Chaoticity in vibrating nuclear billiards

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We study the motion of classical particles confined in a two-dimensional "nuclear" billiard whose walls undergo periodic shape oscillations according to a fixed multipolarity. The presence of a coupling term in the single-particle Hamiltonian between the particle motion and the collective coordinate generates a fully self-consistent dynamics. We consider in particular monopole oscillations and demonstrate that self-consistency is essential in order to induce chaotic single-particle motion. We also discuss the dissipative behavior of the wall motion and its relation with the order-to-chaos transition in the dynamics of the microscopic degrees of freedom. Analogous considerations can be extended to higher multipolarities.

The question about the origin of dissipation of collective motion in finite Fermi systems has been widely and deeply investigated [1], but its solution is still an open problem. One generally considers one- and two-body processes as the main mechanisms which could explain this phenomenon, though their mutual balance is currently a question of debate. The one-body processes include (i) escape of nucleons from the collective potential well into the continuum states and (ii) collisions of nucleons with the nuclear wall generated by the common self-consistent mean field. On the other hand, the two-body processes take into account collisions between nucleons and are effective in the range of excitation energies above the Fermi energy, because of the partial relaxation of the Pauli principle. In this paper we concentrate on one-body processes only and we discuss the relationship between chaoticity at the microscopic level and dissipation of the collective degrees of freedom.

In Ref. [2] Blocki *et al.* analyze the behavior of a gas of several thousands of noninteracting point particles enclosed in a hard-wall container which undergoes periodic and adiabatic shape oscillations, i.e., the wall frequency is much smaller than a typical single-particle frequency. Particle momenta are randomly distributed inside a Fermi sphere, and this represents the only quantum ingredient of the model. In coordinate space particles move on linear trajectories and collide elastically against the walls. The wall motion is not modified by the collisions with the particles; therefore it keeps on oscillating with the same initial frequency pumping energy into the gas. For this system, the authors study the increase of the particles' kinetic energy as a function of time. They also compare their findings with the predictions of the well-known "wall formula" [3], which gives the dissipation rate for each fixed multipolarity. They find that for the quadrupole case the gas does not heat up, whereas for higher multipolarities the gas kinetic energy increases with time. They attribute the different behavior to the fact that for low deformations the particles' motion is regular and corresponds to an integrable situation, whereas for higher multipolarities the scattering of segments of wall with positive curvature

leads to divergence between trajectories and therefore to chaotic motion. Although their results look very interesting, their application to the nuclear case is not straightforward because (i) the self-consistent mean field is absent and (ii) the total energy is not conserved.

A step forward in this direction has been performed by Bauer *et al.* in Ref. [4]. In this work the authors study the damping of collective motion in nuclei within the Vlasov equation, which is the semiclassical approximation to the time dependent Hartree-Fock (TDHF) equation. In the Vlasov equation, which is solved by using the test-particle method [5], self-consistency is taken into account and the total energy is conserved. Moreover, quantum effects like the Pauli principle are present in the initial conditions and during the dynamical evolution because of the Liouville theorem. A multipole-multipole interaction of the Bohr-Mottelson type is adopted for quadrupole and octupole deformations. In both cases the dynamical evolution shows a regular undamped collective motion which coexists with a weakly chaotic single-particle dynamics.

In order to clarify the relationship between chaos at the microscopic level and damping of the collective motion, we consider a classical version of the vibrating potential model for finite nuclei (see, e.g., Ref. [9]). In this model several noninteracting classical particles move in a two-dimensional deep potential well and hit the oscillating surface. Using polar coordinates, the Hamiltonian depends on a set of $\{r_i, \theta_i\}$ variables, describing the motion of the particles, and the collective coordinate α. The Hamiltonian reads

$$
H(r_i, \theta_i, \alpha) = \sum_{i=1}^{A} \left(\frac{p_{r_i}^2}{2m} + \frac{p_{\theta_i}^2}{2mr_i^2} + V(r_i, R(\theta_i)) \right) + \frac{p_\alpha^2}{2M} + \frac{1}{2}M\Omega^2\alpha^2,
\tag{1}
$$

$\{p_{r_i}, p_{\theta_i}, p_\alpha\}$ being the conjugate momenta of $\{r_i, \theta_i, \alpha\}$. $m = 938$ MeV is the nucleon mass, and $M = mAR_0^2$ is the
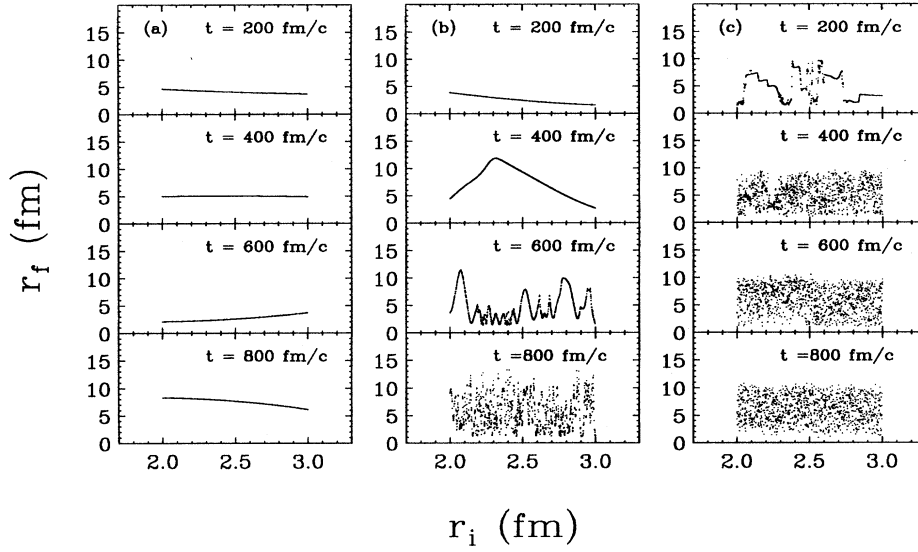
FIG. 1. The final radial coordinate for one particle is drawn as a function of the initial one at different times t = 200, 400, 600, 800 fm/c. In part (a) we display different plots for the case without coupling, whereas in parts (b) and (c) plots result from our model. Calculations are performed respectively with one particle (b) and ten particles (c).

Inglis mass, chosen proportional to the total number of particles A and to the circular billiard radius R_0. Ω is the oscillation frequency of the collective variable α. The potential $V(r,R(\theta))$ is zero inside the billiard and a very steeply rising function on the surface, $V(r,R(\theta))=V_0/(1+\exp\{[R(\theta)-r]/a\})$, with $V_0=1500$ MeV and $a=0.01$ fm. The surface is described by $R(\theta)=R_0[1+\alpha P_L(\cos\theta)]$, where P_L is the Legendre polynomial with multipolarity L. Therefore this potential couples the collective variable motion to the particles' dynamics. The numerical simulation is based on the solution of the Hamilton equations

$$\dot{r}_i=\frac{p_{r_i}}{m}, \quad \dot{p}_{r_i}=\frac{p_{\theta_i}^2}{mr_i^3}-\frac{\partial V}{\partial r_i}, \tag{2}$$

$$\dot{\theta}_i=\frac{p_{\theta_i}}{mr_i^2}, \quad \dot{p}_{\theta_i}=-\frac{\partial V}{\partial R}\frac{\partial R}{\partial \theta_i}, \tag{3}$$

$$\dot{\alpha}=\frac{p_\alpha}{M}, \quad \dot{p}_\alpha=-M\Omega^2\alpha-\frac{\partial V}{\partial R}\frac{\partial R}{\partial \alpha}. \tag{4}$$

We solve the Hamilton equations with an algorithm of Runge-Kutta type with typical time step sizes of 1 fm/c and the total energy is conserved with high accuracy. In our investigation we consider the simplest case, i.e., the monopole mode $L=0$. Because of the spherical symmetry, R does not depend on θ and therefore each angular momentum p_{θ_i} is a constant of motion. Another constant of motion is the total energy E. As a consequence the problem is not integrable, because the number of constants of motion is smaller than the number of degrees of freedom.

Before getting into the analysis of the monopole oscillation, it is appropriate to study the equilibrium conditions for our classical gas. In fact it is well known that in the realistic nuclear case the collective motion takes place around equilibrium. The equilibrium deformation parameter $\bar{\alpha}$ can be calculated by equating the mechanical pressure of the wall $P_w=dE/dS=(M\Omega^2/2\pi)\alpha/(1+\alpha)$ and the pressure $P_p=\rho T=AT/\pi R_0^2(1+\alpha)^2$ exerted by the particles, ρ being

the particle density. One gets $\bar{\alpha}(1+\bar{\alpha})=2T/mR_0^2\Omega^2$, the equation for the equilibrium value $\bar{\alpha}$.

We consider the wall oscillation taking place close to adiabatic conditions. For this purpose we impose a wall frequency smaller than the single-particle one and choose $\Omega=0.05$ c/fm, which corresponds to an oscillation period $\tau_w=2\pi/\Omega\sim125.66$ fm/c. The single-particle oscillation period is equal to $\tau_p=2R_0/v$, v being the maximum particle speed, $v=\sqrt{2T/m}$. We choose $R_0=6$ fm and $T=36$ MeV, and this gives a single-particle period $\tau_p\sim43.3$ fm/c.

The equilibrium value $\bar{\alpha}$ corresponds of course to the thermodynamic limit. Since we are considering a system with a finite number of particles, the actual value of α will fluctuate in time around $\bar{\alpha}$. We have checked that, starting with a set of particle velocities which follows a Maxwell distribution (approximately) and $\alpha=\bar{\alpha}$, the value of α varies in time, but with an average value very close to $\bar{\alpha}$. To obtain a description of the system at the statistical equilibrium, it is therefore necessary to consider the Gibbs ensemble, i.e., a set of N copies of the system, all of them with an initial value $\alpha=\bar{\alpha}$ and differing only by the initial microscopic distribution of particle velocities and positions (all compatible with a Maxwellian). We checked that indeed the value $\langle\alpha\rangle$ obtained by averaging among the different systems, or "events," displays decreasing fluctuations as N increases and stays very close to $\bar{\alpha}$. After we were assured that the numerical calculation produces good equilibrium properties [11], we slightly changed the collective coordinate $\bar{\alpha}$ by a small amount $\delta\alpha=0.15$, and let both the particles and the circular billiard evolve in time.

One possible way to investigate the role played by coupling and see whether it can induce a chaotic dynamics is drawing Poincare's surface of sections either for the single-particle coordinate or for the collective variable. Unfortunately, this is impossible to perform in our case since the number of degrees of freedom is much bigger than 2. An alternative way to visualize a chaotic behavior is to draw the kind of plot shown in Fig. 1. There we display the final radial coordinate at a time t of one chosen particle vs the one at

$t=0$ for three cases: (a) the wall is not coupled to the particles' motion and always oscillates at the same frequency [2] thus giving energy to the gas, (b) coupling is taken into account and the Hamilton equations (2)–(4) are solved respectively for (b) one particle and (c) ten particles. The idea these plots are based on is the following: if the dynamics is regular, two initially close points in space stay close even at later times, but if the dynamics is chaotic the two points will soon separate due to the exponential divergence induced by chaos. In the first case this plot will show a regular curve, whereas in the other one it displays a scattered plot. Let us discuss the results shown in Fig. 1. While the fixed circular billiard is a very well-known integrable case, the wall oscillation changes the dynamics. In case (a) only the angular momentum is conserved but not the energy, and therefore the system is not integrable and we should expect chaotic behavior. In spite of that, we observe regularity because the motion is adiabatic. In other words there exist some adiabatic invariants which guarantee regularity, though for a limited time interval. This result confirms those of Ref. [2]. If we omit any adiabaticity condition by increasing the wall frequency, the dynamics will be chaotic, as we checked explicitly. This is also in agreement with Ref. [6], where the stochasticity parameter is shown to be much less than 1 in the nuclear case and the wall vibration is unable to induce chaotic dynamics of the particle on a nuclear time scale. On the other hand, the inclusion of coupling terms substantially modifies the dynamics even in the hypothesis of small values of the adiabatic parameter τ_p/τ_w; see Figs. 1(b) and 1(c). In both cases the initially regular curves change into scatter plots, which clearly show that the coupling to the wall oscillation randomizes the single-particle motion. In addition, the case (c) shows the appearance of diffusive behavior earlier than the case (b), because of the additional coupling among the particles through the wall. It has to be stressed that in the present case, for a constant fixed value of the collective variable, the particle motion is fully integrable, in sharp contrast with the situation recently investigated in Refs. [7,8]. It is the coupling between the collective and particle motion which enables the chaotic dynamics to appear.

Scattered plots like those shown in cases (b) and (c) have a typical fractal structure, and we found an uncertainty dimension $D=0.899$ independent of time for the case (b) [10,11], which indicates strong chaoticity.

A more quantitative way to characterize the dynamics in a chaotic regime is by means of the largest Lyapunov exponent λ, which gives the average divergence rate between close trajectories in phase space [10]. Let us denote with $d(t)$ the distance between two phase space trajectories, and $d_0=d(0)$. In Fig. 2 we plot the time evolution of the ratio $d(t)/d_0$ for the cases with one and ten particles moving inside the billiard. We consider two close events which differ at $t=0$ only by the position of one particle. By choosing $d_0=10^{-11}$ and integrating forward the equations of motion, we followed the phase space distance $d(t)=\sqrt{(\mathbf{x}^{(1)}-\mathbf{x}^{(2)})^2}$, where $\mathbf{x}=\{r,\theta,p_r,p_\theta\}$. The superscripts refer, respectively, to the trajectories (1) and (2). This procedure is repeated several times, in order to sample different regions of phase space with good accuracy. For each sampling we get a curve like the one shown in Fig. 2, and its slope gives a value of the Lyapunov exponent λ. One can notice the sharp rising of
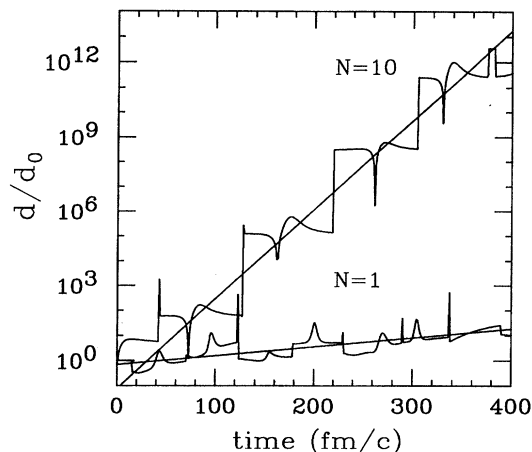

FIG. 2. The evolution of the ratio d/d_0 (see text for details) is displayed vs time for the cases with one and ten particles moving inside the billiard (solid line). The straight line represents a fit whose slope is the largest Lyapunov exponent.

$d(t)$ corresponding to each collision with the wall. A more accurate calculation can be done simply by averaging λ over several different events. We found that $\langle\lambda\rangle$ scales with the number of particles, in particular $\langle\lambda_1\rangle=8\times10^{-3}$ $c/$fm and $\langle\lambda_{10}\rangle=8.8\times10^{-2}$ $c/$fm. Therefore the times for the onset of chaoticity $\tau=1/\langle\lambda\rangle$ are respectively equal to $\tau_1=125.3$ fm/c and $\tau_{10}=11.4$ fm/c. We notice that relevant divergencies on a macroscopic scale are visible at times longer than τ_1 and τ_{10}, as shown in Figs. 1(b) and 1(c).

Now let us concentrate on the dissipative properties of our system. On the left-hand side of Fig. 3 we plot the evolution of the collective variable α vs time, considering only one event, each corresponding to the three cases already shown in Fig. 1. Since many events are needed in order to have a good global picture of the macroscopic system (Gibbs ensemble), an average $\langle\alpha\rangle$ over 4000 different events is also considered and displayed on the right-hand side of Fig. 3. First we analyze the behavior of one single event. We note that the amplitude of the collective motion is a regularly oscillating curve in the uncoupled case [Fig. 3(a)], whereas an irregular pattern appears in the coupled cases with one [Fig. 3(b)] and ten particles [Fig. 3(c)]. This is at variance with the results found in Ref. [4]. Moreover, in all three cases the single event does not show any dissipation. We observe one main difference between cases (b) and (c), i.e., the amplitude of α is bigger in (b). This happens because in our model the wall mass is proportional to the number of particles confined inside the billiard. In case (b) one particle is hitting the wall which has the same inertia, whereas in case (c) the wall has an inertia ten times bigger than the single-particle mass. Therefore in the latter case both the energy transfer and the oscillation amplitude are smaller because collisions are more elastic.

Let us now examine the behavior of an ensemble of events [8], each obtained by assigning random initial conditions to the particles both in coordinate and momentum space. While the ensemble average keeps unchanged the uncoupled case (d)—as expected—some damping appears in cases (e) and (f) corresponding, respectively, to the coupled
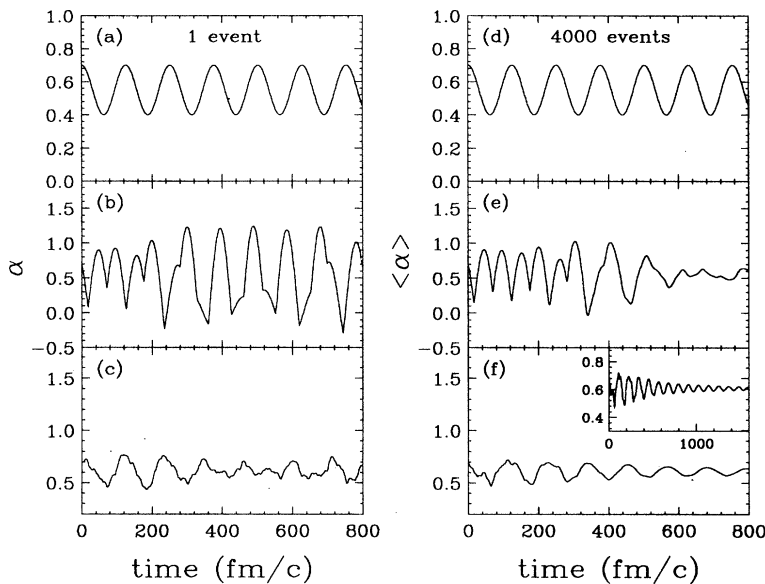
FIG. 3. On the left-hand side we show the behavior of the collective coordinate α as a function of time for one event only. In panel (a) we display the result obtained without coupling, whereas in panels (b) and (c) we show the result from our model respectively with one and ten particles. On the right-hand side we plot an α averaged over 4000 events. The panels (d), (e), and (f) follow the same order as (a), (b), and (c). The inner panel in (f) shows the long time behavior.

motion of one and ten particles. In fact, if the ensemble contains only regular regions, no dissipation is observed in the collective motion, whereas if chaotic regions are populated some damping shows up. In the latter case, once chaos has developed, the particles hit the wall essentially at random, and this produces a stochastic dephasing of the collective motion between different events. If their number is huge, it can be shown that the dissipation rate decays exponentially [11]. This behavior is clearly shown in the inner panel of Fig. 3(f), where we display the time evolution of $\langle \alpha \rangle$ up to 1600 fm/c. We note that the motion of the collective variable is completely damped out for longer times and develops around equilibrium (wiggles are only statistical errors due to the finite number of events). It is important to remark that the damping is slower as we increase the number of particles, and approaches zero in the limit $A \rightarrow \infty$. We notice also that our damping times are longer than the ones predicted by the wall formula [3]. More details are given elsewhere [11].

In conclusion, we have presented a novel approach based on the solution of the Hamilton equations for several classical particles moving in a circular billiard, in order to explain dissipation of the collective variable. In our model the collective variable appears explicitly in the Hamiltonian as an additional degree of freedom. We limited ourselves to the study of monopole oscillations and found that the presence of a coupling term in the single-particle Hamiltonian induces

chaotic motion at microscopic level, even in adiabatic conditions. This result is confirmed by the calculation of the largest Lyapunov exponent. As far as the collective coordinate is concerned, we found irregular behavior and no damping in the single event, the latter point being in line with the work of Bauer et al. [4]. On the other hand, a whole bunch displays dissipation because of incoherence among different events. This incoherence is produced by the chaotic single-particle dynamics, which makes all events belonging to the same ensemble strongly different from each other. This dissipation of the collective energy corresponds to the damping of the nuclear vibration into thermal motion. Therefore we hope to have demonstrated, although at the classical level, that coupling terms are essential in order to induce macroscopic dissipation. Similar considerations can be extended to "nuclear" billiards with higher multipolarities, where we expect an enhanced presence of chaotic trajectories because of strong deformations. Further analysis along these lines is currently in progress. Another step forward could be the quantal treatment of the single-particle degrees of freedom, along the same lines as Ref. [12] which deals with a similar problem in simple systems.

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