Poincaré invariant coupled channel model for the pion-nucleon system

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An exactly Poincaré invariant front form model for the pion-nucleon system is constructed in the space spanned by $|B\rangle$ and $|\mu B\rangle$ states where $B=N,\Delta,N^*,\ldots$ and $\mu=\pi,\eta,\rho,\ldots$. A mass-square operator is constructed in the form $M^2=M_0^2+V$ where M_0 is a noninteracting mass operator and V is an interaction. Assuming that the spin operator is a free spin operator, the most general form for the interaction V is deduced. A fit to the πN elastic scattering amplitudes for pion laboratory kinetic energies up to 700 MeV is carried out, under the assumption that the μB-$\mu' B'$ interactions are separable.

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I. INTRODUCTION

Coupled channel models play an important role in intermediate energy physics, and in fact are essential for treating systems such as the pion-nucleon system and the nucleon-nucleon system. Quantum chromodynamics has taught us that objects such as the $\Delta'(1232)$ are just as elementary as the nucleon; so, for example, in the πN system it is important to include a $\pi\Delta$ channel, while in the NN system it is important to include the $N\Delta$ channel.

The πN model of Blankleider and Walker [1] assumes πN and $\pi\Delta$ channels, while the model of Bhalerao and Liu [2] includes these channels as well as an ηN channel. A multiresonance approach to coupled channel inelastic πN scattering, based on the K matrix, has been developed by Manley and Saleski [3]. A three channel model has recently been used by Batinić *et al.* [4] to fit πN elastic scattering amplitudes, as well as $\pi N\rightarrow\eta N$ total and differential cross sections. A manifestly covariant coupled channel model of the πN system has been constructed by Gross and Surya [5].

The excitation of the nucleon to the Δ state has long been recognized as an important part of the NN interaction [6]. Several authors have constructed coupled channel models of the NN system that take into account the $N\Delta$ channel, and in some instances the $\Delta\Delta$ channel as well [7–9].

Coupled channel models that treat relativity exactly are not all that common. The πN model of Gross and Surya [5] does, since it is manifestly covariant. It is possible to construct exactly Poincaré invariant models using the various *forms* [10] of relativistic quantum mechanics. An instant form model of the $NN\pi$ system, which takes as the elementary degrees of freedom the N, Δ, and π, was formulated some time ago by Betz and Coester [11], and applied by Betz and Lee [12]. An analytically solvable front form model of the $NN\pi$ system is given in the review article of Keister and Polyzou [13].

Here we will construct an exactly Poincaré invariant, front form model of the pion-nucleon system in the space spanned by $|B\rangle$ and $|\mu B\rangle$ states where $B=N,\Delta,N^*,\ldots$ and $\mu=\pi,\eta,\rho,\ldots$. The model we will construct can be called a *free spin model* in that we will assume that the relative angular momentum or spin operator $\mathscr{J}$ is the same as that of a noninteracting system. Even though the model will be constructed in a specific context, it will become clear that the procedure used is quite general; and could be used, for example, to construct an exactly Poincaré invariant model of the NN system in the space spanned by $|NN\rangle$ and $|N\Delta\rangle$ states.

It should be noted that the model assumption of a noninteracting spin operator $\mathscr{J}$ is not in contradiction with the fact that a front form angular momentum operator $\mathbf{J}$ contains interactions. The relation between $\mathbf{J}$, which is the generator of three-rotations, and $\mathscr{J}$ involves the mass operator [13]; so even in the model presented here $\mathbf{J}$ is interacting. There is some indication [14] that it is possible in general to transform a model with an interacting spin operator to one whose spin operator is noninteracting, so it might turn out that the assumption of a noninteracting $\mathscr{J}$ is not a model assumption after all.

The use of the front form in formulating the coupled channel model is pretty much a question of taste. In fact a mass operator with the same functional form in the internal or relative variables could also be developed in the instant and point forms of relativistic quantum mechanics [13]. It is only when the system treated here is probed or embedded in a larger system that the differences between the various forms become important.

The outline of the paper is as follows. The basis states for the πN model are constructed in Sec. II. As pointed out above, our vector space consists of both single particle and two particle states. It is important to specify these states precisely as the behavior of these states under Poincaré transformations plays an essential role in determining the representations of the operators that come into play. In Sec. II we establish the existence of a basis in which the spin operator $\mathscr{J}$ has a simple representation. This simple representation facilitates the construction of a Poincaré invariant model. In Sec. III a Poincaré invariant, mass-square operator is constructed in the form $M^2=M_0^2+V$ where M_0 is a noninteracting mass operator and V is the interaction. Assuming that the space in which the mass operator acts is spanned by $|B\rangle$ and $|\mu B\rangle$ states, and that the spin operator $\mathscr{J}$ is a free spin operator, the most general form for V is deduced. The Lippmann-Schwinger equations that this model gives rise to are analyzed in Sec. IV. Section V gives the results of a separable potential model fit to the πN elastic scattering amplitudes for pion laboratory kinetic energies up to 700 MeV. Here $|N\rangle$, $|\Delta\rangle$, $|\pi N\rangle$, $|\eta N\rangle$, and $|\pi\Delta\rangle$ channels are included.

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In Sec. VI the differences that would arise in developing a coupled channel model in the instant and point forms are pointed out, and a discussion of the results and suggestions for future work are given.

II. BASIS STATES

A Poincaré transformation maps the components of a four-vector from the x frame to the x' frame according to

$$x' = ax + b, \tag{2.1}$$

where a is a Lorentz transformation and b is a spacetime translation. A quantum mechanical state vector in the x frame is mapped to the x' frame according to

$$|\psi'\rangle = U(a,b)|\psi\rangle, \tag{2.2}$$

where $U(a,b)$ is a unitary operator. The set of all (a,b) forms the Poincaré group, while the $U(a,b)$ form a unitary representation of the group.

Basis states that transform simply when acted on by the unitary operators $U(a,b)$ can be obtained by starting with states that transform according to an irreducible representation of SU(2), and then boosting these so-called rest frame states [13]. We will denote Lorentz boosts by $l_g(\lambda)$ where λ is a timelike unit vector, and g distinguishes the different types of boosts. We have in general

$$\lambda = l_g(\lambda)(1,\mathbf{0}), \tag{2.3}$$

with

$$\lambda^2 = 1. \tag{2.4}$$

The inverse of $l_g(\lambda)$ takes us from an arbitrary frame, the x frame, to a frame which depends on g and λ, i.e.,

$$x_{g\lambda} = l_g^{-1}(\lambda)x. \tag{2.5}$$

The so-called *front form boost* $(g=f)$ is given by [13]

$$x^0 = \sqrt{2}\lambda^0 x^0_{f\lambda}, \quad \mathbf{x}_\perp = \sqrt{2}\boldsymbol{\lambda}_\perp x^0_{f\lambda} + \mathbf{x}_{f\lambda\perp}, \tag{2.6}$$

where the components that appear here are light front components defined by

$$(x^\mu) = \left(x^0 = \frac{ct+z}{\sqrt{2}}, x^1 = x, x^2 = y, x^3 = \frac{ct-z}{\sqrt{2}} \right)$$
$$= (x^0, \mathbf{x}_\perp, x^3) = (\tilde{x}, x^3). \tag{2.7}$$

We define single particle, front form states by

$$|p_i s_i h_i\rangle_f = U[l_f(\lambda_i)]|s_i h_i\rangle, \quad i = \pi, \eta, \rho, \ldots, N, \Delta, N^*, \ldots, \tag{2.8a}$$

$$\lambda_i = p_i/m_i, \quad m_i = +(p_i^2)^{1/2}, \tag{2.8b}$$

with s_i the spin of particle i and h_i the z component of this spin. The state $|s_i h_i\rangle$ is a rest frame state for particle i, and it is assumed to rotate according to

$$U(r)|s_i h_i\rangle = \sum_{h_i'} |s_i h_i'\rangle D^{(s_i)}_{h_i' h_i}(r), \tag{2.9}$$

where $D^{(s_i)}(r)$ is the standard SU(2) matrix representation for the spin s_i.

We define meson-baryon, front form states by

$$|p_\mu s_\mu h_\mu, p_B s_B h_B\rangle_f = |p_\mu s_\mu h_\mu\rangle_f \otimes |p_B s_B h_B\rangle_f, \tag{2.10}$$

with the total four-momentum of these states given by

$$p_{\mu B} = p_\mu + p_B. \tag{2.11}$$

We also define relative three-momentum variables $\mathbf{q}_{\mu B}$ as the three-momentum of the meson μ in the μB c.m. frame. We have

$$(\omega_\mu(\mathbf{q}_{\mu B}), \mathbf{q}_{\mu B}) = l_f^{-1}(\lambda_{\mu B})p_\mu, \tag{2.12a}$$

$$(\epsilon_B(\mathbf{q}_{\mu B}), -\mathbf{q}_{\mu B}) = l_f^{-1}(\lambda_{\mu B})p_B, \tag{2.12b}$$

where the energies are given by

$$\omega_\mu(\mathbf{q}_{\mu B}) = (\mathbf{q}^2_{\mu B} + m^2_\mu)^{1/2}, \quad \epsilon_B(\mathbf{q}_{\mu B}) = (\mathbf{q}^2_{\mu B} + m^2_B)^{1/2}, \tag{2.13}$$

and

$$\lambda_{\mu B} = p_{\mu B}/W_{\mu B}(\mathbf{q}_{\mu B}), \tag{2.14}$$

with the invariant total c.m. energy given by

$$W_{\mu B}(\mathbf{q}_{\mu B}) = +(p^2_{\mu B})^{1/2} = \omega_\mu(\mathbf{q}_{\mu B}) + \epsilon_B(\mathbf{q}_{\mu B}). \tag{2.15}$$

Even in the c.m. frame the two particle states (2.10) do not behave simply under three-rotations. In order to get states that rotate simply, we introduce the Melosh rotation [13,15]

$$r_{fc}(\lambda) \equiv l_f^{-1}(\lambda) l_c(\lambda), \tag{2.16}$$

where $l_c(\lambda)$ is a so-called *canonical* or *rotationless* boost defined by

$$x^0 = \lambda^0 x^0_{c\lambda} + \boldsymbol{\lambda}\cdot\mathbf{x}_{c\lambda},$$

$$\mathbf{x} = \mathbf{x}_{c\lambda} + \left(x^0_{c\lambda} + \frac{\boldsymbol{\lambda}\cdot\mathbf{x}_{c\lambda}}{\lambda^0 + 1} \right)\boldsymbol{\lambda}, \tag{2.17}$$

where here x^0, λ^0, and $x^0_{c\lambda}$ are the usual "time" components of the four-vectors, e.g., $x^0 = ct$. From (2.5) and (2.16) it follows that the Melosh rotation relates the front form and canonical rest frames according to

$$x_{f\lambda} = r_{fc}(\lambda) x_{c\lambda}. \tag{2.18}$$

Following Keister and Polyzou [13] we define a Melosh rotated, rest frame, two particle state by

$$|\mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle \equiv \sum_{h'_\mu h'_B} |l_f^{-1}(\lambda_{\mu B}) p_\mu, s_\mu h'_\mu; l_f^{-1}(\lambda_{\mu B}) p_B, s_B h'_B\rangle \times D^{(s_\mu)}_{h'_\mu h_\mu}\{r_{fc}[l_f^{-1}(\lambda_{\mu B})\lambda_\mu]\} \times D^{(s_B)}_{h'_B h_B}\{r_{fc}[l_f^{-1}(\lambda_{\mu B})\lambda_B]\}, \tag{2.19}$$

which can be shown [13,16,17] to rotate according to

$$U(r)|\mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle = \sum_{h'_\mu h'_B} |r\mathbf{q}_{\mu B}, s_\mu h'_\mu, s_B h'_B\rangle \times D^{(s_\mu)}_{h'_\mu h_\mu}(r) D^{(s_B)}_{h'_B h_B}(r). \tag{2.20}$$

The rest frame states (2.19) can be boosted to define the states

$$|\bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle \equiv U[l_f(\lambda_{\mu B})]|\mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle, \tag{2.21}$$

which have the four-momentum (2.11), and where we note that the unit vector $\lambda_{\mu B}$ is given in terms of light front components [see (2.7)] by

$$\lambda_{\mu B} = \left(\bar{p}_{\mu B}, \frac{\mathbf{p}^2_{\mu B\perp} + W^2_{\mu B}(\mathbf{q}_{\mu B})}{2p^0_{\mu B}}\right) \Big/ W_{\mu B}(\mathbf{q}_{\mu B}). \tag{2.22}$$

In order to make the notation for the one baryon states consistent with (2.21) we write

$$|\bar{p}_B; s_B h_B\rangle = |p_B s_B h_B\rangle_f, \tag{2.23}$$

and observe that the states (2.21) and (2.23) are labeled by *external* quantities, $\bar{p}_B$ and $\bar{p}_{\mu B}$, and *internal* quantities, $\mathbf{q}_{\mu B}$, s_μ, h_μ, s_B, and h_B.

The light front spin operator $\mathscr{J}$ acts only on the internal quantities and can be defined by [18]

$$\mathscr{J}|\bar{p}_B; s_B h_B\rangle \equiv U[l_f(\lambda_B)]\mathbf{J}|s_B h_B\rangle,$$

$$\mathscr{J}|\bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle \equiv U[l_f(\lambda_{\mu B})]\mathbf{J}|\mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle, \tag{2.24}$$

where $\mathbf{J}$ is the angular momentum operator, i.e., the generator of the three-rotations r. According to (2.9) and (2.20) the rest frame states rotate just as they do in nonrelativistic quantum mechanics; therefore $\mathbf{J}$ acts on the states $|s_B h_B\rangle$ and $|\mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle$ in the usual way. It follows from this observation and (2.24) that the spin operator has the representation

$$\langle \bar{p}_B; s_B h_B|\mathscr{J} = \sum_{h'_B} (\mathbf{S}_B)_{h_B h'_B}\langle \bar{p}_B; s_B h'_B|, \tag{2.25a}$$

$$\langle \bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B|\mathscr{J} = \sum_{h'_\mu h'_B} \mathscr{J}(\mathbf{q}_{\mu B})_{h_\mu h_B, h'_\mu h'_B} \times \langle \bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h'_\mu, s_B h'_B|, \tag{2.25b}$$

where

$$\mathscr{J}(\mathbf{q}_{\mu B}) = I_\mu \otimes I_B(i\nabla_{\mathbf{q}_{\mu B}} \times \mathbf{q}_{\mu B}) + \mathbf{S}_\mu \otimes I_B + I_\mu \otimes \mathbf{S}_B, \tag{2.26}$$

with I_i and $\mathbf{S}_i$ the unit matrix and spin-matrix vector for particle i, respectively.

In order to calculate matrix elements of operators such as $\mathscr{J}$ we must specify the orthogonality relations for our basis states. We will assume

$$\langle \bar{p}_B; s_B h_B|\bar{p}'_{B'}; s_{B'} h'_{B'}\rangle = \delta_{BB'}(2\pi)^3 2p^0_B \delta^3(\bar{p}_B - \bar{p}'_B)\delta_{h_B h'_B}, \tag{2.27a}$$

$$\langle \bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B|\bar{p}'_{B'} s_{B'} h'_{B'}\rangle = 0, \tag{2.27b}$$

$$\langle \bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B|\bar{p}'_{\mu' B'}; \mathbf{q}'_{\mu' B'}, s_{\mu'} h'_{\mu'}, s_{B'} h'_{B'}\rangle = \delta_{\mu\mu'}\delta_{BB'}(2\pi)^3 2p^0_{\mu B}\delta^3(\bar{p}_{\mu B} - \bar{p}'_{\mu B}) \times \Delta_{\mu B}(\mathbf{q}_{\mu B})\delta^3(\mathbf{q}_{\mu B} - \mathbf{q}'_{\mu B})\delta_{h_\mu h'_\mu}\delta_{h_B h'_B}. \tag{2.27c}$$

Here

$$\Delta_{\mu B}(\mathbf{q}_{\mu B}) = (2\pi)^3 2\omega_\mu(\mathbf{q}_{\mu B})\epsilon_B(\mathbf{q}_{\mu B})/W_{\mu B}(\mathbf{q}_{\mu B}). \tag{2.28}$$

The spin operator (2.26) is of course a very familiar form. It is worth noting that in order to arrive at this result it was necessary to develop basis states that according to (2.21), (2.19), and (2.10) are related to the single particle states in a rather complicated way. This is a general feature of relativistic quantum mechanics, and reflects the complex behavior of angular momentum under Lorentz transformations. In Sec. III we will see that using the basis $\{|\bar{p}_B s_B h_B\rangle, |\bar{p}_{\mu B}; \mathbf{q}_{\mu B}, s_\mu h_\mu, s_B h_B\rangle\}$ facilitates the construction of a Poincaré invariant model for pion-nucleon scattering which assumes that the spin operator $\mathscr{J}$ is given by (2.25) and (2.26).

III. THE MASS OPERATOR

The mass operator is a Casimir operator of the Poincaré group, and as such commutes with all of the generators of the group. In light front dynamics the seven generators $\bar{P} = (P^0, P^1, P^2)$, $\mathbf{B} = (B_1, B_2)$, K_3, and J_3 induce Poincaré transformations which map the null plane $x^0 = (ct + z)/\sqrt{2} = 0$ into itself [13,17,18]. These seven generators are assumed to be noninteracting. The components of $\bar{P}$ are the first three light front components [see (2.7)] of the four-momentum operator, the three operators $\mathbf{B}$ and K_3 generate the front form boosts (2.6), and J_3 generates rotations about the z axis. It can be shown [18,19] that in the basis constructed in Sec. II the representatives of the generators $\bar{P}$, $\mathbf{B}$, and K_3 are given by

$$\langle \bar{p}_c ;\alpha_c|\bar{P}=\langle \bar{p}_c ;\alpha_c|\bar{p}_c , \quad (3.1)$$

$$\langle \bar{p}_c ;\alpha_c|B_r=-ip_c^0 \frac{\partial}{\partial p_c^r}\langle \bar{p}_c ;\alpha_c|, \quad (3.2a)$$

$$\langle \bar{p}_c ;\alpha_c|K_3=-ip_c^0 \frac{\partial}{\partial p_c^0}\langle \bar{p}_c ;\alpha_c|, \quad (3.2b)$$

with $c=B,\mu B$, and $\alpha_B=(s_Bh_B)$, $\alpha_{\mu B}=(\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B)$.

We will construct our mass operator in the form

$$M^2=M_0^2+V, \quad (3.3)$$

where M_0 is the noninteracting mass operator and V is the interaction. The action of M_0 on the basis states constructed in Sec. II is given by

$$M_0|\bar{p}_B ;s_Bh_B\rangle=m_B|\bar{p}_B ;s_Bh_B\rangle,$$

$$M_0|\bar{p}_{\mu B} ;\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle = W_{\mu B}(\mathbf{q}_{\mu B})|\bar{p}_{\mu B} ;\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle, \quad (3.4)$$

where m_B is the physical mass of baryon B and $W_{\mu B}$ is given by (2.15) and (2.13). It follows trivially from (3.1), (3.2), and (3.4) that M_0 commutes with $\bar{P}$, $\mathbf{B}$, and K_3. It can easily be shown [16,17] that M commutes with $\bar{P}$, $\mathbf{B}$, and K_3 if and only if the matrix elements of V have the structure

$$\langle \bar{p}_c ;\alpha_c|V|\bar{p}'_{c'} ;\alpha'_{c'}\rangle=(2\pi)^3 2p_c^0\delta^3(\bar{p}_c-\bar{p}'_{c'})V(\alpha_c ,\alpha'_{c'}). \quad (3.5)$$

It can also be shown that M commutes with the other four generators of the Poincaré group if and only if it commutes with the spin operator $\mathscr{J}$ [18,19]. We are assuming that $\mathscr{J}$ is the noninteracting spin operator given by (2.25) and (2.26), which implies that we must have

$$[\mathscr{J},V]=0, \quad (3.6)$$

since M_0 commutes with $\mathscr{J}$.

In order to work out the consequences of (3.6) it is convenient to work with basis states that are eigenstates of $\mathscr{J}^2$ and $\mathscr{J}_3$. According to (2.25a) the one baryon states are already such states, i.e.,

$$\mathscr{J}^2|\bar{p}_B ;s_Bh_B\rangle=s_B(s_B+1)|\bar{p}_B ;s_Bh_B\rangle,$$

$$\mathscr{J}_3|\bar{p}_B ;s_Bh_B\rangle=h_B|\bar{p}_B ;s_Bh_B\rangle. \quad (3.7)$$

If we define new rest frame, meson-baryon states by

$$|q_{\mu B},ls,j\lambda\rangle\equiv\sum_{\substack{h_\mu h_B\\ h_lh_s}}\int |\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle d\Omega_{\mathbf{q}_{\mu B}}Y_l^{h_l}(\hat{\mathbf{q}}_{\mu B}) \times\langle s_\mu s_Bh_\mu h_B|sh_s\rangle\langle lsh_lh_s|j\lambda\rangle, \quad (3.8)$$

where the $\langle\,|\,\rangle$ are Clebsch-Gordan coefficients, then it follows from (2.20) that these states rotate according to

$$U(r)|q_{\mu B},ls,j\lambda\rangle=\sum_{\lambda'}|q_{\mu B},ls,j\lambda'\rangle D^{(j)}_{\lambda'\lambda}(r). \quad (3.9)$$

As in (2.21) we can boost the states (3.8) and define

$$|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle=U[l_f(\lambda_{\mu B})]\,|q_{\mu B},ls,j\lambda\rangle. \quad (3.10)$$

According to (3.10), (3.8), (2.24), and (3.9) we have

$$\mathscr{J}^2|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle=j(j+1)|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle,$$
$$\mathscr{J}_3|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle=\lambda|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle. \quad (3.11)$$

It now follows from (3.5), (3.6), (3.7), and (3.11) that the matrix elements of the interaction V are given by

$$\langle \bar{p}_B ;s_Bh_B|V|\bar{p}'_{B'} ;s_{B'}h'_{B'}\rangle =(2\pi)^3 2p_B^0\delta^3(\bar{p}_B-\bar{p}'_{B'})\delta_{s_Bs_{B'}}\delta_{h_Bh'_{B'}}V_{BB'}, \quad (3.12a)$$

$$\langle \bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda|V|\bar{p}'_{B'} ;s_{B'}h'_{B'}\rangle =(2\pi)^3 2p_{\mu B}^0\delta^3(\bar{p}_{\mu B}-\bar{p}'_{B'})\delta_{js_{B'}}\delta_{\lambda h'_{B'}}V^{ls}_{\mu B,B'}(q_{\mu B}), \quad (3.12b)$$

$$\langle \bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda|V|\bar{p}'_{\mu' B'} ;q'_{\mu' B'},l's',j'\lambda'\rangle =(2\pi)^3 2p_{\mu B}^0\delta^3(\bar{p}_{\mu B}-\bar{p}'_{\mu' B'})\delta_{jj'}\delta_{\lambda\lambda'} \times V^{ls,l's';j}_{\mu B,\mu' B'}(q_{\mu B},q'_{\mu' B'}). \quad (3.12c)$$

Assuming that the spin operator $\mathscr{J}$ is given by (2.25) and (2.26), Eqs. (3.12a)–(3.12c) define the most general Poincaré invariant interaction that we can have in the model space spanned by the vectors $\{|\bar{p}_B ;s_Bh_B\rangle,|\bar{p}_B ;\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle\}$.

The above matrix elements can be further constrained by assuming that V does not couple states with different parities. If we let $\mathscr{P}_\mu$ and $\mathscr{P}_B$ be the intrinsic parities of the μ meson and the baryon B, respectively, we see that the matrix elements (3.12a), (3.12b), and (3.12c) vanish unless $\mathscr{P}_B=\mathscr{P}_{B'}$, $(-1)^l\mathscr{P}_\mu\mathscr{P}_B=\mathscr{P}_{B'}$, and $(-1)^l\mathscr{P}_\mu\mathscr{P}_B=(-1)^{l'}\mathscr{P}_{\mu'}\mathscr{P}_{B'}$, respectively. Also, if we assume that

$$M|\bar{p}_B ;s_Bh_B\rangle=m_{0B}|\bar{p}_B ;s_Bh_B\rangle, \quad (3.13)$$

where m_{0B} is the "bare" mass of baryon B, then it follows from (3.3), (3.4), (3.12a), and (2.27a) that

$$V_{BB'}=\delta_{BB'}(m_{0B}^2-m_B^2). \quad (3.14)$$

The matrix elements of V in the original basis $\{|\bar{p}_B ;s_Bh_B\rangle,|\bar{p}_B ;\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle\}$ can be obtained by inverting (3.8). Using (2.21) and (3.10) we find that

$$|\bar{p}_{\mu B} ;\mathbf{q}_{\mu B},s_\mu h_\mu ,s_Bh_B\rangle =\sum_{\substack{lh_l,sh_s,\\ j\lambda}}|\bar{p}_{\mu B} ;q_{\mu B},ls,j\lambda\rangle\langle lsh_lh_s|j\lambda\rangle \times\langle s_\mu s_Bh_\mu h_B|sh_s\rangle Y_l^{h_l^*}(\hat{\mathbf{q}}_{\mu B}), \quad (3.15)$$

which when combined with (3.12b), for example, leads to

$$\langle \bar{p}_{\mu B};\mathbf{q}_{\mu B},s_\mu h_\mu,s_B h_B|V|\bar{p}'_{B'};s_{B'},h'_{B'}\rangle$$
$$=(2\pi)^3 2p^0_{\mu b}\delta^3(\bar{p}_{\mu B}-\bar{p}'_{B'})V^{h_\mu h_B,h'_{B'}}_{\mu B,B'}(\mathbf{q}_{\mu B}), \tag{3.16}$$

with

$$V^{h_\mu h_B,h'_{B'}}_{\mu B,B'}(\mathbf{q}_{\mu B})=\sum_{lh_l,sh_s}Y^{h_l}_l(\hat{\mathbf{q}}_{\mu B})\langle s_\mu s_B h_\mu h_B|sh_s\rangle$$
$$\times\langle lsh_l h_s|s_{B'}h'_{B'}\rangle V^{ls}_{\mu B,B'}(q_{\mu B}). \tag{3.17}$$

A familiar result is obtained from (3.17) if we consider the πNN or $\pi N\Delta$ vertices. Parity and angular momentum conservation imply that only the $l=1,s=s_B,h_s=h_B$ terms in (3.17) survive. If we define complex unit vectors by

$$\mathbf{e}_1=-\mathbf{e}^*_{-1}=-(1/\sqrt{2})(1,i,0),\quad \mathbf{e}_0=(0,0,1), \tag{3.18}$$

then we find

$$V^{0,h_N;h'_B}_{\pi N,B}(\mathbf{q}_{\pi N})=\sqrt{\frac{3}{4\pi}}\,V^{1,1/2}_{\pi N,B}(q_{\pi N})\hat{\mathbf{q}}_{\pi N}\cdot(\mathbf{S}_{NB})_{h_N h'_B},$$
$$B=N,\Delta, \tag{3.19}$$

where

$$(\mathbf{S}_{NB})_{h_N h'_B}=\sum_{h_l=-1}^{1}\langle 1,1/2,h_l h_N|s_B h'_B\rangle\mathbf{e}_{h_l},\quad B=N,\Delta. \tag{3.20}$$

It can be shown that $\mathbf{S}_{NN}=-\boldsymbol{\sigma}/\sqrt{3}$ where the components of $\boldsymbol{\sigma}$ are the Pauli matrices, and that $\mathbf{S}_{N\Delta}$ is the well known spin transition matrix associated with the $\pi N\Delta$ vertex [20].

IV. THE LIPPMANN-SCHWINGER EQUATIONS

In order to calculate scattering amplitudes with the interaction V it is necessary to solve the equation for the T operator, i.e.,

$$T(z)=V+VG_0(z)T(z), \tag{4.1}$$

where

$$G_0(z)=(z-M_0^2)^{-1}, \tag{4.2}$$

and z is a complex parameter, which when calculating physical amplitudes becomes $z=W^2+i\epsilon$. It turns out that in solving this equation it is convenient to first eliminate the single baryon channels, and thereby obtain an effective potential that acts in the subspace of meson-baryon states. To this end we introduce complementary, orthogonal projection operators Λ_1 and Λ_2, where Λ_1 projects onto the subspace spanned by the single baryon states (2.23), and Λ_2 projects onto the subspace spanned by the meson-baryon states (2.21) or (3.10). We define

$$O_{ij}=\Lambda_i O\Lambda_j, \tag{4.3}$$

where O is any operator of interest. From (4.1) we obtain

$$T_{ij}(z)=V_{ij}+\sum_{k=1}^{2}V_{ik}G_0(z)T_{kj}(z). \tag{4.4}$$

According to (2.27a) the projector onto the subspace of single baryon states can be written in the form

$$\Lambda_1=\sum_{Bh_B}\int|\bar{p}_B;s_B h_B\rangle\frac{d^3p_B}{(2\pi)^3 2p^0_B}\langle\bar{p}_B;s_B h_B|$$
$$[d^3p_B=dp^0_B\theta(p^0_B)d\mathbf{p}_{B\perp}], \tag{4.5}$$

which when used in (4.4), along with (3.4), (3.12a), and (3.14), leads to

$$\langle\bar{p}_B;s_B h_B|T_{12}(z)=\frac{z-m_B^2}{z-m_{0B}^2}\langle\bar{p}_B;s_B h_B|$$
$$\times V_{12}[1+G_0(z)T_{22}(z)]. \tag{4.6}$$

Upon putting this into (4.4) with $i=j=2$, we find

$$T_{22}(z)=V_{22}+W(z)+[V_{22}+W(z)]G_0(z)T_{22}(z), \tag{4.7}$$

where

$$W(z)=\sum_{Bh_B}\int V_{21}|\bar{p}_B;s_B h_B\rangle\frac{d^3p_B}{(2\pi)^3 2p^0_B(z-m^2_{0B})}$$
$$\times\langle\bar{p}_B;s_B h_B|V_{12}. \tag{4.8}$$

Equation (4.7) is an operator equation in the meson-baryon subspace whose formal solution can be obtained by using the well known two-potential formalism [21], i.e.,

$$T_{22}(z)=t_{22}(z)+[1+t_{22}(z)G_0(z)]\tau(z)[1+G_0(z)t_{22}(z)], \tag{4.9}$$

where $t_{22}(z)$ and $\tau(z)$ satisfy the equations

$$t_{22}(z)=V_{22}+V_{22}G_0(z)t_{22}(z), \tag{4.10}$$

$$\tau(z)=W(z)+W(z)G(z)\tau(z), \tag{4.11}$$

with

$$G(z)=G_0(z)+G_0(z)t_{22}(z)G_0(z). \tag{4.12}$$

If we write out these equations in the basis for the meson-baryon subspace, i.e., $\{|\bar{p}_{\mu B};q_{\mu B},ls,j\lambda\rangle\}$, and exploit the fact that $W(z)$ is a sum of separable terms, we can show that

$$\langle\bar{p}_{\mu B};q_{\mu B},ls,j\lambda|T(z)|\bar{p}'_{\mu' B'};q'_{\mu' B'},l's',j'\lambda'\rangle$$
$$=(2\pi)^3 2p^0_{\mu B}\delta^3(\bar{p}_{\mu B}-\bar{p}'_{\mu' B'})$$
$$\times\delta_{jj'}\delta_{\lambda\lambda'}T^j_{cc'}(q_{\mu B},q'_{\mu' B'};z), \tag{4.13}$$

where

$$c=\{\mu,B,l,s\}, \tag{4.14}$$

a set of channel labels, and

$$T^j_{cc'}(q,q';z)=t^j_{cc'}(q,q';z)+\sum_{B''B'''} F_{cB''}(q;z)\delta_{s_{B''}j} \times\Gamma^{(j)}_{B''B'''}(z)\delta_{s_{B'''}j}F^*_{c'B'''}(q';z). \quad (4.15)$$

The first term on the right hand side of (4.15) is the solution of the coupled integral equations

$$t^j_{cc'}(q,q';z)=V^j_{cc'}(q,q')+\sum_{c''}\int_0^\infty V^j_{cc''}(q,q'') \times\frac{q''^2dq''}{\Delta_{\mu''B''}(q'')[z-W^2_{\mu''B''}(q'')]}\,t^j_{c''c'}(q'',q';z), \quad (4.16)$$

while the second term can be obtained by quadratures involving the the solution of (4.16). We have

$$F_{cB''}(q;z)=V_{cB''}(q)+\sum_{c'}\int t^j_{cc'}(q,q';z) \times\frac{q'^2dq'}{\Delta_{\mu'B'}(q')[z-W^2_{\mu'B'}(q')]}\,V_{c'B''}(q) \quad (j=s_{B''}), \quad (4.17)$$

and

$$\Gamma^{(j)}(z)=[(z-m^2_{0B})\delta_{BB'}-\Sigma^{(j)}_{BB'}(z)]^{-1}, \quad (4.18)$$

where the self-energy matrix $\Sigma^{(j)}(z)$ is given by

$$\Sigma^{(j)}_{BB'}(z)=\sum_{c''}\int V^*_{c''B}(q'')\,\frac{q''^2dq''}{\Delta_{\mu''B''}(q'')[z-W^2_{\mu''B''}(q'')]} \times F_{c''B'}(q'';z) \quad (s_B=s_{B'}=j). \quad (4.19)$$

The dimensionality of these matrices is equal to the number of baryons with spin j.

V. NUMERICAL RESULTS

In calculating the pion-nucleon elastic scattering amplitudes we deal with states of well defined total angular momentum j, isospin i, and parity, labeled in the usual way, i.e., $X_{2i,2j}$, where $X=S,P,D,\ldots$, corresponding to $l_{\pi N}=0,1,2,\ldots$. Since the pion is a pseudoscalar particle the parity is $(-1)^{1+l_{\pi N}}$. The states that we will include are shown in Table I, as well as the particle channels that are coupled in each state. The quantities in parentheses are the relative orbital angular momenta in the meson-baryon channel other than the pion-nucleon channel. We note that in the D_{13} channel there are actually two possible relative orbital angular momenta. We will retain only the $l_{\pi\Delta}=0$ channel. We also note that if we had included a $\pi\Delta$ channel in the S_{11} state it would have $l_{\pi\Delta}=2$; therefore there is some justification in ignoring it since it is kinematically suppressed.

Since both the π and η are pseudoscalar particles the total spin s of the meson-baryon states is the same as s_B, the spin

TABLE I. States and particle channels of the pion-nucleon system.

State	Particle channels
S_{11}	$\pi N, \eta N$ ($l_{\eta N}=0$)
S_{31}	$\pi N, \pi\Delta$ ($l_{\pi\Delta}=2$)
P_{11}	$N, \pi N, \pi\Delta$ ($l_{\pi\Delta}=1$)
P_{13}, P_{31}	πN
P_{33}	$\Delta, \pi N$
D_{13}	$\pi N, \pi\Delta$ ($l_{\pi\Delta}=0.2$)

of the baryon; therefore the index s in (4.14) is redundant. Also, since for each state in Table I there is a unique value of l for each meson-baryon channel, we can drop the relative orbital angular momentum indices. In each state we assume separable meson-baryon potentials of the form

$$V^{l,l';j}_{\mu B,\mu' B'}(q,q')=V_{cc'}(q,q')=h_c(q)\lambda_{cc'}h(q'), \quad (5.1)$$

where in all states $X_{2i,2j}$, $c=1$ labels the πN channel, while $c=2$ labels the second meson-baryon channel in the S_{11}, S_{31}, P_{11}, and D_{13} states. In (5.1) $\lambda_{cc}=\pm 1$, and the form factors are assumed to be

$$h_c(q)=C_c(q/\beta_c)^l[1+(q/\alpha_c)^2]^{1/4}/[1+(q/\beta_c)^2]^{K_c}, \quad (5.2)$$

where it turns out that the factor $[1+(q/\alpha_c)^2]^{1/4}$ is only necessary in the S_{11} state. We also have two vertex functions, corresponding to the πNN and $\pi N\Delta$ vertices. They are assumed to be

$$V_{\pi N,B}(q)=C_{NB}(q/\beta_{NB})/[1+(q/\beta_{NB})^2]^{K_{NB}}, \quad B=N,\Delta. \quad (5.3)$$

The parameters where determined by a least squares fit to the SAID-SP95 analysis of the pion-nucleon scattering data [22]. The resulting parameters are given in Table II, while the phase shifts and inelasticities are shown in Figs. 1–11. The masses of the particles were taken to be $m_\pi=138.03$ MeV, $m_N=938.92$ MeV, $m_\Delta=m_\pi+m_N$, and $m_\eta=556.3$ MeV. The mass of the Δ, which differs from the actual mass of 1232 MeV, was chosen so as to put the inelastic threshold at the correct energy in the S_{31}, P_{11}, and D_{13} channels; while the η mass was taken as an adjustable parameter.

In fitting the P_{11} channel the nucleon channel is included (see Table I), so the second term on the right hand side of (4.15) makes a contribution with $B''=B'''=N$. In this case the self-energy matrix (4.18) is a single function, and the bare nucleon mass can be eliminated by requiring that

$$m^2_N-m^2_{0N}-\Sigma^{(1/2)}_{NN}(m^2_N)=0. \quad (5.4)$$

For our fit it turns out that $m_{0N}=948.07$ MeV, so the difference between the bare and physical mass is only 0.97%. As a result of (5.4), the πN elastic scattering amplitude in the P_{11}

TABLE II. Separable potential and vertex function parameters. See Eqs. (5.1)–(5.3).

State	C_1	α_1 (fm^{-1})	β_1 (fm^{-1})	C_2	α_2 (fm^{-1})	β_2 (fm^{-1})	λ_{12}	λ_{11}	λ_{22}	K_1	K_2	C_{NB}	β_{NB} (fm^{-1})	K_{NB}
S_{11}	2.710		4.414	17.12		2.375	33.87	−1.0	−1.0	1	1			
S_{31}	124.6	0.4321	4.682	2.930	0.4081	9.004	7.338	1.0	−1.0	1	2			
P_{11}	2.872		11.17	53.12		11.93	63.21	1.0	1.0	2	3	114.0	1.922	2
P_{13}	31.04		3.072					1.0		2				
P_{31}	86.08		5.743					1.0		2				
P_{33}												233.9	2.566	2
D_{13}	44.99		3.754	348.6		0.8868	1.076	−1.0	1.0	2	2			

channel has a pole when $W_{\pi N}(q)=m_N$. With the normalization determined by (2.27c), (3.8), and (3.10), it turns out that

$$T_{\pi N,\pi N}[q,q;W^2_{\pi N}(q)] \xrightarrow[W_{\pi N}(q)\to m_N]{} -\frac{12\pi m_\pi^2 g^2_{\pi NN}}{W^2_{\pi N}(q)-m_N^2} \quad (P_{11}\ \text{channel}), \quad (5.5)$$

where $g_{\pi NN}$ is the pion-nucleon coupling constant. In fitting the P_{11} channel the parameters were constrained so that $g^2_{\pi NN}/4\pi=13.5$, which is a reasonable value for the coupling constant.

As Table II shows the separable potential (5.1) was not included in the P_{33} channel; only the $B=\Delta$ term in (4.8) contributed in this channel. In carrying out the fit, the parameters $C_{N\Delta}$ and $\beta_{N\Delta}$ in (5.3) were varied; as well as the bare Δ mass, which turned out to be 1291 MeV. In the P_{33} channel, as in the P_{11} channel, the self-energy matrix (4.18) is a single function. For our fit the real part of this function vanishes at $W_{\pi N}=1232$ MeV, which can be interpreted as the physical mass; so the difference between the bare and physical masses is 4.8%. A nice feature of the model considered here is that it is relatively straightforward to analytically continue the P_{33} elastic amplitude through the elastic right hand cut, which begins at $W_{\pi N}=m_\pi+m_N$, onto the second Riemann sheet. By so doing it has been found that the P_{33} elastic amplitude has a pole on the unphysical sheet at $W_{\pi N}=1208-i52$ MeV.

VI. DISCUSSION

The simplicity of the momentum space integral equations given in Sec. IV, as well as the quality of the fits obtained with the separable potential model, indicate that the mass operator defined by (3.3), (3.4), (3.8), and (3.12) provides a reasonable framework for treating a coupled channel system such as the pion-nucleon system. This mass operator has been developed in the context of the front form of relativistic quantum mechanics. In specifying the *internal* part of the mass operator it is not necessary to make a commitment to a particular form of relativistic quantum mechanics, where by *internal* we are referring, e.g., to the factor $V(\alpha_c,\alpha'_{c'})$ in (3.5). However, in establishing the exact Poincaré invariance of the model it is necessary to specify the relation between the rest frames, in which the internal part of the mass operator is specified, and an arbitrary frame. This relation involves specifying, in addition to the mass, three *external* variables which make it possible to determine the total four-momentum of the system. In the front form the external variables are taken to be the first three front form components [see (2.7)] of the four-momentum, i.e., $\bar{p}=(p^0,p^1,p^2)$. In the instant form the external variables are the total three-momentum $\mathbf{p}$, while in the point form they are the velocities $\mathbf{p}/m$. The external variables are kinematic and conserved in the respective forms of relativistic quantum mechanics. This is reflected, e.g., in the δ function in (3.5). The choice of a form also entails a choice of the Lorentz boost that is used to relate the rest frame quantities to an arbitrary frame. As we have seen the front form employs the front form boost defined by (2.6). The instant and point forms use the canonical

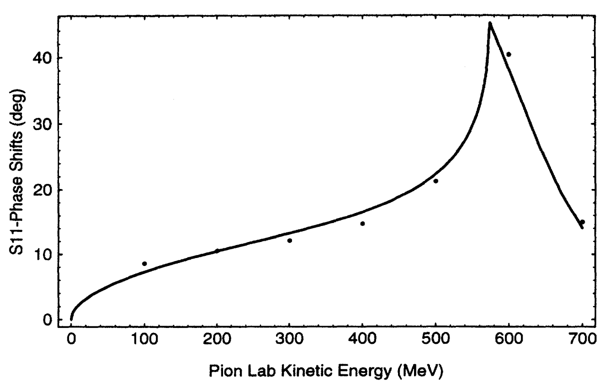

FIG. 1. Fit of the S_{11} phase shifts to the SAID-SP95 analysis.

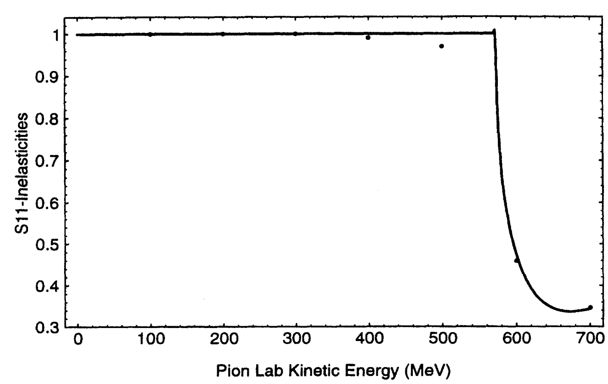

FIG. 2. Fit of the S_{11} inelasticities to the SAID-SP95 analysis.

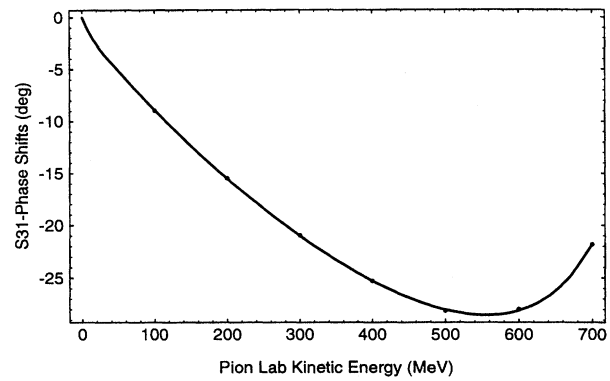

FIG. 3. Fit of the S_{31} phase shifts to the SAID-SP95 analysis.

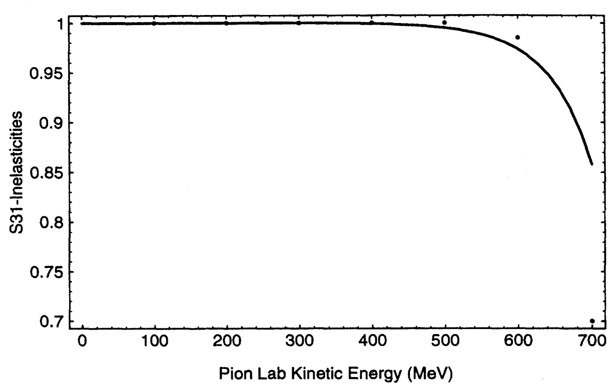

FIG. 4. Fit of the S_{31} inelasticities to the SAID-SP95 analysis.

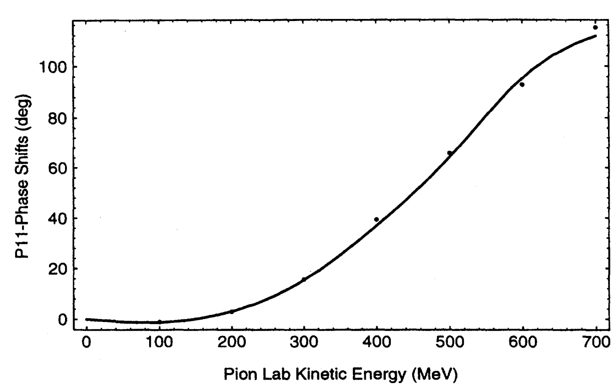

FIG. 5. Fit of the P_{11} phase shifts to the SAID-SP95 analysis.

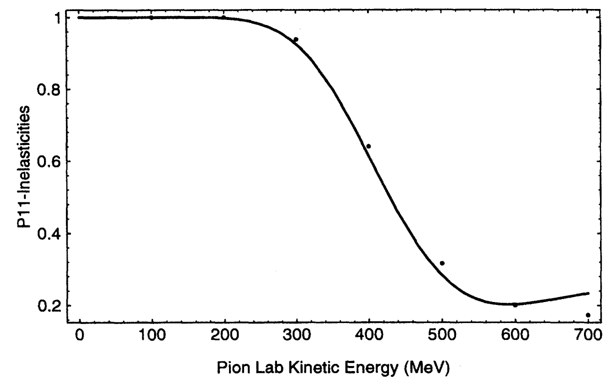

FIG. 6. Fit of the P_{11} inelasticities to the SAID-SP95 analysis.

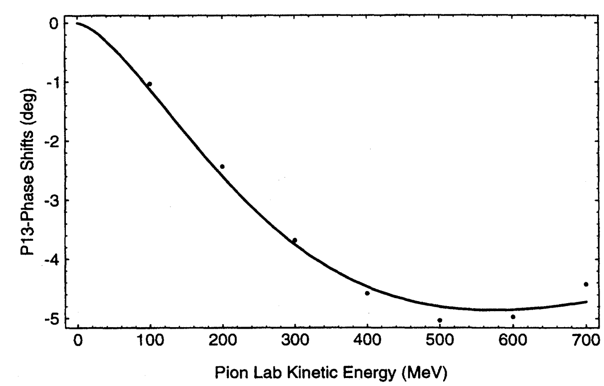

FIG. 7. Fit of the P_{13} phase shifts to the SAID-SP95 analysis.

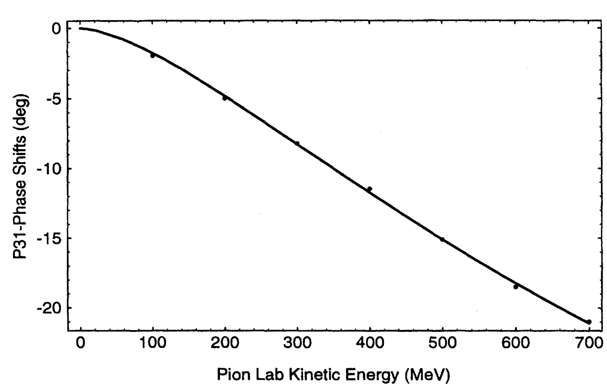

FIG. 8. Fit of the P_{31} phase shifts to the SAID-SP95 analysis.

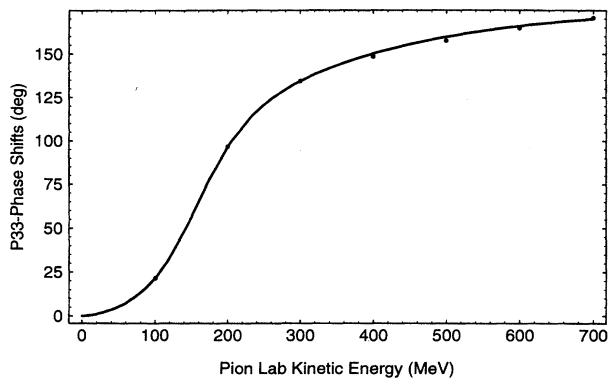

FIG. 9. Fit of the P_{33} phase shifts to the SAID-SP95 analysis.

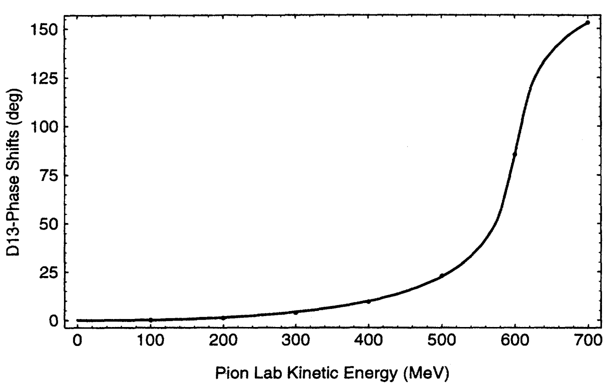

FIG. 10. Fit of the D_{13} phase shifts to the SAID-SP95 analysis.

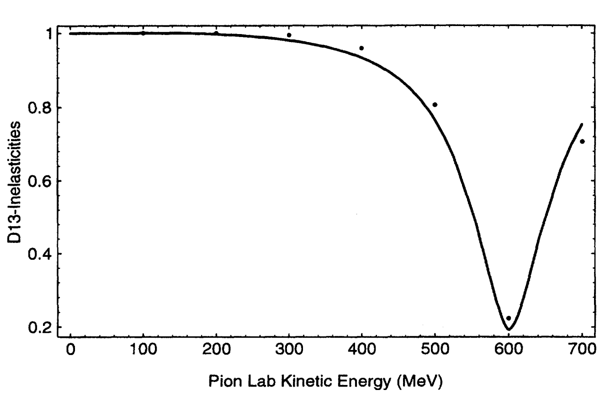

FIG. 11. Fit of the D_{13} inelasticities to the SAID-SP95 analysis.

boost (2.17). Thus, even though it is possible to assume the same dependence on the relative momentum variables, i.e., the $\mathbf{q}$'s, in all of the forms, we see that according to (2.12a) and (2.12b) the relation of the $\mathbf{q}$'s to the single particle momentum is not the same in every form. By looking at (2.24) we see that the spin operator $\mathscr{J}$ is also form dependent, since in the instant and point forms the front form boost l_f is replaced by the canonical boost l_c. The spin operators associated with the various forms are related by Melosh rotations [13]. As pointed out in Sec. I, these differences come into play when the two particle system is probed or part of a larger system.

As also pointed out in Sec. I, even though we have developed a Poincaré invariant coupled channel model in the context of the pion-nucleon system, the method used can obviously be applied in a broader context. The πN model itself can be extended to incorporate other mesons, e.g., $\rho(770)$, $\omega(783)$, and K. The method can also be applied directly to the NN system, which makes it possible to construct Poincaré invariant coupled channel models involving the NN, $N\Delta$, and $\Delta\Delta$ channels.

A natural question arises in regard to the type of model formulated here. Given that the most general interaction consistent with Poincaré invariance and the assumption of a free spin operator is defined by equations such as (3.12a)–(3.12c), how are the remaining arbitrary functions to be determined? Of course there are other invariances (parity, isospin, strangeness, etc.) that must be taken into account; however even after this is done there is still arbitrariness. There seem to be only two alternatives; either be content with purely phenomenological forms, as was done here, or take guidance from field theory.

The manifestly covariant model of the pion-nucleon system developed by Gross and Surya [5] takes much of its input from Lorentz invariant field theory vertices. Other authors have also used quantum field theory as a starting point for models of the pion-nucleon system [23–25]. In general, in obtaining interactions from quantum field theory for use in instant or front form models it is necessary to use truncations that destroy the exact Poincaré invariance of the field theory. The author has developed methods that correct for this [16,17,26,27], which makes it possible to construct exactly Poincaré invariant models that take their input from quantum field theory. With these methods the relation between the basis states which are used to formulate the exactly Poincaré invariant model and the elementary direct product states plays an essential role. For the front form, this relation is given, e.g., by (2.21), (2.19), and (2.10). The elementary direct product states (2.10) correspond to the Fock space basis states of the quantum field that provides the underlying model for the interactions. These techniques have already led to a reasonable front form, one boson exchange model of the nucleon-nucleon system, which only includes the NN channel [17], as well as to a simple instant form model of the pion-nucleon system [27]. These models are being extended to include coupling to other particle channels. The specific model developed here is also being extended to include a photon-nucleon channel, which will make it possible to construct a Poincaré invariant model of pion photoproduction.

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