathrm{nH})}\hat{B}_E^{1/2}=\lambda_j(\overline{\overline{\Psi}}_j|\hat{B}_E^{1/2}, \tag{67}$$

that is,

$$(\tilde{\overline{\overline{\Psi}}}_j|\hat{O}_E^{(H)}=\lambda_j(\tilde{\overline{\overline{\Psi}}}_j|, \tag{68}$$

where

$$(\tilde{\overline{\overline{\Psi}}}_j|\equiv(\overline{\overline{\Psi}}_j|\hat{B}_E^{1/2}. \tag{69}$$

This state is the eigenbra of $\hat{O}_E^{(H)}$ corresponding to the eigenket defined in Eq. (60). More particularly, if λ_γ is a physical eigenvalue, the eigenbra corresponding to the eigenket (63) is

$$(\tilde{\overline{\overline{\Psi}}}_\gamma|=\sum_\alpha c_{\alpha\gamma}\overline{\overline{(\alpha|}}\hat{B}_E^{1/2}. \tag{70}$$

It can be verified that

$$\overline{\overline{(\alpha|}}\hat{B}_E^{1/2}=\widetilde{(\alpha|}=\overline{(\alpha|}\;\hat{B}_E^{-1/2}, \tag{71}$$

i.e., this bra is the dual vector of the ket $\widetilde{|\alpha)}$ defined in Eq. (64). As expected, in this case, the eigenbra (70) is simply the dual vector of the eigenket (63). The two spaces $\bar{E}$ and $\overline{\overline{E}}$ defined in the previous subsection for kets and bras, respectively, are replaced here only by the spaces $\tilde{E}$ and its dual $\tilde{E}^*$.

In conclusion, the operator $\hat{O}_E^{(H)}$ (59) is a Hermitian operator whose spectrum in E is exactly the same as that of $\hat{O}_E^{(\mathrm{nH})}$ in E and so contains (a) the $\bar{N}$ physical eigenvalues of $\hat{O}_C$ in C and (b) the $N-\bar{N}$ unphysical zero eigenvalues. The physical subspace $\tilde{E}$ of E is spanned by a set of $\bar{N}$ states in a one-to-one correspondence with the basis states of C and such that corresponding matrix elements of $\hat{O}_E^{(H)}$ and $\hat{O}_C$ are equal. The spectrum of $\hat{O}_E^{(H)}$ in $\tilde{E}$ coincides with that of $\hat{O}_C$ in C.

IV. N-BODY STRUCTURE OF A BARYON OPERATOR: THE LIPKIN HAMILTONIAN

In the previous section, we have derived the matrix elements defining the baryon image of a quark operator. An important problem which we discuss in this section is that related to the n-body structure of this baryon operator. As an example, we will refer to the image of the Lipkin Hamiltonian (3).

The derivation of the image operator treated in Sec. III has referred to the case of the correspondence between the states (13) and (18), namely, the case $\Omega=2$, although, in principle, applicable also to larger values of Ω. The case $\Omega=1$, instead, the case of the correspondence between the states (10) and (17), is particularly simple a case. In this case, in fact, corresponding states have the same overlaps so that the image operator $\hat{H}_{E,1}$ of the Lipkin Hamiltonian is simply defined by the equality

$$(\mu_1\mu_2\mu_3|\hat{H}_{E,1}|\mu_1'\mu_2'\mu_3')=\langle\mu_1\mu_2\mu_3|\hat{H}_C|\mu_1'\mu_2'\mu_3'\rangle. \tag{72}$$

$\hat{H}_{E,1}$ turns out to be the one-body Hermitian operator

$$\hat{H}_{E,1}=\frac{3\Delta}{2}\sum_{p_1p_2p_3\sigma_2\sigma_3}(f^\dagger_{+p_1\sigma_2p_2\sigma_3p_3}f_{+p_1\sigma_2p_2\sigma_3p_3}-f^\dagger_{-p_1\sigma_2p_2\sigma_3p_3}f_{-p_1\sigma_2p_2\sigma_3p_3})-\frac{12\chi_2}{\Omega}\sum_{p_1p_2p_3\sigma_3}(f^\dagger_{+p_1+p_2\sigma_3p_3}f_{-p_1-p_2\sigma_3p_3}+f^\dagger_{-p_1-p_2\sigma_3p_3}f_{+p_1+p_2\sigma_3p_3})-\frac{36\chi_3}{\Omega^2}\sum_{p_1p_2p_3}(f^\dagger_{+p_1+p_2+p_3}f_{-p_1-p_2-p_3}+f^\dagger_{-p_1-p_2-p_3}f_{+p_1+p_2+p_3}). \tag{73}$$

For $\Omega=2$, let us call $\hat{\hat{H}}_{E,2}$ the Hermitian image of the Lipkin Hamiltonian which is defined by Eq. (62). One can find infinite combinations of one-body plus two-body baryon operators satisfying this equation. However, wishing the image Hamiltonian to be a good baryon image for both $\Omega=1$ and $\Omega=2$ one is forced to take

$$\hat{\hat{H}}_{E,2}=\hat{H}_{E,1}+\hat{H}_{E,2}, \tag{74}$$

where $\hat{H}_{E,1}$ has just been defined in Eq. (73) and $\hat{H}_{E,2}$ is a two-body baryon operator defined by the matrix elements

$$(i|\hat{H}_{E,2}|j)=(i|\hat{\hat{H}}_{E,2}|j)-(i|\hat{H}_{E,1}|j). \tag{75}$$

A similar procedure has to be extended to any Ω leading to the general result that

$$\hat{\hat{H}}_{E,\Omega}=\hat{H}_{E,1}+\hat{H}_{E,2}+\cdots+\hat{H}_{E,\Omega}. \tag{76}$$

As a general result, then, the image Hamiltonian is a baryon operator containing up to Ω-body terms even if $\hat{H}_C$ is at most one body. The presence of these many-body terms results from the need to simulate in the baryon space the complicated underlying quark exchange dynamics.

V. SUMMARY AND CONCLUSIONS

We have discussed a mapping procedure from a space of colorless three-quark clusters into a space of elementary baryons and illustrated it within a three-color extension of the Lipkin model. Special attention has been addressed to the problem of the formation of unphysical states in the mapped space.

The mechanism of the mapping proposed has required us, first, to establish a correspondence between the quark cluster space and the baryon space. Therefore the baryon image of a generic quark operator has been defined in both its Hermitian and non-Hermitian forms and its spectrum analyzed. As a general result, this spectrum (equal in the two cases) has been found to consist of a physical part containing the same eigenvalues of the quark operator in the cluster space and an unphysical part consisting only of zero eigenvalues. The last ones emerge as a product of the mapping mechanism.

This derivation of the baryon image has not passed through the preservation either of the commutation relations of the operators or of the matrix elements in the quark and baryon spaces. However, a further correspondence has been established between quark and baryon spaces such as to guarantee the equality of corresponding matrix elements. A physical subspace of the baryon space has been so defined.

We have examined the n-body structure of the image operator and considered, as an example, the case of the Lipkin Hamiltonian. In a correspondence involving Ω clusters, the need of up to Ω-body terms in the baryon operator, even in the case of a one-body quark operator, has been discussed and the definition of these terms provided.

With reference to the mapping procedure elaborated by PADF, we notice that the baryon Hamiltonian which has been constructed in that work, for $\Omega=2$, has been found able to reproduce the spectrum of the Lipkin Hamiltonian in the cluster space. In the cases examined, however, unphysical eigenvalues have also appeared spread all over the spectrum, in the Hermitian case, or pushed up in energy, in the non-Hermitian case. This result clearly differs from that of the present procedure, characterized by zero energy for all the unphysical eigenstates and, therefore, by a better definite separation between physical and unphysical eigenstates of the mapped Hamiltonian.

Although explicitly referring to the correspondence between systems of $3n$ fermions and n fermions, an important aspect of the mapping procedure discussed in this paper consists in its applicability to quite different scenarios like, for instance, the correspondence between systems of $2n$ fermions and n bosons. This can be clearly noticed in Secs. III B and III C where the formalism of the procedure has been kept quite general just on purpose. This ductility, together with its simplicity, makes this procedure available for the most various applications.

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