Nucleon structure in a relativistic quark model

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We study nucleon structure in the relativistic quark model based on the Bakamjian-Thomas construction of the Poincaré generators for an arbitrary quantization surface. The one body, single particle approximation to the current operators is used to calculate electromagnetic matrix elements. The Lorentz symmetry breaking resulting from such an approximation is fully investigated. The results for the light front and instant quantization limits are detailed. A suggestion for the resolution of the quark model inability to simultaneously describe the positive neutron electric form factor, $G_E^n(Q^2)$ at small Q^2 and the negative slope of the neutron to proton structure function ratio at large x is presented.

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I. INTRODUCTION

The prospects for new precision data from CEBAF recently have led to an increased activity in quark model phenomenology [1]. Even though current interest focuses on strangeness, high mass baryonic resonances, and exotic state electroproduction [2], there are still many unresolved issues in the structure of the low-lying states. The simplest systems, π, ρ, N, and Δ still present many challenges to a quark model analysis. In particular, the following topics remain open problems: treating nonvalence degrees of freedom, ambiguities in the quark-quark potential responsible for the $\bar{q}q$ binding; and reconciling the simple spin-spin interaction as the source of π-ρ mass splitting with that of chiral symmetry breaking. The complexity of this is compounded further when attempts are made to develop a consistent relativistic quark approach. In this paper we concentrate on this last issue and present a consistent relativistic description of the the nucleon as a three-valence-quark bound state. Specifically we shall construct the nucleon state as an element of the unitary representation of the Poincaré group. The phenomenological description of the nucleon as a few-body bound state can be extended to fulfill the requirements of Poincaré symmetry by an explicit construction of interaction-dependent generators of the group. The problem of formulating relativistic dynamics for a fixed number of particles originated with the pioneering work of Dirac [3] and has been extensively studied over the years on both classical and quantum levels. To find a representation of the Poincaré group the quantization surface is first defined and the generators are split into an interaction-independent subgroup of the symmetries of the quantization surface and interaction-dependent Hamiltonians that describe the evolution of the system in time defined as the direction perpendicular to the quantization surface. For a sensible phenomenology a key requirement is the existence of a set of relative and center of mass variables such that the generators of the symmetry group can be written in a macroscopic form representing the motion of a system as whole with the bound state mass determined by an operator that depends only on the relative variables. For example, if the microscopic energy operator expressed in the individual quark momenta $\mathbf{p}_a$, position ρ_a, and spin $\mathbf{s}_a$ variables is given by

$$P^0=\sum_a \sqrt{m_a^2+\mathbf{p}_a}+V(\mathbf{p}_a,\rho_a,\mathbf{s}_a), \tag{1.1}$$

with V representing the interquark potential, the macroscopic Hamiltonian should have the form of

$$P^0=\sqrt{\mathbf{P}^2+M^2(\mathbf{k}_a,\mathbf{x}_a,\mathbf{s}_a')}, \tag{1.2}$$

with $\mathbf{P}$ standing for the total bound state momentum and M being the mass operator depending on the relative variables, but not on $\mathbf{P}$ or the conjugated operator which defines the position of the center of mass of the bound state. As discussed in Ref. [4] for given V there is no unique way in which the relative variables can be defined in terms of the individual variables so that the energy operator and all remaining generators of the Poincaré group take this macroscopic form. Furthermore the most general relation may depend in the interaction itself. A particular set of relative and center of mass variables that bring the generators to the macroscopic form has been introduced by Bakamjian and Thomas in Ref. [5] and Gartenhaus and Schwartz in Ref. [6] and was applied extensively by Osborn [7] and Close and Copely [7] to derive the low energy sum rules. In this paper we shall use such a construction, often referred to as the Bakamjian-Thomas (BT) construction, to derive the Poincaré invariant formulation of the Isgur-Karl quark model for baryons [8,9]. In the last few years many approaches to the nucleon structure in relativistic quark models have been studied [10–13]. Most of them are, however, based on writing an ansatz for the wave function such that in the nonrelativistic limit it is an eigenstate of a nonrelativistic quark model Hamiltonian. Other models merely try to fit the wave function to few measured form factors. These approaches do not allow for a deeper understanding of the relativistic quark dynamics. In particular there has been a long-standing problem related to the possibility of a simultaneous description of the negative neutron charge radius and the negative slope of the neutron to proton structure function ratio as the Bjorken scaling variable $x\to 1$ [14]. As we shall demonstrate these two phenomena have a common dynamical origin and can be understood without any artificial prescriptions for generating relativistic nucleon wave functions.

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The paper is organized into five sections. In the next section we discuss the Bakamjian-Thomas construction in the context of the constituent quark model and construct the nucleon wave function with proper transformation properties under the Poincaré group. We also address ambiguities in the BT construction arising from possible different quark coupling schemes, cluster separability and relations between the relativistic wave functions in the BT construction and those often used in other models. In Sec. III we examine the limitations of the single-particle, free current approximation and systematically study the effects from Lorentz symmetry breaking due to this approximation. Generalizing the BT construction so that the quantization surface and/or frame dependence can be studied, we connect the canonical, instant relativistic wave function quantized on the three-dimensional spacelike surface to the light cone wave function corresponding to a quantization on a light front. We show that there are domains of validity for the single-particle current approximation in both instant and light front quantization. The analysis of the deep inelastic structure functions is also given in Sec. III. The main findings are summarized in Sec. IV with many mathematical details relocated to the Appendix.

II. RELATIVISTIC NUCLEON WAVE FUNCTION

A. Bakamjian-Thomas construction

As discussed in Sec. I, for a system of N interacting constituents Poincaré algebra cannot be transformed uniquely from the microscopic multiparticle to a macroscopic single-particle representation corresponding to a group of transformations of the entire N particle system. A particular example of such a transformation is the Bakamjian-Thomas (BT) [5,7] construction. This has an advantage of directly connecting with nonrelativistic dynamics and is therefore well suited for formulating a relativistic quark model while preserving many features of the nonrelativistic approach. The BT construction proceeds as follows. First consider a system of N noninteracting constituents with masses m_a, $a=1,\ldots,N$. The Poincaré algebra expressed in terms of the individual particle variables, position $\mathbf{r}_a$, momentum $\mathbf{p}_a$, and spin $\mathbf{s}_a$, is given by a set of operators

$$P^0=\mathscr{E}=\sum_a E_a=\sum_a \sqrt{m_a^2+\mathbf{p}_a^2},$$

$$\mathbf{P}=\sum_a \mathbf{p}_a,$$

$$\mathbf{J}=\sum_a (\mathbf{r}_a\times\mathbf{p}_a+\mathbf{s}_a),$$

$$\mathbf{K}=\sum_a \frac{1}{2}\{\mathbf{r}_a,E_a\}-\frac{\mathbf{s}_a\times\mathbf{p}_a}{E_a+m_a}. \tag{2.1}$$

Here P^0 and $\mathbf{P}$ are the total energy and momentum of the N particle system, respectively, $\mathbf{J}$ are the generators of angular momentum, and $\mathbf{K}$ are the generators of Lorentz boosts. Defining relative position $\mathbf{x}_a$, momentum $\mathbf{k}_a$, and spin $\mathbf{s}'_a$ through the Gartenhaus-Schwartz transformation [6,7] of the corresponding individual particle variables

$$(\mathbf{k}_a,\mathbf{x}_a,\mathbf{s}'_a)=\lim_{\alpha\to\infty}\exp\left[i\alpha\frac{1}{2}\left\{\mathbf{K},\frac{\mathbf{P}}{P^0}\right\}\right](\mathbf{p}_a,\mathbf{r}_a,\mathbf{s}_a)\exp\left[i\alpha\frac{1}{2}\left\{\mathbf{K},\frac{\mathbf{P}}{P^0}\right\}\right], \tag{2.2}$$

the algebra of Poincaré generators expressed in terms of $\mathbf{k}_a$, $\mathbf{x}_a$, and $\mathbf{s}'_a$ has the desired single-particle form

$$P^0=\mathscr{E}=\sqrt{\mathbf{P}^2+\mathscr{M}(\mathbf{k}_a)^2},$$

$$\mathbf{P}=\mathbf{P},$$

$$\mathbf{J}=\mathbf{R}\times\mathbf{P}+\mathbf{S}\ ,\quad \mathbf{S}=\sum_a (\mathbf{x}_a\times\mathbf{k}_a+\mathbf{s}'_a),$$

$$\mathbf{K}=\frac{1}{2}\{\mathbf{R},\mathscr{E}\}-\frac{\mathbf{S}\times\mathbf{P}}{\mathscr{E}+\mathscr{M}}, \tag{2.3}$$

with

$$\mathscr{M}=\sqrt{\mathscr{E}^2-\mathbf{P}^2}=\sum_a \omega_a(\mathbf{k}_a)=\sum_a \sqrt{m_a^2+\mathbf{k}_a^2} \tag{2.4}$$

being the invariant mass of the free N constituents. The center of mass position operator $\mathbf{R}=\mathbf{R}(\mathbf{p}_a,\mathbf{r}_a,\mathbf{s}_a)$ is given in Ref. [7]. The relative position and momentum variables are constrained by

$$\sum_a m_a\mathbf{x}=0\ ,\sum_a \mathbf{k}_a=0, \tag{2.5}$$

and together with the spin operator, $\mathbf{s}'_a$ and the center of mass variables satisfy the canonical commutation relations

$$[x_{ia},k_{jb}]=i\left(\delta_{ab}-\frac{m_b}{\Sigma_c m_c}\right)\delta_{ij},\quad [s'_{ia},s'_{jb}]=i\epsilon_{ijk}s'_{ka}\delta_{ab},\quad [R_i,P_j]=i\delta_{ij}, \tag{2.6}$$

with all other commutators vanishing. In particular, for the momentum and spin operators the transformation defined in Eq. (2.2) gives

$$\mathbf{k}_a=\mathbf{p}_a+\frac{\mathbf{p}_a\cdot\mathbf{P}}{\mathcal{M}(\mathcal{M}+\mathcal{E})}\mathbf{P}-\frac{E_a}{\mathcal{M}}\mathbf{P},\quad \mathbf{p}_a=\mathbf{k}_a+\frac{\mathbf{k}_a\cdot\mathbf{P}}{\mathcal{M}(\mathcal{M}+\mathcal{E})}\mathbf{P}+\frac{\omega_a}{\mathcal{M}}\mathbf{P} \tag{2.7}$$

and

$$[\mathbf{s}'_a]_{\sigma\sigma'}=\sum_{\lambda\lambda'}D^{s\dagger}(\mathbf{p}_a,\mathbf{k}_a)_{\sigma\lambda}[\mathbf{s}_a]_{\lambda\lambda'}D^{s}(\mathbf{p}_a,\mathbf{k}_a)_{\lambda'\sigma'},$$

$$[\mathbf{s}_a]_{\lambda\lambda'}=\sum_{\sigma\sigma'}D^{s}(\mathbf{p}_a,\mathbf{k}_a)_{\lambda\sigma}[\mathbf{s}'_a]_{\sigma\sigma'}D^{s\dagger}(\mathbf{p}_a,\mathbf{k}_a)_{\sigma'\lambda'}, \tag{2.8}$$

with E_a, $\mathcal{E}$, ω_a, and $\mathbf{P}$ given in Eqs. (2.1) and (2.4). For spin-1/2 constituents the Wigner rotations D in Eq. (2.8) are given by

$$D^{1/2}(\mathbf{p}_a,\mathbf{k}_a)_{\lambda\sigma}=D(\mathbf{p}_a,\mathbf{k}_a)_{\lambda\sigma}=\frac{[(\omega_a+m_a)(\mathcal{E}+\mathcal{M})+\mathbf{P}\cdot\mathbf{k}_a]\delta_{\lambda\sigma}+i[\vec{\sigma}]_{\lambda\sigma}\cdot[\mathbf{P},\mathbf{k}_a]}{\sqrt{2(\omega_a+m_a)(\mathcal{E}+\mathcal{M})(\omega_a\mathcal{E}+\mathbf{P}\cdot\mathbf{k}_a+m_a\mathcal{M})}}$$

$$=\frac{[(E_a+m_a)(\mathcal{E}+\mathcal{M})-\mathbf{P}\cdot\mathbf{p}_a]\delta_{\lambda\sigma}+i[\vec{\sigma}]_{\lambda\sigma}\cdot[\mathbf{P},\mathbf{p}_a]}{\sqrt{2(E_a+m_a)(\mathcal{E}+\mathcal{M})(E_a\mathcal{E}-\mathbf{P}\cdot\mathbf{p}_a+m_a\mathcal{M})}}. \tag{2.9}$$

The expression for $\mathbf{x}_a$ in terms of the individual variables can be found in Ref. [7].

Interactions are incorporated into the generators through the replacement of the free mass $\mathcal{M}=\mathcal{M}(\mathbf{k}_a)$ by an interaction dependent operator

$$\mathcal{M}(\mathbf{k}_a)\to M=\mathcal{M}(\mathbf{k}_a)+V. \tag{2.10}$$

The structure of the Poincaré algebra is preserved, provided V is a function of the scalar products of the relative variables, i.e., $[V,\mathbf{S}]=0$. The replacement of $\mathcal{M}$ with M in the single-particle representation of the Poincaré algebra preserves the interaction-free relation between the relative and individual particle variables. This is the essence of the BT construction.

Since physical states belong to a unitary representation of the Poincaré group, in the BT construction with $N=3$ interacting constituent quarks the nucleon state can be represented as

$$|\mathbf{P}_N,M_N,\lambda_N,t_N\rangle=\sum_{\sigma_a,\alpha_a,c_a}\int[d\mathbf{k}_a]\frac{d\mathbf{P}}{(2\pi)^3 2\mathcal{E}}\psi_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{P},\mathbf{k}_a,\sigma_a,\alpha_a,c_a)|\mathbf{P},\mathbf{k}_a,\sigma_a,\alpha_a,c_a\rangle, \tag{2.11}$$

where the invariant measure is given by

$$[d\mathbf{k}_a]\equiv f(\mathbf{k}_a)[d\mathbf{k}_a]_{\rm NR}=f(\mathbf{k}_a)\delta^3\left(\sum_a\mathbf{k}_a\right)\prod_a d^3\mathbf{k}_a, \tag{2.12}$$

with the phase space factor

$$f(\mathbf{k}_a)=\frac{\mathcal{M}}{(2(2\pi)^3)^2\omega_1\omega_2\omega_3} \tag{2.13}$$

and $\mathcal{M}$ and ω_a defined in Eq. (2.4). The δ function represents the momentum constraint given by Eq. (2.5) and assures that only two of the three relative momenta $\mathbf{k}_a$ are independent. The labels α_a and c_a stand for the quark flavor and color quantum numbers, respectively, while λ_N and t_N denote the spin and isospin component of the nucleon. The wave function ψ in Eq. (2.11) can furthermore be written as

$$\psi_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{P},\mathbf{k}_a,\sigma_a,\alpha_a,c_a)=(2\pi)^3 2\sqrt{\mathcal{E}E_N}\,\delta^3(\mathbf{P}_N-\mathbf{P})\frac{1}{\sqrt{6}}\epsilon_{c_1c_2c_3}\psi_{M_N\lambda_N t_N}(\mathbf{k}_a,\sigma_a,\alpha_a), \tag{2.14}$$

with

$$E_N=\sqrt{M_N^2+\mathbf{P}_N^2}. \tag{2.15}$$

ψ_{M_N} is then an eigenstate of the mass operator

$$M\psi_{M_N\lambda_N t_N}(\mathbf{k}_a,\sigma_a,\alpha_a)=\sum_{\sigma_a',\alpha_a'}\int[d\mathbf{k}_a']M(\mathbf{k}_a,\sigma_a,\alpha_a;\mathbf{k}_a',\sigma_a',\alpha_a')\psi_{M_N\lambda_N t_N}(\mathbf{k}_a',\sigma_a',\alpha_a'). \tag{2.16}$$

The kernel in Eq. (2.16) is the matrix element of M in the quark basis states expressed in the relative variables,

$$(2\pi)^3 2\mathscr{E}\delta(\mathbf{P}-\mathbf{P}')M(\mathbf{k}_a,\sigma_a,\alpha_a;\mathbf{k}_a',\sigma_a',\alpha_a')=\langle\mathbf{P},\mathbf{k}_a,\sigma_a,\alpha_a|M(\mathbf{x}_a,\mathbf{k}_a,\vec{\tau}_a)|\mathbf{P}',\mathbf{k}_a',\sigma_a',\alpha_a'\rangle, \tag{2.17}$$

where $\vec{\tau}_a$ are the Pauli matrices acting on the isospin α_a quark indices. Writing this kernel as

$$M(\mathbf{k}_a,\sigma_a,\alpha_a;\mathbf{k}_a',\sigma_a',\alpha_a')\equiv f^{-1/2}(\mathbf{k}_a')M_{\rm NR}(\mathbf{k}_a,\sigma_a,\alpha_a;\mathbf{k}_a',\sigma_a',\alpha_a')f^{-1/2}(\mathbf{k}_a) \tag{2.18}$$

and substituting into Eq. (2.16) leads to the following equation for $M_{\rm NR}$:

$$M_{\rm NR}\psi^{\rm NR}_{M_N\lambda_N t_N}(\mathbf{k}_a,\sigma_a,\alpha_a)=\sum_{\sigma_a',\alpha_a'}\int[d\mathbf{k}_a']_{\rm NR}M_{\rm NR}(\mathbf{k}_a,\sigma_a,\alpha_a;\mathbf{k}_a',\sigma_a',\alpha_a')\psi^{\rm NR}_{M_N\lambda_N t_N}(\mathbf{k}_a',\sigma_a',\alpha_a'), \tag{2.19}$$

with

$$\psi^{\rm NR}_{M_N\lambda_N t_N}\equiv f^{1/2}(\mathbf{k}_a)\psi_{M_N\lambda_N t_N}(\mathbf{k}_a\sigma_a,\alpha' a). \tag{2.20}$$

The normalization of the wave function $\psi^{\rm NR}$ follows from the covariant normalization of the states

$$\langle\mathbf{P}'M_N\lambda_N' t_N'|\mathbf{P}M_N\lambda_N t_N\rangle=(2\pi)^3 2E_N\delta^3(\mathbf{P}'-\mathbf{P})\delta_{\lambda_N'\lambda_N}\delta_{t_N' t_N}, \tag{2.21}$$

$$\langle\mathbf{P}',\mathbf{k}_a',\sigma_a',\alpha_a',c_a'|\mathbf{P},\mathbf{k}_a,\sigma_a,\alpha_a,c_a\rangle=(2\pi)^3 2\mathscr{E}\delta^3(\mathbf{P}'-\mathbf{P})f^{-1}(\mathbf{k}_a)\prod_a^{N-1}\delta^3(\mathbf{k}_a'-\mathbf{k}_a)\prod_a^{N}\delta_{\sigma_a'\sigma_a}\delta_{\alpha_a'\alpha_a}\delta_{c_a'c_a}, \tag{2.22}$$

and is given by

$$\sum_{\sigma_a,\alpha_a}\int[d\mathbf{k}_a]_{\rm NR}|\psi^{\rm NR}_{M_N\lambda_N t_N}(\mathbf{k}_a\sigma_a,\alpha_a)|^2=1 \tag{2.23}$$

which is identical to the normalization of a nonrelativistic nucleon function. In order to make further connection with the nonrelativistic constituent quark model we take $M_{\rm NR}$ to be identical to the quark model Hamiltonian in the center of mass frame expressed in terms of the relative variables. The simple version of the Isgur-Karl Hamiltonian adopted here is given by [8,9]

$$M_{\rm NR}=\sum_a\omega(\mathbf{k}_a)+\sum_{a<b}U(\mathbf{x}_a-\mathbf{x}_b)+\sum_{a<b}\frac{\beta^4}{6m}(\mathbf{x}_a-\mathbf{x}_b)^2+\frac{2}{3}(2\pi)^{3/2}\frac{\delta}{\beta^3}\sum_{a<b}\mathbf{S}_a\cdot\mathbf{S}_b\delta^3(\mathbf{x}_a-\mathbf{x}_b), \tag{2.24}$$

with U representing the difference between the "true" and harmonic oscillator confining potential. The nucleon wave function, corresponding to the $J^P=1/2^+$, $I=1/2$ ground state, is obtained through diagonalization of the above mass operator in the harmonic oscillator basis including up to 2ω states, $\omega=\beta^2/m$. In this basis the general solution for the nucleon wave function can be written as

$$\psi^{\rm NR}_{M_N\lambda_N t_N}=\frac{1}{\sqrt{2}}\cos\phi[\phi_+\xi_++\phi_-\xi_-][\cos\theta\psi_0+\sin\theta\psi_2]+\frac{1}{2}\sin\phi[(\phi_+\xi_-+\phi_-\xi_+)\psi_{2-}+(\phi_-\xi_--\phi_+\xi_+)\psi_{2+}], \tag{2.25}$$

with $\phi_\pm$ and $\xi_\pm$ being the usual spin-1/2 and isospin-1/2 wave functions,

$$\phi_+=\phi_{+\lambda_N}(\sigma_1,\sigma_2,\sigma_3)=\frac{1}{\sqrt{6}}\chi(\sigma_1)\vec{\tau}i\tau_2\chi(\sigma_2)\chi(\sigma_3)\vec{\tau}\chi(\lambda_N),\quad \phi_-=\phi_{-\lambda_N}(\sigma_1,\sigma_2,\sigma_3)=\frac{1}{\sqrt{2}}\chi(\sigma_1)i\tau_2\chi(\sigma_2)\chi(\sigma_3)\xi(\lambda_N), \tag{2.26}$$

$\chi(\sigma_a)$ being the Pauli spinors, and

$$\xi_\pm = \xi_{\pm t_N}(\alpha_a) = \phi_{\pm \lambda_N \to t_N}(\sigma_a \to \alpha_a). \tag{2.27}$$

The normalized, harmonic oscillator orbitals in Eq. (2.25) are given by

$$\psi_0 = \left(\frac{\sqrt{3}}{\pi\beta^2}\right)^{3/2} \exp\left(-\frac{1}{6\beta^2}\sum_{a<b}(\mathbf{k}_a - \mathbf{k}_b)^2\right),$$

$$\psi_2 = \frac{\sqrt{3}}{\beta^2}\left[\sum_{a<b}(\mathbf{k}_a - \mathbf{k}_b)^2 - \beta^2\right]\psi_0,$$

$$\psi_{2+} = \frac{\sqrt{2}}{3\beta^2}\frac{1}{\sqrt{6}}[2(\mathbf{k}_1 - \mathbf{k}_2)^2 - (\mathbf{k}_1 - \mathbf{k}_3)^2 - (\mathbf{k}_2 - \mathbf{k}_3)^2]\psi_0,$$

$$\psi_{2-} = \frac{\sqrt{2}}{3\beta^2}\frac{1}{\sqrt{2}}[(\mathbf{k}_1 - \mathbf{k}_3)^2 - (\mathbf{k}_2 - \mathbf{k}_3)^2]\psi_0. \tag{2.28}$$

In the language of the SU(6) spin-flavor group the angles θ and ϕ specify the magnitudes of the mixture of the ground-state symmetric **56** representation with the **56′** 2ω symmetric and **70** 2ω mixed symmetric representations, respectively. In the harmonic oscillator model, $U=0$, $\omega_a \to \mathbf{k_a}^2/2m_a$ the mixing angles, θ,ϕ in Eq. (2.25) are determined by the matrix elements of the spin-spin interaction whose strength in turn is fitted to the N-Δ mass splitting. This leads to $\delta \sim 300$ MeV and mixing angles $\theta \sim -20°$ and $\phi \sim -14°$ [9]. More realistic, phenomenological interactions including Coulomb potential at short distances, linear instead of harmonic oscillator (HO) confinement, and relativistic dispersion relation will change the HO model parameters [1]. We thus allow for the wave function parameters β, m_a, θ, and ϕ to be varied around their harmonic oscillator values $\beta \sim m_a \sim 300$ MeV.

B. Nucleon wave function in the individual particle basis

In order to calculate current matrix elements it is necessary to express the nucleon wave function in terms of the individual particle basis states rather then in the basis labeled by the relative and c.m. variables. These transition matrix elements utilize Eqs. (2.7) and (2.8) and are given by

$$\begin{aligned}\langle \mathbf{p_a}, \lambda_a | \mathbf{P}, \mathbf{k}_a, \sigma_a \rangle &= \prod_a (2\pi)^3 2\mathscr{E}_a(\mathbf{p}_a)\delta^3(\mathbf{p}_a - \mathbf{p}_a(\mathbf{k}_a, \mathbf{P}))\Omega_{\lambda_a\sigma_a}(\mathbf{k}_a, \mathbf{P}) \\ &= (2\pi)^3 2\mathscr{E}\delta^3\left(\mathbf{P} - \sum_a \mathbf{p}_a\right) f^{-1}(\mathbf{k}_a)\prod_a^{N-1}\delta^3(\mathbf{k}_a - \mathbf{k}_a(\mathbf{p}_a))\Omega_{\lambda_a\sigma_a}(\mathbf{p}_a),\end{aligned} \tag{2.29}$$

with

$$\Omega_{\lambda_a\sigma_a} = \prod_a D(\mathbf{p}_a, \mathbf{k}_a)_{\lambda_a\sigma_a}. \tag{2.30}$$

The nucleon state then becomes

$$|\mathbf{P}_N M_N \lambda_N t_N\rangle = \sum_{\lambda_a\alpha_a c_a}\int\left[\prod_a \frac{d\mathbf{p}_a}{(2\pi)^3 2E_a}\right]\psi_{\mathbf{P}_N M_N \lambda_N t_N}(\mathbf{p}_a, \lambda_a \alpha_a, c_a)|\mathbf{p}_a, \lambda_a, \alpha_a, c_a\rangle, \tag{2.31}$$

with

$$\begin{aligned}\psi_{\mathbf{P}_N M_N \lambda_N t_N}(\mathbf{p}_a, \lambda_a, \alpha_a, c_a) &= \sum_{\sigma_a}\Omega_{\lambda_a\sigma_a}(\mathbf{p}_a)\psi_{\mathbf{P}_N M_N \lambda_N t_N}(\mathbf{k}_a(\mathbf{p}_a), \sigma_a, \lambda_a) \\ &= \sum_{\sigma_a}\left[\prod_a \frac{1}{[(P\cdot p_a) - m_a\mathscr{M}]}\bar{u}(p_a, \lambda_a)u(P, \sigma_a)\right]\psi_{\mathbf{P}_N M_N \lambda_N t_N}(\mathbf{P}(\mathbf{p}_a), \mathbf{k}_a(\mathbf{p}_a), \sigma_a, \alpha_a, c_a).\end{aligned} \tag{2.32}$$

Here u are the free Dirac spinors with the four-vectors p_a and P given by $p_a=(E_a(\mathbf{p}_a),\mathbf{p}_a)$, $P=(\mathscr{E}(\mathbf{P}),\mathbf{P})$. The relative variables in ψ are expressed in terms of the individual ones through Eq. (2.7) which in particular gives

$$(\mathbf{k}_a-\mathbf{k}_b)^2=-(p_a-p_b)^2+\frac{[(p_a-p_b)\cdot P]^2}{\mathscr{M}^2}. \quad (2.33)$$

The important property of the relative variables $\mathbf{k}_a$ in Eq. (2.7) and the matrix elements in Eq. (2.29) is that they are symmetric under the permutation of the three quarks. Thus the nucleon wave function in the individual particle representation has the same symmetry as the wave function in the relative variable representation. The mass operator, however, and the wave function do not have cluster separability in the sense that the dynamics of quarks in a two-quark subsystem depends on the interactions with the third quark (i.e., no three-body forces). It is possible to construct a mass operator in the BT formalism that does obey cluster separability. For that it is necessary, however, to introduce a different set of relative variables; in fact three sets of two relative variables each $(\mathbf{k}_c,\mathbf{K}_c)$, $c=1,2,3$, are required [15]. In each set the variable $\mathbf{k}_c$ describes the relative motion of the a and b quarks in the rest frame of the (ab) subsystem and the variable $\mathbf{K}_c$ represents the relative motion of the (ab) cluster and the c quark in the rest frame of the three quarks. In the presence of interactions the BT construction with the cluster coupling scheme is defined by an interaction-independent relation between the relative variables introduced above and the individual particle variables in analogy to the original BT construction with the "democratic" coupling described in the previous section [16,17]. This allows one to express relative variables in each of the three sets in terms of relative variables of another set or in terms of individual particle variables or relative variables $\mathbf{k}_a$ corresponding to the "democratic" coupling scheme. The details are summarized in the Appendix. The general expression of the mass operator which is symmetric under the permutation of the quarks and has manifestly separable two-quark clusters can also be written in a form given by Eq. (2.10) but with interaction potential satisfying

$$V=\sum_{\mathrm{perm}(abc)} V_{abc}, \quad V_{abc}=\sqrt{M_{ab}^2+\mathbf{K}_c^2}-\sqrt{\mathscr{M}_{ab}^2+\mathbf{K}_c^2}, \quad (2.34)$$

with

$$M_{ab}=\mathscr{M}_{ab}+W_{ab}, \quad (2.35)$$

$\mathscr{M}_{ab}$ being the free mass of the (ab) cluster,

$$\mathscr{M}_{ab}=\sqrt{m_a^2+\mathbf{k}_c^2}+\sqrt{m_b^2+\mathbf{k}_c^2}, \quad (2.36)$$

and $W_{ab}=W_{ab}(\mathbf{x}_c,\mathbf{k}_c,\hat{\mathbf{s}}_a,\hat{\mathbf{s}}_b)$ representing the interaction between quarks in the (ab) cluster. Here $\mathbf{x}_c$ is a position variable conjugated to $\mathbf{k}_c$ and $\hat{\mathbf{s}}_{a,b}$ are the spin variables given in the Appendix. As already mentioned, since only one of the three sets of relative variables is independent it is necessary to specify a particular coupling scheme, say, $(ab)c=(12)3$, and the corresponding set of variables $(\mathbf{k}_3,\mathbf{K}_3)$ to construct the representation space of the Poincaré algebra. Since the mass operator with the potential given by Eq. (2.34) has a complicated structure it is, however, much more difficult to find the exact mass eigenstates. The nucleon state can be generally written as

$$|\mathbf{P}_N M_N \lambda_N t_N\rangle=\sum_{\hat{\sigma}_a,\alpha_a c_a}\int[d\mathbf{k}_3 d\mathbf{K}_3]\frac{d\mathbf{P}}{(2\pi)^3\mathscr{E}}\frac{1}{\sqrt{6}}\epsilon_{c_1c_2c_3}\hat{\psi}_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a)|\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a,c_a\rangle, \quad (2.37)$$

where we have used a caret symbol to distinguish between cluster and "democratic" coupling schemes. The norm in Eq. (2.37) is given in the Appendix. Since there is no simple relation between the mass operator with cluster separability and the constituent quark model Hamiltonian, there is also no direct connection between the constituent quark model wave functions and the relativistic wave function $\hat{\psi}$ in the cluster coupling scheme. It would appear that a choice

$$\hat{\psi}(\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a)\propto\psi^{\mathrm{NR}}(\mathbf{k}_a,\hat{\sigma}_a,\alpha_a), \quad (2.38)$$

with ψ^{NR} given by Eq. (2.25), could be used, in the cluster coupling scheme, to represent the nucleon wave function just like it was used in the "democratic" scheme. However, unlike $\Omega_{\lambda_a\sigma_a}(\mathbf{p}_a)$ the matrix elements

$$\langle\mathbf{p}_a,\lambda_a|\mathbf{P},\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a\rangle\propto\hat{\Omega}_{\lambda_a\hat{\sigma}_a}(\mathbf{p}_a) \quad (2.39)$$

are not symmetric under the permutation of the three quarks. Thus, the nucleon wave function given by Eq. (2.38) when expressed in the individual particle basis will not have the desired permutational symmetry. For example, if, as commonly used, ψ^{NR} is restricted to the ground-state HO wave function given by the first line in Eq. (2.25) with $\theta=\phi=0$, for the proton it may be written as

$$\hat{\psi}=\psi_0\frac{1}{\sqrt{3}}\Bigg\{(uud)\phi_{+\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)+(udu)\left[-\frac{1}{2}\phi_{+\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)+\frac{\sqrt{3}}{2}\phi_{-\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)\right]$$
$$+(duu)\left[-\frac{1}{2}\phi_{+\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)-\frac{\sqrt{3}}{2}\phi_{-\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)\right]\Bigg\}. \quad (2.40)$$

The overall symmetry in the isospin-spin-orbital indices requires

$$-\frac{1}{2}\,\phi_{+\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)+\frac{\sqrt{3}}{2}\,\phi_{-\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)=\phi_{+\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_3,\hat{\sigma}_2), \tag{2.41}$$

which is satisfied by the nonrelativistic wave functions given in Eq. (2.26). Transforming to the individual particle basis, in analogy to Eq. (2.32), $\hat{\psi}=\hat{\psi}(\mathbf{p}_a,\lambda_a)$ is given by Eq. (2.40) with

$$\phi_{\pm\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)\rightarrow\phi_{\pm\lambda_N}(\mathbf{p}_a,\lambda_1,\lambda_2,\lambda_3)=\sum_{\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3}\hat{\Omega}_{\lambda_1,\lambda_2,\lambda_3;\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3}(\mathbf{p}_a)\,\phi_{\pm\lambda_N}(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3), \tag{2.42}$$

and using the explicit form of $\hat{\Omega}$ given in the Appendix it can be shown that $\phi_{\pm\lambda_N}(\mathbf{p}_a;\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)$ no longer satisfies Eq. (2.41). Inspecting Eq. (2.40) it may seem appropriate to independently use three different transformation matrices, $\hat{\Omega}^c$, $c=1,2,3$, for the three components of the wave function corresponding to the $(uu)_{S=1}u$, $(ud)_{S=1}u$, and $(du)_{S=1}u$ coupling schemes [11]. This transformation is fully symmetric but not unitary and therefore destroys orthogonality and leads to mixture of states which in the relative variable representation would correspond to nucleon excitations. It thus seems that only the "democratic" coupling scheme, discussed previously, allows for a systematic construction of the quark representation for baryon wave functions which may also be directly related to the nonrelativistic constituent quark model wave functions.

Finally, we should mention that there also exists models based on yet another generalization of the nonrelativistic quark model wave function [12,13]. As shown in the Appendix the wave function in the cluster coupling scheme can be written in terms of free Dirac spinors. Generalized wave functions are usually constructed by writing

$$\phi(\mathbf{p}_i,\lambda_i)\propto\bar{u}(p_1,\lambda_1)\Gamma_{12}C\mathbf{u}^T(p_2,\lambda_2)\bar{u}(p_3,\lambda_3)\Gamma_3 u(P,\lambda_N), \tag{2.43}$$

with Γ_{12},Γ_3 being combinations of the Dirac gamma matrices and the particle momenta such that for $\mathbf{p}_a/m_a\rightarrow 0$ the wave function reduces to that of the nonrelativistic quark model. Because such model wave functions are usually not eigenstates of a mass operator and often entail a large number of free parameters, the effectiveness and insight of this approach are quite limited.

C. Current matrix elements and covariance

Under a Poincaré transformation, hadronic states should transform as elements of a unitary representation while a matrix element is expected to transform covariantly (i.e., consistent with the tensorial rank of the matrix element operator). Covariance under a Lorentz transformation Λ for a matrix element of a vector current J^μ implies

$$J^\mu(P',P)=\langle P'|J^\mu(0)|P\rangle=\Lambda^\mu_\nu J^\nu(\Lambda^{-1}P',\Lambda^{-1}P). \tag{2.44}$$

If $U(\Lambda)$ are the unitary operators corresponding to the transformation Λ in the space of physical states $|P\rangle$, Eq. (2.44) requires

$$U^{-1}(\Lambda)J^\mu(0)U(\Lambda)=\Lambda^\mu_\nu J^\nu(0) \tag{2.45}$$

or, in infinitesimal form,

$$[J^\mu,M_{\alpha\beta}]=i[\delta^\mu_\alpha J_\beta(0)-\delta^\mu_\beta J_\alpha(0)], \tag{2.46}$$

with $M_{\mu\nu}$ being the generators of the Lorentz group. These conditions are not satisfied by the free field, one-body current frequently used to calculate matrix elements and to define form factors because, as described in a previous subsection, the generators, $M_{\mu\nu}$ contain interactions. Thus two-particle and perhaps even more complex currents must be included to restore covariance. Consequently, hadronic matrix elements expressed in terms of four-vectors representing the particle momenta and polarizations will violate covariance. The nonrelativistic expansion of the electromagnetic currents in the presence of internal interactions has been extensively studied [4,18]; however, a solution to all orders in V/m required to maintain covariance is still lacking. Violation of covariance in the matrix elements of the current will be manifest through presence of additional terms in Lorentz decomposition of the matrix elements and in a spurious physical form factors dependence on particle momenta. Since the hadronic states, which by definition do not transform covariantly, naturally introduce a separation between the energy and momentum components of the particle four-momenta

$$P^\mu=(E,\mathbf{P})=(n\cdot P,P-(n\cdot P)n)\ ,\quad n^\mu=(1,\mathbf{0}), \tag{2.47}$$

it is convenient to allow for the vector n to explicitly appear in the Lorentz decomposition of matrix elements. Expressing all four vectors in terms of their longitudinal, parallel to n, and transverse, perpendicular to n, components will permit identifying spurious momentum dependences of the physical form factors and unphysical form factors as remnant terms proportional to the vector n in the Lorentz decomposition of matrix elements. This decomposition also has a deeper geometrical interpretation. The vector n specifies the orientation of a $3N$-dimensional quantization surface in a $4N$-dimensional direct product space which contains world lines of the N particles. Defining the components of n as in Eq. (2.47) corresponds to a particular orientation of the quantization surface or equivalently to a particular choice of the quantization scheme [19]. In such a scheme the wave functions are defined as probability amplitudes depending only on the ordinary three-momenta and the evolution of the systems is determined by the energy operator which transforms the states in the ordinary Minkowski time. It is, however, possible to choose other quantization schemes corresponding to a surface defined by n whose components are different from those in Eq. (2.47). In general, for an arbitrary n the construction of the generators of the Poincaré group can be formulated in a fame-independent, quantization-dependent way. For a vector n defining a spacelike surface with $n^2<0$ the individual particle momenta are defined as the transverse p_{Ta} and longitudinal p_{La} (to n) components of Lorentz four-vectors:

$$p^{\mu}_{Ta}=p^{\mu}_{a}-p_{La}\frac{n^{\mu}}{|n|}, \quad p_{La}=\frac{n\cdot p_a}{|n|},$$
$$p_a^2=m_a^2=p_{La}^2(p_{Ta}^2)+p_{Ta}^2. \qquad (2.48)$$

Since all transverse four-vectors have only three independent components these again can be denoted in a three-vector form, i.e., $p_{Ta}=(0,\mathbf{p}_a)$. The ten generators P^{μ}, $M_{\mu\nu}$ of the Poincaré group are also projected into their longitudinal and transverse components with the interactions contained in the longitudinal components only. In this construction the wave functions will have a frame-independent form but they will explicitly depend on the vector n. Although the spurious n dependence of matrix elements results from the Lorentz symmetry breaking of the current operator, it permits quantitatively assessing the extent covariance is violated in a model calculation and, as detailed in the next section, to establish a potentially useful model criterion. For fixed values of external particle momenta P^{μ}, sensitivity of observables to n^{μ} orientation corresponds to sensitivity to different quantization schemes. However, since all spurious dependence of form factors enters through scalar products of physical four-vectors with n, an equivalent description of the Lorentz symmetry breaking can be obtained by fixing the components of n^{μ} while changing the reference frame. This corresponds to changing the particle's momentum components with the physical scalar products, $P_i\cdot P_j$ fixed. These two alternative approaches can in particular be used to show the equivalence of relativistic quark model calculations in the light front quantization and in the infinite momentum frame approach.

III. ELECTROMAGNETIC STRUCTURE OF THE NUCLEON

A. Elastic form factors

In this section the wave function, Eq. (2.32), and the formalism outline above are utilized to calculate the static form factors of the nucleon. Allowing for the vector n^{μ} to explicitly appear in the matrix element of the electromagnetic current yields the most general form

$$\langle \mathbf{P}'\lambda'|J^{\mu}_{\rm em}|\mathbf{P}\lambda\rangle=\bar u(\mathbf{P}'\lambda')\Gamma^{\mu}u(\mathbf{P},\lambda)\equiv M^{\mu}_{\lambda'\lambda},$$
$$\Gamma^{\mu}\equiv\gamma^{\mu}F_1+\frac{i\sigma^{\mu\nu}}{2M}q_{\nu}F_2+\frac{n^{\mu}|\Sigma|}{\Sigma\cdot n}F_3+\frac{n^{\mu}\Sigma^2 n\cdot\gamma}{(\Sigma\cdot n)^2}F_4+\frac{\Sigma^{\mu}n\cdot\gamma}{\Sigma\cdot n}F_5, \qquad (3.1)$$

with $\Sigma^{\mu}\equiv P'^{\mu}+P^{\mu}$, $|\Sigma|=2\sqrt{M^2+Q^2/4}$, $Q^2=-q^2=-(P'-P)^2$, and $M=M_N$ being the nucleon mass. The form factors F_i, $i=1,\cdots,5$ will in general depend not only on Q^2 but also on the scalar products of the final and initial nucleon momenta with n. Here the analysis is restricted to spacelike momentum transfer with $q\cdot n=0$. The elastic matrix element satisfies current conservation and this limits the number of spurious form factors to three (F_i, $i=3,\ldots,5$). Besides Q^2 the only independent scalar product that can be formed from the vectors q, Σ and n is $\Sigma\cdot n$ which implies

$$F_i=F_i(Q^2,\mathbf{\Sigma}^2)=F_i\left(Q^2,\Sigma^2-\frac{(\Sigma\cdot n)^2}{n^2}\right). \qquad (3.2)$$

The three vectors n, q, and Σ are linearly independent and together with

$$v_{\mu}\equiv\epsilon_{\mu\nu\rho\sigma}\frac{n^{\nu}}{|n|}\frac{\Sigma^{\rho}}{|\Sigma|}\frac{q^{\sigma}}{Q}, \qquad (3.3)$$

with $Q\equiv\sqrt{Q^2}$, can be used to define the basis. Since $n^2>0$ and $q\cdot n=0$ without loss of generality, n and q may be chosen to define the timelike and one spacelike axis of the coordinate system, respectively,

$$\frac{n^{\mu}}{|n|}=(1,0,0,0)^T, \quad \frac{q^{\mu}}{Q}=(0,0,1,0)^T. \qquad (3.4)$$

Since $\Sigma\cdot n=\Sigma_L\neq 0$ and $\Sigma\cdot q=P'^2-P^2=0$, the vector Σ defines another spacelike direction chosen to be the third axis,

$$\Sigma^{\mu}=(\Sigma_L,0,0,|\mathbf{\Sigma}|)^T, \qquad (3.5)$$

with $\Sigma_L=\sqrt{\Sigma^2+\mathbf{\Sigma}^2}=2\sqrt{M^2+Q^2/4+\mathbf{\Sigma}^2/4}$. Finally the components of v are given by

$$v^{\mu}=(0,1,0,0)^T. \qquad (3.6)$$

The form factors can now be extracted from $M^0_{\lambda'\lambda}$, $M^1_{\lambda'\lambda}$ and $M^3_{\lambda'\lambda}$ ($M^2_{\lambda'\lambda}=0$ because of current conservation) calculated for different nucleon spin projections. Since $\Sigma_L=\Sigma_L(|\mathbf{\Sigma_T}|)$, the matrix elements and form factors become functions of Q^2 and $|\mathbf{\Sigma}|$ alone. The unphysical dependence on $|\mathbf{\Sigma}|$ will be studied in two limiting cases $|\mathbf{\Sigma}|=0$ and $|\mathbf{\Sigma}|\to\infty$. As seen from the choice of the basis the first case corresponds to the instant quantization, while the $|\mathbf{\Sigma}|\to\infty$ limit corresponds to an infinite momentum frame limit in which the initial and final nucleons move with a large velocity in the third axis. The results obtained in this limit are equivalent to those of the light cone quantization on the $z^+=z^0+z^3=0$ surface. In terms of the matrix elements $M^{\mu}_{\lambda'\lambda}$ the five form factors F_i can be determined from

$$M^0_{++}=2M\left(1+\frac{\boldsymbol{\Sigma}^2}{2M(\Sigma_L+2M)}\right)(F_1+F_5)-2M\,\eta F_2+\frac{|\Sigma|}{2\Sigma_L}\left(\Sigma_L+2M-\frac{\boldsymbol{\Sigma}^2-Q^2}{\Sigma_L+2M}\right)F_3+\frac{\Sigma^2}{2\Sigma_L^2}\left(\Sigma_L+2M+\frac{\boldsymbol{\Sigma}^2-Q^2}{\Sigma_L+2M}\right)F_4, \tag{3.7}$$

$$M^0_{+-}=iQ\Sigma_L\left[\frac{F_1+F_5}{\Sigma_L+2M}+\frac{F_2}{2M}-\frac{|\Sigma|}{\Sigma_L(\Sigma_L+2M)}F_3+\frac{\Sigma^2}{\Sigma_L^2(\Sigma_L+2M)}F_4\right], \tag{3.8}$$

$$M^3_{++}=\Sigma_L\left[F_1-\eta\frac{2M}{\Sigma_L+2M}F_2+\frac{2M}{\Sigma_L}\left(1+\frac{\boldsymbol{\Sigma}^2}{2M(\Sigma_L+2M)}\right)F_5\right], \tag{3.9}$$

$$M^3_{+-}=iQ\left[F_1+\left(1+\frac{\boldsymbol{\Sigma}^2}{2M(\Sigma_L+2M)}\right)F_2+\frac{\boldsymbol{\Sigma}^2}{\Sigma_L(\Sigma_L+2M)}F_5\right], \tag{3.10}$$

$$M^1_{++}=-iQ[F_1+F_2], \tag{3.11}$$

with $\eta\equiv Q^2/(4M^2)$. In particular in the $|\boldsymbol{\Sigma}|\to 0$ limit

$$M^0_{++}\to 2M[F_1-\eta F_2]+(\text{terms with } F_3,\ F_4,\ \text{and } F_5), \tag{3.12}$$

while in the $|\boldsymbol{\Sigma}|\to\infty$ limit

$$M^+_{++}\equiv M^0_{++}+M^3_{++}\to 2|\boldsymbol{\Sigma}|[F_1+F_5]+O(1),$$
$$M^+_{+-}\equiv M^0_{+-}+M^3_{+-}\to\frac{2iQ}{M}F_2+O\left(\frac{1}{\boldsymbol{\Sigma}}\right). \tag{3.13}$$

If the unphysical form factors are ignored, then in the instant quantization M^0_{++} is directly proportional to the nucleon electric form factor $G_E=F_1-\eta F_2$ while $M^+_{++(-)}$ in the infinite momentum frame or light cone quantization are proportional to the Pauli and Dirac form factors $F_{1(2)}$, respectively. However, in general both M^0_{++} and M^+_{++} receive contributions from the unphysical form factors. It is worth noting that for spin-0 particles like the pion the matrix elements of the J^+ component do not involve unphysical form factors. But because, as follows from Eq. (3.13), this does not happen in general, it cannot be argued that the use of the J^+ component eliminates all spurious contributions. Ignoring the unphysical form factors, the J^+ component alone can be used to determine $F_{1,2}$ and subsequently $G_{E,M}$. Interestingly, from Eq. (3.11) it follows that the $J^1=v\cdot J$ component does not contain spurious form factors in any quantization. Thus it also could be used to define G_M in infinite momentum frame as it is done in the instant quantization. In the nonrelativistic limit of the light cone quantization the two definitions of G_M become equivalent.

Using the wave function given by Eqs. (2.32), (2.25)–(2.28) for $\theta=0$ we have varied the quark mass m, the oscillator parameter β, and the mixing angle ϕ between the **56** and **70** SU(6) representations to obtain the best fit to the

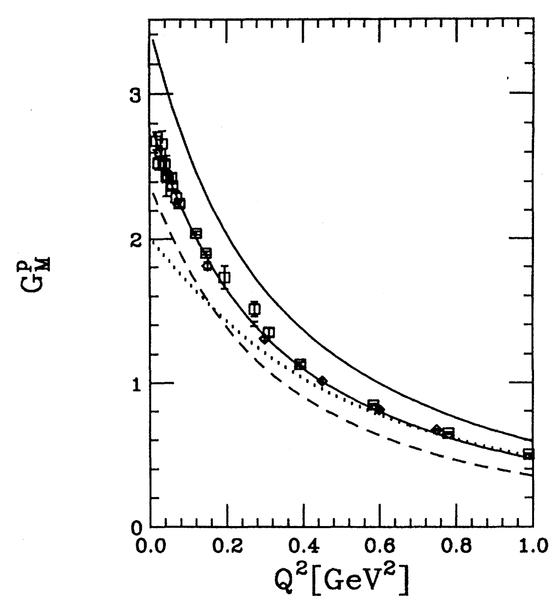

FIG. 1. Proton magnetic form factor. The assignment of curves is explained in the text.

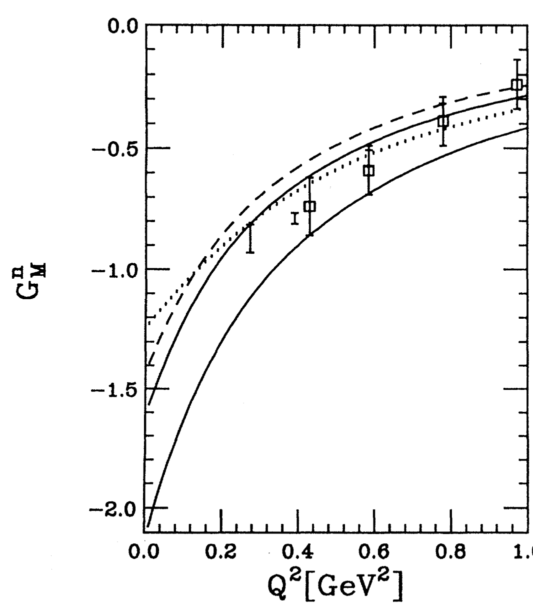

FIG. 2. Neutron magnetic form factor.

proton and neutron electric and magnetic form factors. Restriction to $\theta=0$, i.e., elimination of the orbital symmetric 2ω, **56′** contribution, does not change significantly the behavior of form factors at momentum transfer $Q^2\lesssim 1$ GeV^2. Higher orbital excitations will become important at larger momentum transfer but at the same time will require a more detailed knowledge of the mass operator. The addition of the mixed symmetric 2ω excitations of the **70** representation is crucial for the proper description of the neutron electric form factor, G_E^n at low Q^2 [20] and also, as will be shown in the next section, for the right behavior of the neutron to proton structure function ratio $F^n(x)/F^p(x)$ in the valence region as $x\rightarrow 1$. In Fig. 1 the proton magnetic form factor G_E^p is shown for $Q^2\leqslant 1$ GeV^2. For quark mass $m_{LC}=200$ MeV, oscillator parameter $\beta_{LC}=400$ MeV, and mixing angle $\phi=-14°$ the magnetic form factor, as extracted from Eq. (3.13) in the infinite momentum frame, is represented by the lower solid line. The upper solid line corresponds to G_M^p as calculated in the infinite momentum frame limit from Eq. (3.11). The doted line shows the results in the instant quantization ($|\mathbf{\Sigma}|=0$) with the above values for the quark mass, β, and ϕ. The dashed line is the instant result with the parameters $m_{IN}=50$ MeV and $\beta_{IN}=350$ MeV obtained as the best overall fit to proton and neutron electric and magnetic form factors. Figure 2 displays the results obtained for the neutron magnetic form factor, G_M^n with the same curve assignment as in Fig. 1. As seen from Figs. 1 and 2 only the use of the J^+ component of the current in the light cone limit leads to a good description of the magnetic form factors for parameters which roughly agree with those used to fit the low mass spectrum with the mass operator given by Eq. (2.24). For $\phi=0$ the magnetic moments $G_M^{(p,n)}(0)$, derived from Eq. (3.11) for the instant quantization and from Eq. (3.13) for the light cone case, are given by the following:

instant,

$$G_M^{p,n}(0)=e_{u,d}\int [d\mathbf{k}]_{\rm NR}|\psi_0(\mathbf{k}_a)|^2\frac{M_N}{E_1}\left[1-\frac{1}{E_1+m}\left(\frac{\mathbf{k}_1^2}{3E_1}-\frac{\mathbf{k}_1\cdot\mathbf{k}_3}{2E}\right)\right]+e_{d,u}\int [d\mathbf{k}]_{\rm NR}\psi_0(\mathbf{k}_a)^2\frac{M_N}{E_3}\left[1-\frac{\mathbf{k}_3^2}{E_3+m}\left(\frac{1}{3E_3}-\frac{1}{E}\right)\right]; \tag{3.14}$$

light cone:

$$\begin{aligned}G_M^{p,n}(0)=&e_{u,d}\int [d\mathbf{k}]_{NR}|\psi_0(\mathbf{k}_a)|^2\left[2\frac{(m+k_1^+)(k_1^+-E)+\frac{1}{2}\mathbf{k}_{\perp 1}^2}{(E_1+m)k_1^+}+2\frac{k_2^+(m+k_2^+)+\dfrac{k_2^+}{2k_1^+}\mathbf{k}_{\perp 1}\cdot\mathbf{k}_{\perp 2}}{(E_2+m)k_2^+}\right.\\&\left.-\frac{k_3^+(m+k_3^+)+\dfrac{k_3^+}{2k_1^+}\mathbf{k}_{\perp 1}\cdot\mathbf{k}_{\perp 3}}{(E_3+m)k_3^+}\right]+e_{d,u}\int [d\mathbf{k}]_{\rm NR}\psi_0(\mathbf{k}_a)^2\left[2\frac{(m+k_3^+)(k_3^+-E)+\frac{1}{2}\mathbf{k}_{\perp 3}^2}{(E_3+m)k_3^+}\right.\\&\left.-4\frac{k_2^+(m+k_2^+)+\dfrac{k_2^+}{2k_3^+}\mathbf{k}_{\perp 3}\cdot\mathbf{k}_{\perp 2}}{(E_2+m)k_2^+}-2\frac{k_1^+(m+k_1^+)+\dfrac{k_1^+}{2k_3^+}\mathbf{k}_{\perp 3}\cdot\mathbf{k}_{\perp 1}}{(E_1+m)k_1^+}\right]\\=&e_{u,d}\int [dx_ad^2\mathbf{k}_{\perp a}]|\psi_0(x_a,\mathbf{k}_{\perp a})|^2\left[\frac{\Pi_aE_a}{E}\right]g_{M1}(x_a,\mathbf{k}_{\perp a})+e_{d,u}\int [dx_ad^2\mathbf{k}_{\perp a}]|\psi_0(x_a,\mathbf{k}_{\perp a})|^2\left[\frac{\Pi_aE_a}{E}\right]g_{M3}(x_a,\mathbf{k}_{\perp a}),\end{aligned} \tag{3.15}$$

with

$$E_a=\sqrt{m^2+\mathbf{k}_a^2},\quad E=\sum_a E_a,\quad k_a^+=E_a+k_a^z,\quad \mathbf{k}_{\perp a}=(k_a^x,k_a^y), \tag{3.16}$$

and ψ_0 given by the first line in Eq. (2.28). The last two lines in Eq. (3.15) are explicitly written in terms of the light cone momenta obtained through a change of variables $k_a^z\rightarrow x_a$ given by

$$x_a=\frac{k_a^+}{E}=\frac{E_a(\mathbf{k}_a)+k_a^z}{\Sigma_aE_a(\mathbf{k}_a)} \tag{3.17}$$

and

$$[dx_ad^2\mathbf{k}_{\perp a}]=\delta\left(\sum_a x_a-1\right)\delta^2\left(\sum_a \mathbf{k}_{\perp a}\right)\prod_a\left[\frac{dx_a}{x_a}d^2\mathbf{k}_{\perp a}\right] \tag{3.18}$$

and with $g_{M1(3)}$ given by the two terms in the parentheses in the first two lines in Eq. (3.15) respectively.

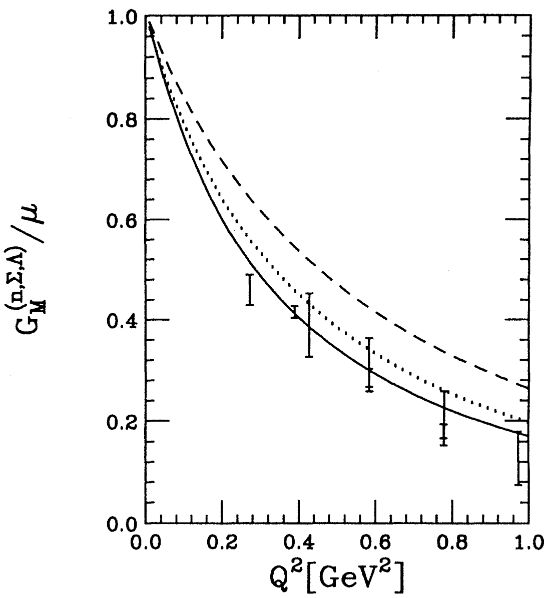

FIG. 3. Λ (dashed line), Σ^0 (dotted line) magnetic form factors compared with the neutron magnetic form factor (solid line) in the instant quantization.

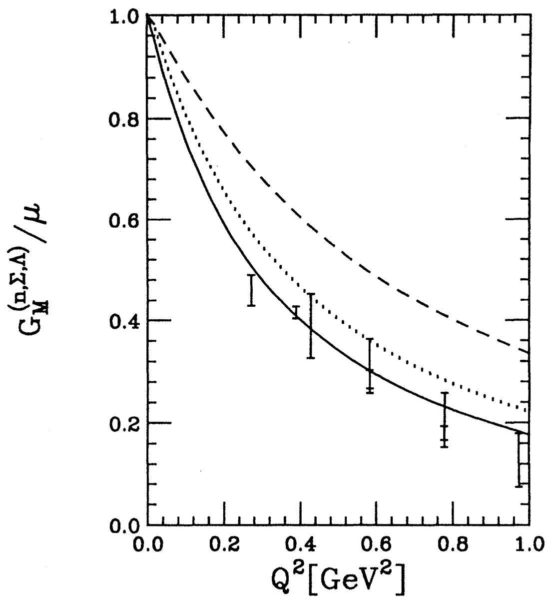

FIG. 4. Λ (dashed line), Σ^0 (dotted line) magnetic form factors compared with the neutron magnetic form factor (solid line) in the light cone quantization.

In the nonrelativistic limit $|\mathbf{k}_a|/m \rightarrow 0$, Eqs. (3.14), (3.15) give $G_M^{p,n}(0) = M_N/m, -2M_N/3m$ for the instant quantization while for the light cone case, $G_M^{p,n}(0) = 3,-2$, and is both nucleon and quark mass independent. In the instant case the relativistic corrections significantly reduce the nonrelativistic $G_M^{p,n}(0)$ and with the typical quark mass and oscillator parameter lead to magnetic moments 30% smaller than measured. Even for $m = 50$ MeV, which gives the best overall fit to the nucleon electromagnetic properties in the instant quantization, the magnetic moments are still about 18% too small. The effects of the mixture with the **70** representation do not change these results significantly. In Figs. 3 and 4 the neutron magnetic form factor (solid line) is also compared with our predictions for the Λ (dashed line) and Σ^0 (dotted line) magnetic form factors in the instant and light cone quantizations respectively. The effects of the unphysical form factors are even more significant for the electric form factors than for the magnetic ones. In Figs. 5 and 6 the electric form factors are shown. In Fig. 3 the upper solid line is the $|\mathbf{\Sigma}| \rightarrow \infty$ result for proton electric form factor G_E^p, calculated with $F_{1,2}$ form factors extracted from Eqs. (3.7)–(3.11) i.e., with an explicit account for the possibility of the unphysical form factors in the matrix element. The lower solid line is the light cone result as defined by the matrix element of J^+ which in turn include the unphysical form factor F_5 but at the same time is consistent with the normalization of the wave function given by Eq. (2.23). The upper dashed line is the instant result (with the smaller quark mass) for the matrix element of J^0 which again include unphysical form factors but agrees with the wave function normalization. The lower dashed line is the instant result calculated with the form factors $F_{1,2}$ extracted from Eqs. (3.7)–(3.11). For the neutron electric form factor in Fig. 6 the solid and dashed curves leading to $G_N^E(0) = 0$ are given by the J^+ and J^0 matrix elements in the light cone and instant quantization, respectively, while the other two curves are obtained after extracting $F_{1,2}$ from the five matrix elements. It is clear that only the matrix elements of the charge components of the current, $n \cdot J = J^0, J^+$ for instant and light cone quantization respectively, lead to sensible results. The negative charge radius of the neutron given by $G_E^n(Q^2) > 0$ at small Q^2 comes from Wigner rotations in Eq. (2.32) but mostly from the admixture of the **70** representation. The negative sign, $\phi \sim -14°$ of the mixing angle resulting from the spin-spin

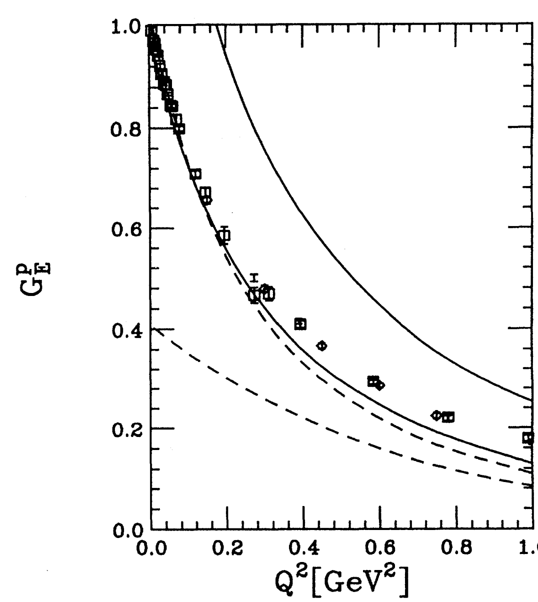

FIG. 5. Proton electric form factor.

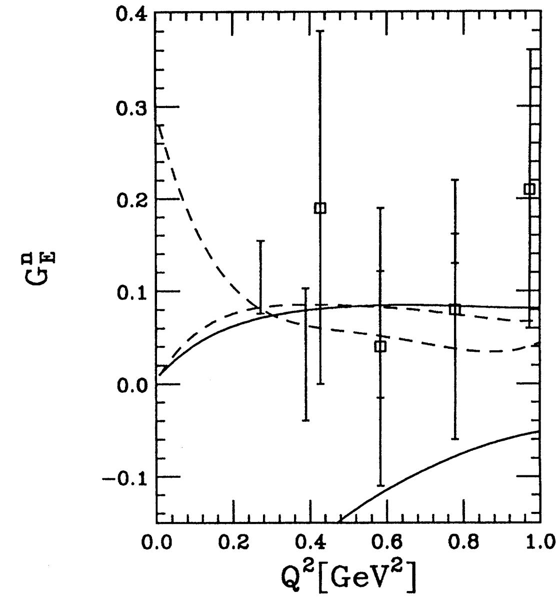

FIG. 6. Neutron electric form factor.

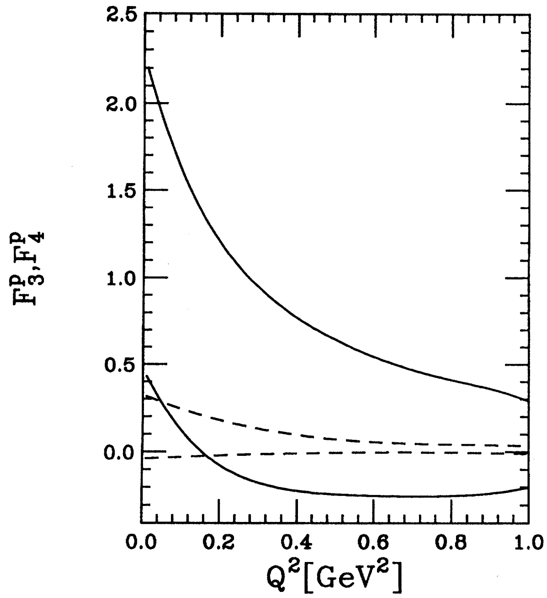

FIG. 7. Unphysical form factors in the light cone (solid lines) and instant (dashed lines) quantizations.

term in the mass operator is crucial to obtain the correct behavior. Figure 7 displays the unphysical form factors $F_{3,4}$ for the light cone (solid line) and instant (dashed line) limits, respectively. While the light cone quantization has provided the correct description of the magnetic moments it also suffers from the largest Lorentz symmetry breaking in the matrix element. It is thus clear that one should not trust the one-body approximations for all components of the current in the light cone quantization; however, the results for the magnetic moments indicate that it may be a good approximation for the J^+ component alone. From Fig. 7 it also follows that at larger momentum transfer the one-body approximation in the instant quantization may be accurate but, as already mentioned, a good description of the form factors may then require a more detailed knowledge of the higher harmonic oscillator components of the nucleon wave function. We have also computed the nucleon axial charge. For the pure **56** representation G_A computed from the $n \cdot A$ component of the axial current is given by the following:

instant,

$$G_A(0)=\frac{5}{3}\int [d\mathbf{k}]_{\rm NR}|\psi_0(\mathbf{k}_a)|^2\frac{(m+E_1)^2-\frac{1}{3}\mathbf{k}_1^2}{2E_1(m+E_1)}\ ; \tag{3.19}$$

light cone,

$$G_A(0)=\frac{5}{3}\int [d\mathbf{k}]_{\rm NR}|\psi_0(\mathbf{k}_a)|^2\frac{(m+k_1^+)^2-\mathbf{k}_{\perp 1}^2}{2(E_1+m)k_1^+}$$

$$=\frac{5}{3}\int [dx_a d^2\mathbf{k}_{\perp a}]|\psi_0(x_a,\mathbf{k}_{\perp a})|^2\left[\frac{\Pi_a E_a}{E}\right]\frac{(m+k_1^+)^2-\mathbf{k}_{\perp 1}^2}{2(E_1+m)k_1^+}\ . \tag{3.20}$$

This for $m=200$ MeV and $\beta=400$ MeV gives $G_A(0)\sim 1.1$ and 1.0 for the instant and light cone quantizations, respectively. Again the relativistic effects are large and significantly alter the nonrelativistic $G_A(0)=5/3$ limit.

B. Deep inelastic structure functions

The twist-2 nucleon structure function $F_2^{p,n}(x,\mu^2)$ that is measured in unpolarized deep inelastic scattering can be defined in terms of the following matrix elements [21]

$$\frac{1}{2}\sum_{\lambda_N}\langle P,\lambda_N|i\bar{q}_\alpha(0)\gamma^+(\overleftrightarrow{D}_+)^n q_\beta(0)|P,\lambda_N\rangle_\mu=\delta_{\alpha\beta}a_{n+1}(M_N)^{n+1}, \tag{3.21}$$

$$\frac{F_2^{p,n}(x,\mu^2)}{x}=\sum_{\alpha\beta}\frac{\langle p(n)|Q^2_{\alpha\beta}|p(n)\rangle}{8\pi}\int dz^- e^{x\frac{z^-P^+}{2}}\langle P|\bar{q}_\alpha(z^-)\gamma^+ q_\beta(0)|P\rangle, \tag{3.22}$$

where $q_\alpha(z^-)=q_\alpha(z^-,z^+,\mathbf{z}_\perp\sim 1/\mu)$ and α is the flavor index. If the quark field operators $q_i(z^-)$ are quantized on the light front surface $z^+=0$ and expanded in terms of creation and annihilation operators, the valence contribution to the structure functions is given by

$$\frac{1}{x}F_2^{p,n}(x,\mu^2)=\sum_\alpha e_\alpha^2 q_\alpha^{p,n}(x), \tag{3.23}$$

with $q_\alpha^{p,n}(x)$ denoting the valence quark distributions in the proton and neutron, respectively, and e_a being the valence quark electric charges. The quark distributions are determined from the valence nucleon light cone wave function

$$q_\alpha^{p,n}(x)=\sum_{\lambda_N\lambda_a\alpha_a c_a}\int [dx_a d^2\mathbf{k}_{\perp a}]|\psi_{M_N\Lambda_N t_n}(x_a,\mathbf{k}_{\perp a},\lambda_a,\alpha_a,c_a)|^2\delta(x_a-x)\delta_{\alpha_a\alpha}, \tag{3.24}$$

with p and n corresponding to $t_n=+1/2$ and $-1/2$, respectively. For the soft wave function the integral over transverse momentum is dominated by $\mu\lesssim\beta$; thus we can set the integration limits to be infinite while still computing the low energy structure functions $F_2(x,\mu\sim\beta)$. To compare with experimental data the soft structure functions should be evolved to the appropriate momentum scale of an experiment, typically of the order of $\mu^2\sim Q^2\sim 10$–$20\ \mathrm{GeV}^2$:

$$q_\alpha^N(x,Q)=\sum_i\int_x^1\frac{dy}{y}P_{\alpha i}^N\left(\frac{x}{y},\frac{\mu}{\mu_0}\right)q_i^N(y,\mu_0). \tag{3.25}$$

In general the left hand side involves both valence and nonvalence distributions. Perturbative, Altarelli-Parisi evolution equations may be used (for not too small x) if the starting distributions $q_i^N(y,\mu_0)$ are evaluated at μ_0 high enough to justify perturbative expansion. This is clearly not the case for the soft structure functions and phenomenological splitting functions $P_{\alpha i}^N$ have to be introduced to match the different scales [22]. We shall, however, be primarily interested in the ratio of the nucleon and proton structure functions in the valence region for which the evolution effects are small and may be neglected. For the nucleon wave function constructed in Sec. II with the parameters $m=m_{LC}$, $\beta=\beta_{LC}$, and $\phi=-14°$ obtained from the fits to the electromagnetic form factors the ratio $R^{np}(x)$

$$R^{np}(x)=\frac{F_2^n(x)}{F_2^p(x)}=\frac{q_u^p(x)+4q_d^p(x)}{4q_u^p(x)+q_d^p(x)}=\frac{1}{4}+\frac{15r_{du}}{4(4+r_{du})}, \tag{3.26}$$

with $r_{du}=r_{du}(x)=q_d^p(x)/q_u^p(x)$ given by the dashed line in Fig. 8. The resulting positive slope of $R^{np}(x)$ for large x, contradictory to the experimental data, can easily be understood. After expressing Eq. (3.24) in terms of the relative three momenta $\mathbf{k}_a$ related to x_a through Eq. (3.17),

$$q_\alpha^N(x)=\sum_{\lambda_N\sigma_a\alpha_a c_a}\int [d\mathbf{k}]_{\mathrm{NR}}\frac{E}{\Pi_a E_a}|\psi_{M_N\Lambda_N t_n}(\mathbf{k}_a,\sigma_a,\alpha_a,c_a)|^2\delta\left(x-\frac{\sqrt{m_a^2+\mathbf{k}_a^2}+k_a^z}{E}\right)\delta_{\alpha_a\alpha}, \tag{3.27}$$

it becomes clear that the quark distributions $q(x)$ are solely determined by the momentum distribution of the valence quarks. The interaction-independent relation in the Bakamjian-Thomas construction, between the individual and relative particle momenta is equivalent to an interaction-independent boost. Thus the light cone wave function is given by a free Lorentz transformation of the rest frame wave function to the infinite momentum frame. This is equivalent to the change of variables given by Eq. (3.17). The quark distributions in the scaling variable x, in the Bakamjian-Thomas construction, correspond to distributions of the combination $E_a(\mathbf{k}_a)+k_a^z$ in the rest frame wave function. The negative charge radius of the neutron implies that in the neutron there is an excess of the d quarks over the u at larger distances. This in turn implies that the neutron wave function generates a u quark momentum distribution peaking at larger momentum than the d quark distribution. Since Eq. (3.27) relates large momenta to large x this also implies that for the proton the d quark distribution will peak at larger x than the u quark distribution, thus $r_{du}>1$ and therefore in Eq. (3.26) the ratio will not approach 1/4 as $x\to 1$. This example shows the limitation of the Bakamjian-Thomas construction which includes only the free quark kinetic energy in the boost transformation which relates wave functions in different frames or quantization schemes.

In practice it is possible to model the light cone wave function so that both $dG_E^n(0)/dQ^2>0$ and $R^{np}(x)\to 1/4$ as $x\to 1$. One such wave function has been proposed in Ref.

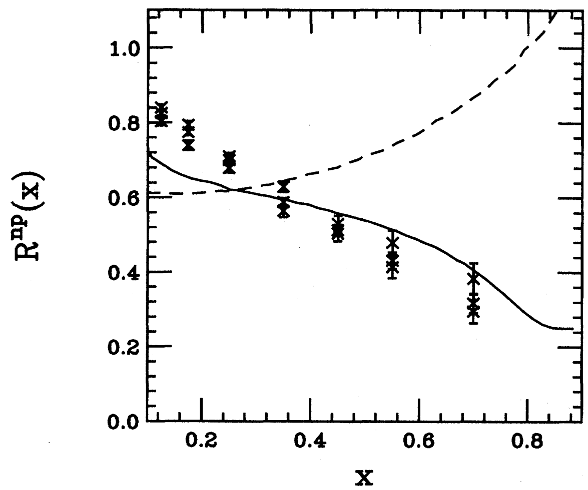

FIG. 8. Neutron to proton structure function ratio.

[23]. It was obtained through an *ad hoc* procedure used to express the individual particle momenta in the nucleon rest frame wave function where $\mathbf{p}_a=\mathbf{k}_a$ by the light cone variables $x_a,\mathbf{k}_a$. The results obtained in Ref. [23] are very sensitive to the details of the prescription, in particular whether the free invariant three-quark mass or the physical nucleon mass is used to scale the light cone momenta k_a^+. Furthermore model wave functions as the one in Ref. [23] will not in general satisfy the requirements of Poincaré symmetry and an arbitrary prescription does not give physical insight for the dynamical features which lead to the above mentioned problem with R^{np}.

There is a physical reason why a simple boost approximation cannot properly describe R^{np}. The spin-spin interaction in Eq. (2.24) responsible for the N-Δ mass splitting and the spatially asymmetric quark wave function give an asymmetry in the quark spin potential energy which is defined as an average over the nucleon wave function of the spin-spin interaction of a quark with the spectators,

$$V_q^{p,n}=\langle p,n|H_q|p,n\rangle, \tag{3.28}$$

where

$$H_q=\frac{2}{3}(2\pi)^{3/2}\frac{\delta}{\beta^3}\mathbf{S}_q\cdot\sum_{q'\neq q}\mathbf{S}_{q'}\delta^3(\mathbf{x}_{q'}-\mathbf{x}_q). \tag{3.29}$$

This gives

$$V_u^p=E_d^n=-\frac{1}{12}(M_\Delta-M_N)+O(\phi^2),$$

$$V_d^p=E_u^n=-\frac{1}{3}(M_\Delta-M_N)+O(\phi^2), \tag{3.30}$$

and $\Delta V^p=V_u^p-V_d^p\sim 75$ MeV. If the potential energy contribution to the quark energy is included in Eq. (3.27) so that

$$\delta\left(x-\frac{E_a+k_a^z}{E}\right)\to\delta\left(x-\frac{E_a+V_a^N+k_a^z}{E}\right), \tag{3.31}$$

then because $\Delta V^p>0$ the position of the peaks in the u and d distributions in the proton may be interchanged with respect to the case when only the kinetic energy E_a is used. There is also an asymmetric contribution from the potential energy associated with the confining interaction which should be included in Eq. (3.31). For the pure harmonic oscillator model this contribution is by an order of magnitude smaller than the spin contributions and may be neglected. The result for the ratio $R^{np}(x)$ with the spin potential energy taken into account is shown by the solid line in Fig. 8. With the magnitude of ΔV^p fixed by the mass operator which properly splits N and Δ masses and leads to HO configuration mixing that properly describes G_N^E the structure function ratio can now also be well reproduced.

IV. SUMMARY AND CONCLUSIONS

We have studied the nucleon structure in the relativistic quark model based on the BT construction of the Poincaré group with the Isgur-Karl Hamiltonian as the underlying mass operator in the rest frame. The construction allows for an arbitrary orientation of the quantization surface and to study the sensitivity of electromagnetic current matrix elements to the quantization surface and/or choice of the frame which arise from the Lorentz symmetry breaking of the single particle current. These effects are manifested through an appearance of unphysical form factors and spurious momentum dependence in the physical form factors that are both quite sizable. If it is valid that only some components of the full current should be approximated by the respective free current components [10], then the J^+ component in the light cone quantization provides good results. In particular, we have obtained a quantitative description of nucleon electromagnetic structure at low moments using the wave function parameters determined from the spectrum fits. The corresponding procedure of approximating the J^0 charge component in the instant quantization is however not sufficient to determine the magnetic form factors. This suggests in the instant quantization some dynamical effects in the $v\cdot J$ component should be included. Conversely, Lorentz symmetry violations are found to be more significant in the light cone quantization. In particular at larger momentum transfer the single-particle current approximation in the instant quantization seems to be preferred. These findings should only be cautiously generalized as they may be particular to the case studied. It is well known that even the matrix elements of the J^+ component in the light cone quantization require dynamical corrections [24]. Finally we have resolved the deficiency in simultaneously describing the negative neutron charge radius and the negative slope of $R^{np}(x)$ for large x. The resolution relies on a proper consideration of the total quark energy which has so far been omitted in previous relativistic constituent quark models which were based on an *ad hoc*, Poincaré violating, parametrization of the relativistic nucleon wave function. In our approach it can be shown that qualitative agreement with data can be obtained for a quite general wave function while quantitative agreement can be achieved using a specific **56⊕70** representation with the parameters determined from the $N-\Delta$ mass splitting.

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APPENDIX

We start by listing the relations between the individual particle variables and the relative variables associated with the cluster coupling scheme of the BT construction discussed in Sec. II. In the (12)3 coupling corresponding with the relative variables $\mathbf{k}_3,\mathbf{K}_3$, individual momenta are given by

$$\mathbf{p}_{1,2}=\mathbf{p}'_{1,2}+\frac{\mathbf{p}'_{1,2}\cdot\mathbf{P}}{\mathscr{M}(\mathscr{M}+\mathscr{E})}+\frac{\omega_{1,2}}{\mathscr{M}}\mathbf{P}_3,$$

$$\mathbf{p}_3=-\mathbf{K}_3-\frac{\mathbf{K}_3\cdot\mathbf{P}}{\mathscr{M}(\mathscr{M}+\mathscr{E})}+\frac{\omega_3}{\mathscr{M}}\mathbf{P} \tag{A1}$$

where

$$\mathbf{p}'_{1,2}=\mathbf{p}'_{1,2}(\mathbf{k}_3,\mathbf{K}_3)=\pm\mathbf{k}_3\pm\frac{\mathbf{k}_3\cdot\mathbf{K}_3}{\mathscr{M}_{12}(\mathscr{M}_{12}+\mathscr{E}_{12})}+\frac{\omega_{1,2}}{\mathscr{M}_{12}}\mathbf{K}_3,\quad \mathbf{p}'_1+\mathbf{p}'_2=\mathbf{K}_3, \tag{A2}$$

and

$$\mathscr{M}_{12}=\omega_1+\omega_2\ ,\quad \mathscr{M}=\omega_1+\omega_2+\omega_3\ ,\quad \mathscr{E}_{12}=\sqrt{\mathscr{M}_{12}^2+\mathbf{K}_3^2},$$

$$\omega_{1,2}=\sqrt{m_{1,2}^2+\mathbf{k}_3^2}\ ,\quad \omega_3=\sqrt{m_3^2+\mathbf{K}_3^2}. \tag{A3}$$

Permutation of the three-quark indices leads to two additional relations between $\mathbf{p}_a$ and the other two sets $\mathbf{k}_a,\mathbf{K}_a$, $a=1,2$. All three relations can be used to express $\mathbf{k}_a$, $\mathbf{K}_a$, $a=1,2$, in terms of $\mathbf{k}_3,\mathbf{K}_3$. The momenta $\mathbf{p}_{1,2}$ in Eq. (A2) are obtained by a product of two free Lorentz boosts applied to the relative variables $\mathbf{k}_3$ and $-\mathbf{k}_3$, respectively. The first boost transforms $\pm\mathbf{k}_3$ to $\mathbf{p}'_{1,2}$ corresponding to the momenta of the first two particles in the center of mass of the three particles. The second boost transforms $\mathbf{p}'_{1,2}$ to $\mathbf{p}_{1,2}$ corresponding to the momenta in the frame in which the total momentum of the three quarks is $\mathbf{P}$. Accordingly the spin operators $\hat{\mathbf{s}}_a$ in the (12)3 coupling scheme are related to the individual spin operators $\mathbf{s}_a$ through the following transformations

$$[\mathbf{s}_{1,2}]_{\lambda\lambda'}=\sum_{\hat{\sigma}\hat{\sigma}'}[D(\mathbf{p}_{1,2},\mathbf{p}'_{1,2})D(\mathbf{p}'_{1,2},\pm\mathbf{k}_3)]_{\lambda\hat{\sigma}}[\hat{\mathbf{s}}_{1,2}]_{\hat{\sigma}\hat{\sigma}'}[D(\mathbf{p}_{1,2},\mathbf{p}'_{1,2})D(\mathbf{p}'_{1,2},\pm\mathbf{k}_3)]^{\dagger}_{\hat{\sigma}'\lambda'}[\mathbf{s}_3]_{\lambda\lambda'}$$

$$=\sum_{\hat{\sigma}\hat{\sigma}'}D(\mathbf{p}_3,-\mathbf{K}_3)_{\lambda,\hat{\sigma}}[\hat{\mathbf{s}}_3]_{\hat{\sigma}\hat{\sigma}'}D^{\dagger}(\mathbf{p}_3,-\mathbf{K}_3)_{\hat{\sigma}'\lambda'}\ , \tag{A4}$$

where the Wigner rotations are now given by

$$D(\mathbf{p}'_{1,2},\pm\mathbf{k}_3)_{\hat{\lambda}\hat{\sigma}}=\frac{[(\omega_{1,2}+m_{1,2})(\mathscr{E}_{12}+\mathscr{M}_{12})+\mathbf{K}_3\cdot(\pm\mathbf{k}_3)]\delta_{\hat{\lambda}\hat{\sigma}}+i[\sigma]_{\hat{\lambda}\hat{\sigma}}\cdot[\mathbf{K}_3,\pm\mathbf{k}_3]}{\sqrt{2(\omega_{1,2}+m_{1,2})(\mathscr{E}_{12}+\mathscr{M}_{12})[\omega_{1,2}\mathscr{E}_{12}+\mathbf{K}_3\cdot(\pm\mathbf{k}_3)+m_{1,2}\mathscr{M}_{12}]}},$$

$$=\frac{[(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}_{12}+\mathscr{M}_{12})-\mathbf{K}_3\cdot\mathbf{p}'_{1,2}]\delta_{\hat{\lambda}\hat{\sigma}}+i[\sigma]_{\hat{\lambda}\hat{\sigma}}\cdot[\mathbf{K}_3,\mathbf{p}'_{1,2}]}{\sqrt{2(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}_{12}+\mathscr{M}_{12})(\hat{E}_{1,2}\mathscr{E}_{12}-\mathbf{K}_3\cdot\mathbf{p}'_{1,2}+m_{1,2}\mathscr{M}_{12})}}, \tag{A5}$$

$$D(\mathbf{p}_{1,2},\mathbf{p}'_{1,2})_{\lambda\hat{\lambda}}=\frac{[(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}+\mathscr{M})+\mathbf{P}\cdot\mathbf{p}'_{1,2}]\delta_{\lambda\hat{\lambda}}+i[\sigma]_{\lambda\hat{\lambda}}\cdot[\mathbf{P},\mathbf{p}'_{1,2}]}{\sqrt{2(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}+\mathscr{M})(\hat{E}_{1,2}\mathscr{E}+\mathbf{P}\cdot\mathbf{p}'_{1,2}+m_{1,2}\mathscr{M})}}$$

$$=\frac{[(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}+\mathscr{M})+-\mathbf{P}\cdot\mathbf{p}'_{1,2}]\delta_{\lambda\hat{\lambda}}+i[\sigma]_{\lambda\hat{\lambda}}\cdot[\mathbf{P},\mathbf{p}'_{1,2}]}{\sqrt{2(\hat{E}_{1,2}+m_{1,2})(\mathscr{E}+\mathscr{M})(\hat{E}_{1,2}\mathscr{E}-\mathbf{P}\cdot\mathbf{p}'_{1,2}+m_{1,2}\mathscr{M})}}, \tag{A6}$$

$$\hat{E}_{1,2}=\sqrt{\mathbf{p}_{1,2}'^2+m_{1,2}^2}=\omega_{1,2}\frac{\mathscr{E}_{12}}{\mathscr{M}_{12}}+\frac{\mathbf{K}_3\cdot(\pm\mathbf{k}_3)}{\mathscr{M}_{12}}=E_{1,2}\frac{\mathscr{E}}{\mathscr{M}}-\frac{\mathbf{P}\cdot\mathbf{p}_{1,2}}{\mathscr{M}}, \tag{A7}$$

and

$$D(\mathbf{p}_3,-\mathbf{K}_3)_{\lambda\hat{\sigma}}=\frac{[(\omega_3+m_3)(\mathscr{E}+\mathscr{M})-\mathbf{P}\cdot\mathbf{K}_3]\delta_{\lambda\hat{\sigma}}-i[\boldsymbol{\sigma}]_{\lambda\hat{\sigma}}\cdot[\mathbf{P},\mathbf{K}_3]}{\sqrt{2(\omega_3+m_3)(\mathscr{E}+\mathscr{M})(\omega_3\mathscr{E}-\mathbf{P}\cdot\mathbf{K}_3+m_3\mathscr{M})}}\,\frac{[(E_3+m_3)(\mathscr{E}+\mathscr{M})-\mathbf{P}\cdot\mathbf{p}_3]\delta_{\lambda\hat{\sigma}}+i[\boldsymbol{\sigma}]_{\lambda\hat{\sigma}}\cdot[\mathbf{P},\mathbf{p}_3]}{\sqrt{2(E_3+m_3)(\mathscr{E}+\mathscr{M})(E_3\mathscr{E}-\mathbf{P}\cdot\mathbf{p}_3+m^2\mathscr{M})}}. \tag{A8}$$

The transformation matrix between the quark basis states is given by

$$\langle\mathbf{p}_a,\lambda_a|\mathbf{P},\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a\rangle(2\pi)^3 2\mathscr{E}\delta^3\left(\mathbf{P}-\sum_a\mathbf{p}_a\right)g_{12}^{-1}(\mathbf{k}_3)g^{-1}(\mathbf{k}_3,\mathbf{K}_3)\,\delta^3(\mathbf{k}_3-\mathbf{k}_3(\mathbf{p}_a))\,\delta^3(\mathbf{K}_3-\mathbf{K}_3(\mathbf{p}_a))\hat{\Omega}_{\lambda_a\hat{\sigma}_a}(\mathbf{p}_a), \tag{A9}$$

with

$$\hat{\Omega}_{\lambda_a\hat{\sigma}_a}(\mathbf{p}_a)=[D(\mathbf{p}_1,\mathbf{p}_1')D(\mathbf{p}_1',\mathbf{k}_3)]_{\lambda_1\hat{\sigma}_1}[D(\mathbf{p}_2,\mathbf{p}_2')D(\mathbf{p}_2',-\mathbf{k}_3)]_{\lambda_2\hat{\sigma}_2}D(\mathbf{p}_3,-\mathbf{K}_3)_{\lambda_3\hat{\sigma}_3} \tag{A10}$$

and the phase space factors

$$g_{12}(\mathbf{k}_3)=\frac{\mathscr{M}_{12}}{[2(2\pi)^3]\omega_1\omega_2},\quad g(\mathbf{k}_3,\mathbf{K}_3)=\frac{\mathscr{M}}{[2(2\pi)^3]\mathscr{E}_{12}\omega_3}. \tag{A11}$$

The three-particle phase space satisfies

$$[d^3\mathbf{k}_3d^3\mathbf{K}_3]\frac{d^3\mathbf{P}}{(2\pi)^3 2\mathscr{E}}=g_{12}(\mathbf{k}_3)g(\mathbf{k}_3\mathbf{K}_3)d^3\mathbf{k}_3d^3\mathbf{K}_3\frac{d^3\mathbf{P}}{(2\pi)^3 2\mathscr{E}}=f(\mathbf{k}_a)[d^3\mathbf{k}_a]_{\mathrm{NR}}\frac{d^3\mathbf{P}}{(2\pi)^3 2\mathscr{E}}=\prod_a\frac{d^3\mathbf{p}_a}{(2\pi)^3 2E_a}, \tag{A12}$$

and the wave function $\hat{\psi}$ is given by

$$\begin{aligned}|\mathbf{P}_N M_N\lambda_N t_N\rangle&=\sum_{\hat{\sigma}_a\alpha_a c_a}\int[d\mathbf{k}_3 d\mathbf{K}_3]\frac{d\mathbf{P}}{(2\pi)^3\mathscr{E}}\hat{\psi}_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{P},\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a,c_a)|\mathbf{P},\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a,c_a\rangle\\&=\sum_{\lambda_a\alpha_a c_a}\int\left[\prod_a\frac{d\mathbf{p}_a}{(2\pi)^3 2E_a}\right]\hat{\psi}_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{p}_a,\lambda_a\alpha_a,c_a)|\mathbf{p}_a,\lambda_a,\alpha_a,c_a\rangle,\end{aligned} \tag{A13}$$

with

$$\hat{\psi}_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{p}_a,\lambda_a\alpha_a,c_a)=\sum_{\hat{\sigma}_a}\hat{\Omega}_{\lambda_a\hat{\sigma}_a}\hat{\psi}_{\mathbf{P}_N M_N\lambda_N t_N}(\mathbf{P},\mathbf{k}_3,\mathbf{K}_3,\hat{\sigma}_a,\alpha_a,c_a) \tag{A14}$$

and $\mathbf{k}_3$, $\mathbf{K}_3$, $\mathbf{P}$ expressed in terms of $\mathbf{p}_a$. In general the wave function $\hat{\psi}(\mathbf{p}_a)$ can be written in terms of the free Dirac spinors. In particular for the two-spinor wave function which appears in the ground-state HO contribution to the nucleon wave function,

$$\hat{\psi}_{+\lambda_N}(\hat{\sigma}_a)=\frac{1}{\sqrt{6}}\chi(\hat{\sigma}_1)\vec{\tau}i\tau_2\chi(\hat{\sigma}_2)\chi(\hat{\sigma}_3)\vec{\tau}\chi(\lambda_N), \tag{A15}$$

$$\hat{\psi}_{-\lambda_N}(\hat{\sigma}_a)=\frac{1}{\sqrt{2}}\chi(\hat{\sigma}_1)i\tau_2\chi(\hat{\sigma}_2)\chi(\hat{\sigma}_3)\xi(\lambda_N), \tag{A16}$$

the transformation $\hat{\Omega}$ gives

$$\hat{\psi}_{\pm\lambda_N}(\mathbf{p}_a,\lambda_a)=N_\pm\bar{u}(p_1,\lambda_1)\Gamma_{\pm,12}C\bar{u}(p_2,\lambda_2)^T\bar{u}(p_3,\lambda_3)\Gamma_{\pm,3N}u(P,\lambda_N), \tag{A17}$$

where $p_a^\mu=(E_a,\mathbf{p}_a)$, $P^\mu=(\mathscr{E},\mathbf{P})$, $\mathscr{E}=\sqrt{\mathscr{M}^2+\mathbf{P}^2}$. The spinors are normalized according to $\bar{u}(\mathbf{p}_a,\lambda_a')u(\mathbf{p}_a,\lambda_a)=2m_a\delta_{\lambda_a'\lambda_a}$. $C=i\gamma^0\gamma^2$ is the charge conjugation matrix and the matrices $\Gamma_{\pm,12}$ and $\Gamma_{\pm,3N}$ are given by

$$\Gamma_{-,12}=\gamma_5,\quad \Gamma_{+,12}=\gamma^\mu-p_1^\mu\frac{m_1-m_2+\mathscr{M}_{12}}{\mathscr{M}_{12}(\mathscr{M}_{12}+m_1+m_2)}-p_2^\mu\frac{m_1-m_2-\mathscr{M}_{12}}{\mathscr{M}_{12}(\mathscr{M}_{12}+m_1+m_2)},$$

$$\Gamma_{-,3N}=I,\quad \Gamma_{+,3N}=\gamma_\mu\gamma_5+\frac{2}{\mathscr{M}+\mathscr{M}_{12}-m_3}P_\mu\gamma_5. \tag{A18}$$

The normalization constants are

$$N_-=-\frac{1}{\sqrt{2}}N,\quad N_+=-\frac{1}{\sqrt{6}}N, \tag{A19}$$

with

$$N=\frac{1}{[\mathscr{M}_{12}^2-(m_1-m_2)^2)((\mathscr{M}+m_3)^2-\mathscr{M}_{12}^2]}. \tag{A20}$$

The expression for $\hat{\psi}_{+\lambda_N}(\mathbf{p}_a,\lambda_a)$ is derived in the following way. Since the wave function $\hat{\psi}_+$ corresponds to the quarks (12) coupled to $S=1$ state, Eq. (A15) may be written as

$$\hat{\psi}_{+\lambda_N}(\hat{\sigma}_a)=-N_+\sum_{\hat{\lambda}_{12}}\left[\bar{u}(\omega_1,\mathbf{k}_3,\hat{\sigma}_1)\left(\gamma^i-2\frac{k_3^i}{\omega_1+\omega_2+m_1+m_2}\right)C\bar{u}(\omega_2,-\mathbf{k}_3,\hat{\sigma}_2)^T\epsilon^i(M_{12},\mathbf{0},\hat{\lambda}_{12})\right]$$
$$\times\left[\epsilon^{*\mu}(E_{12},\mathbf{K}_3,\hat{\lambda}_{12})\bar{u}(\omega_3-\mathbf{K}_3,\lambda_3)\left(\gamma_\mu\gamma_5+\frac{2}{\mathscr{M}+\mathscr{M}_{12}-m_3}\hat{P}_\mu\gamma_5\right)u(\mathscr{M},\mathbf{0},\lambda_N)\right]. \tag{A21}$$

The first set of parentheses represents the coupling of the one and two quarks to a spin-1 state with zero total momentum, mass $\mathscr{M}_{12}$, and polarization $\hat{\lambda}_{12}$. The second set of parentheses represents a coupling of a spin-1 particle with momentum $\mathbf{K}_3$ and energy $\mathscr{E}_{12}$, $\mathscr{E}_{12}=\sqrt{\mathbf{M}_{12}^2+\mathbf{K}_3}$ and the third quark to a spin-1/2, nucleon state with mass $\mathscr{M}$, zero total momentum $\hat{P}=(\mathscr{M},\mathbf{0})$, and spin projection λ_N. The polarization vectors, $\epsilon^\mu(E,\mathbf{p},\lambda)$ of a spin-1 particle are given by

$$\epsilon^0(E,\mathbf{P},\lambda)=\frac{\epsilon^i(M,\mathbf{0},\lambda)P^i}{M},$$

$$\epsilon^i(E,\mathbf{P},\lambda)=\epsilon^i(M,\mathbf{0},\lambda)+\frac{\epsilon^j(M,\mathbf{0},\lambda)P^j}{M(E+M)}P^i, \tag{A22}$$

and

$$\epsilon^i(M,\mathbf{0},+1)=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-i\\0\end{pmatrix},\quad \epsilon^i(M,\mathbf{0},-1)=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\\0\end{pmatrix},\quad \epsilon^i(M,\mathbf{0},0)=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{A23}$$

Using the above representation for $\hat{\psi}_{+\Lambda_N}(\hat{\sigma}_a)$ and the relations

$$D(\mathbf{p}',\mathbf{p})_{\lambda\lambda'}\bar{u}(p,\lambda')=\bar{u}(p',\lambda)S(p\to p'),$$

$$S(p\to p')\gamma^\mu S^{-1}(p\to p')=\Lambda^{-1\mu}_{\ \ \nu}\gamma^\nu,\quad \Lambda^{-1\mu}_{\ \ \nu}=\Lambda^{\ \mu}_{\nu},$$

$$\sum_\lambda\epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda)=\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}\right), \tag{A24}$$

where S is the Dirac spinor representation of the Lorentz transformation Λ,

$$S(p\to p')=E'+E\sqrt{2(EE'+\mathbf{pp}'+m^2)}\left[I+\frac{\mathbf{p}'-\mathbf{p}}{E'+E}\gamma^0\gamma^i\right], \tag{A25}$$

the expression given in Eq. (A17) can be derived in a straightforward way.

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