Baryon mappings applied to the three-color delta model

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A recently proposed baryon mapping of colorless three-quark clusters is applied to a three-color delta model in the infinite nuclear matter limit, to assess its usefulness in the presence of spatial three-quark correlations. Treating the resulting baryon Hamiltonian in Hartree-Fock approximation exactly reproduces the energy per quark of the model through second order in the density. Deviations show up in higher orders, however, most likely reflecting the need for an improved description of the short-range repulsion between baryons beyond that of the Hartree-Fock approximation.

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I. INTRODUCTION

It is generally accepted that the strong interaction is responsible both for building nucleons and for the residual interaction between nucleons that leads to nuclear stucture. Recent experiments suggest, however, that these two features of the strong interaction are not completely decoupled, as is traditionally assumed in nuclear physics. This has motivated much recent effort to derive nuclear properties directly from the established theory of strong interactions—QCD. To date, however, significant progress along these lines has only been made within the framework of constituent quark models, and even there only for few baryon systems [1].

A possible advance in the description of multinucleon systems starting from constituent quarks has been provided by the development of baryon mapping methods [2–5], analogous to the boson mappings used in traditional nuclear physics [6]. Before these methods can be applied with confidence to real nuclear systems, however, it is important to test them in the context of exactly solvable models that incorporate to as large an extent as possible the correlation structures of relevance to realistic systems.

Several such tests have already been reported for a variety of models [2,4,5]. None, however, contained spatial three-quark correlations, which are clearly a central ingredient of a proper quark description of nuclei. In this work, we report an application of baryon mapping techniques to a three-color delta model of interacting quarks in the infinite nuclear matter limit [7,8]. Although this model involves quarks moving in one dimension (whereas real nuclei exist in a three-dimensional world) and does not contain the physics of quark confinement, it nevertheless has several features that make it attractive as a testing ground of baryon mapping methods: (1) in the limit of very low density, it leads to spatially localized three-quark clusters (nucleons), (2) in the limit of very high density, it leads to a free quark gas, and (3) it can be solved exactly.

The current analysis is a natural follow-up to an earlier application of boson mapping techniques to a two-color version of the same model [9]. As we will see, mapping techniques seem to be even more suitable for the three-color model, in large part because of the role played by the Pauli principle at the baryon level.

The structure of the paper is as follows. In Sec. II, we briefly describe the three-color delta model, and discuss its exact solution at low densities. In Sec. III, we review the consistent baryon mapping developed in Ref. [5] and then apply it to the model Hamiltonian. The resulting baryon Hamiltonian cannot be diagonalized exactly. Thus, in Sec. IV, we consider an approximate treatment of the baryon system based on the (nonunitary) Hartree-Fock method [2]. It turns out that effects due to the residual interaction between baryons are suppressed at low densities, simplifying the analysis dramatically in that regime. When all is said and done, we find that our results for the energy per particle at low densities (ρ) are in perfect agreement with those obtained by exact solution through second order in ρ. Deviations begin to show up at third order, however, most likely reflecting the need for an improved variational approximation beyond the Hartree-Fock approximation to treat the strong repulsion between baryons originating in quark exchange. In Sec. V we extend the analysis to the high-density regime and in Sec. VI summarize the principal conclusions of this work.

II. THREE-COLOR QUARK MODEL

A. The model

In this model, a system of N nonrelativistic quarks, with no spin-flavor degrees of freedom, move in a one-dimensional box of length L subject to an attractive delta-function interaction. Subsequently, the infinite-matter limit is achieved by letting $N,L\rightarrow\infty$ in such a way that $\rho=N/L$ remains finite. Exact solutions for this model have been reported by Koltun [7].

For our purposes, it is most convenient to formulate the model in second-quantized momentum-space representation, defined in terms of the creation and annihilation operators $q^\dagger_{1k}$ and q_{1k}. Here, the first subscript denotes the SU(3) color quantum number (which can take three possible values) and the second the linear momentum k. The linear momentum is quantized for finite L and continuous in the infinite-matter limit. The fermion nature of the quarks is expressed by the anticommutation relations

$$\{q^\dagger_{1k},q^\dagger_{2k'}\}=\{q_{1k},q_{2k'}\}=0,$$
$$\{q^\dagger_{1k},q_{2k'}\}=\delta_{12}\delta_{kk'}. \tag{1}$$

In this representation, the quark number operator is given by

$$\hat{N}_q=\sum_1\sum_k q^\dagger_{1k}q_{1k}, \tag{2}$$

and the Hamiltonian by

$$\hat{H}=\sum_1\sum_k\frac{k^2}{2M}q^\dagger_{1k}q_{1k}-\frac{G}{2}\sum_{1\neq 2}\sum_{ijkl}q^\dagger_{1i}q^\dagger_{2j}q_{2l}q_{1k}\delta_{i+j,k+l}, \tag{3}$$

where G is related to the strength of the delta-function potential g in configuration space by $G=g/L$.

A delta-function interaction has the special property that it is only felt by particles at the same point in space. As a consequence, the two interacting quarks must be in a space-symmetric and color-antisymmetric configuration. This can be made more explicit by rewriting the Hamiltonian (3) as

$$\hat{H}=\sum_1\sum_k\frac{k^2}{2M}q^\dagger_{1k}q_{1k}-\frac{G}{4}\sum_{12345}\sum_{ijkl}\epsilon_{123}\epsilon_{145}q^\dagger_{2i}q^\dagger_{3j}q_{5l}q_{4k}\delta_{i+j,k+l}, \tag{4}$$

where ϵ_{123} is the usual antisymmetric tensor. This is the form of the Hamiltonian that we will use in the subsequent analysis.

In what follows, we first work with the above discrete-momentum Hamiltonian (4) and then pass to the infinite-matter limit by making the replacements

$$\sum_k\rightarrow\frac{L}{2\pi}\int dk,$$
$$\delta_{ij}\rightarrow\frac{2\pi}{L}\delta(i-j). \tag{5}$$

B. Exact solution at low densities

As noted earlier, one of the most attractive features of the three-color delta model in one dimension is that it can be solved exactly in the infinite-matter limit, using the Bethe ansatz. The end result is the following set of integral equations for the ground-state energy per particle [7]:

$$\rho=3\int_{-K}^{K}F(k,K)dk,$$
$$\frac{E}{N}\frac{1}{Mg^2}=-\frac{1}{3}+\frac{3}{2M^2g^2\rho}\int_{-K}^{K}F(k,K)k^2dk, \tag{6}$$

where the density function for triplet clusters $F(k,K)$ satisfies the relation

$$2\pi F(k,K)=3-2\int_{-K}^{K}d\omega F(\omega,K)\left[\frac{Mg}{(Mg)^2+(k-\omega)^2}+\frac{2Mg}{(2Mg)^2+(k-\omega)^2}\right]. \tag{7}$$

These equations scale with the dimensionless parameter $\tilde{\rho}=\rho/Mg$. From this, it is straightforward to obtain the energy per particle at low densities as an expansion in powers of this parameter, either numerically or analytically. Here, we present the analytic results through seventh order in $\tilde{\rho}$:

$$\frac{E}{N}\frac{1}{Mg^2}=-\frac{1}{3}+\frac{\pi^2\tilde{\rho}^2}{486}+\frac{\pi^2\tilde{\rho}^3}{729}+\frac{\pi^2\tilde{\rho}^4}{1458}+\frac{2\pi^2(45-\pi^2)\tilde{\rho}^5}{295245}+\frac{\pi^2(45-4\pi^2)\tilde{\rho}^6}{354294}+\frac{\pi^2(5670-1233\pi^2+22\pi^4)\tilde{\rho}^7}{95659380}+O(\tilde{\rho}^8). \tag{8}$$

III. BARYON MAPPING APPLIED TO THE MODEL

In this section, we first briefly review the subject of baryon mapping methods and then discuss their application to the momentum-space Hamiltonian (4) of the model. We follow the method developed in Ref. [5], which gives a consistent procedure for mapping any colorless Hamiltonian involving up to three-body interactions onto colorless baryons.

A. The baryon mapping

The fundamental problem in trying to treat systems of interacting quarks is the difficulty in describing strong three-body correlations with traditional many-body techniques. Baryon mapping techniques were introduced several years ago in order to provide a practical means of doing precisely this.

The basic idea is to replace the original quark problem by an *equivalent* one involving interacting baryons. Assuming that such a mapping can be found, it will lead from the original multiquark Hamiltonian to an effective Hamiltonian for the baryons, which rigorously incorporates the physics of the quark Pauli principle and, furthermore, can be treated using traditional fermion many-body techniques.

In Ref. [5], it was shown how to consistently map colorless one-, two-, and three-quark operators onto colorless baryon operators so as to preserve all of the (anti)commutation relations of the original quark space. For our purposes, it suffices to consider the mapping of colorless one- and two-body operators only, since the Hamiltonian (4) does not contain any three-body terms.

According to Ref. [5], an arbitrary colorless one-body operator

$$A_{ab}=\sum_{1} q^{\dagger}_{1a}q_{1b} \tag{9}$$

is mapped according to

$$A_{ab}\rightarrow 3\sum_{cd} \Lambda^{\dagger}_{acd}\Lambda_{bcd}\,. \tag{10}$$

The operators $\Lambda^{\dagger}_{ijk}$ and Λ_{ijk}, that appear in (10) create and annihilate colorless baryons, respectively. They only depend on the noncolor quantum numbers of the original quarks; all the color information has been separated out and then explicitly summed over. These operators are fully symmetric under the interchange of their indices, i.e.,

$$\Lambda_{ijk}=\Lambda_{ikj}=\Lambda_{jik}=\Lambda_{jki}=\Lambda_{kij}=\Lambda_{kji}\,, \tag{11}$$

and satisfy the anticommutation relation

$$\{\Lambda_{ijk},\Lambda^{\dagger}_{lmn}\}=\frac{1}{6}\,S(ijk,lmn), \tag{12}$$

where

$$S(ijk,lmn)=\delta_{il}\delta_{jm}\delta_{kn}+\delta_{im}\delta_{jn}\delta_{kl}+\delta_{in}\delta_{jl}\delta_{km}+\delta_{il}\delta_{jn}\delta_{km} +\delta_{im}\delta_{jl}\delta_{kn}+\delta_{in}\delta_{jm}\delta_{kl}\,. \tag{13}$$

An important application of (10) is to the quark number operator, given in (2), which is readily seen to map according to

$$\hat{N}_q\rightarrow 3\sum_{abc} \Lambda^{\dagger}_{abc}\Lambda_{abc}=3\hat{N}_B\,, \tag{14}$$

where $\hat{N}_B$ is the number operator for *colorless baryons*. Thus, the mapping gives the expected result that the number of colorless baryons is one-third the total number of quarks in the system.

Next we turn to a colorless two-body operator

$$B_{abcd}=\sum_{12345} \epsilon_{123}\epsilon_{145}q^{\dagger}_{2a}q^{\dagger}_{3b}q_{5c}q_{4d}\,. \tag{15}$$

It too can be mapped onto colorless baryons using the formalism of Ref. [5]. The resulting expression is

$$B_{abcd}\rightarrow 12\sum_{e} \Lambda^{\dagger}_{abe}\Lambda_{cde}+9\sum_{efgh} \Lambda^{\dagger}_{aef}\Lambda^{\dagger}_{bgh}(\Lambda_{cde}\Lambda_{fgh} +\Lambda_{efg}\Lambda_{cdh}). \tag{16}$$

Note that the baryon image of a two-body operator is finite but non-Hermitian. Non-Hermiticity is the means by which quark Pauli effects are transmitted to the baryon space in the Dyson approach of Ref. [5]. A diagrammatic representation of the physics contained in the mapping of two-body operators (16) is presented in Fig.1.

It is important to emphasize that the most general colorless two-body operator need not be of the form (15). It is only for a delta potential that the two interacting quarks must have distinct colors and be in a color $\bar{3}$ configuration. More general interactions can include components in the color 6 channel; they too can be readily treated with the same formalism developed in Ref. [5].

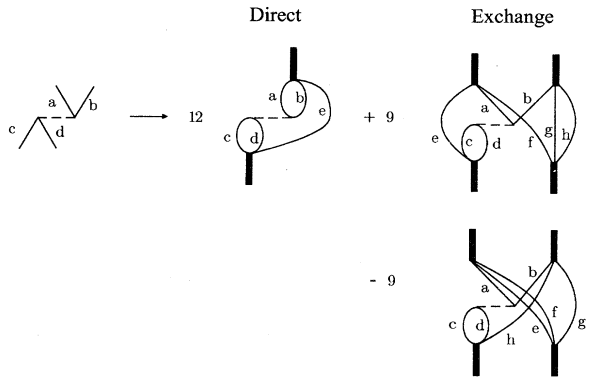

FIG. 1. Diagrammatic representation of the baryon mapping (16) of the colorless two-quark operator (15). Thin solid lines refer to quarks, thick solid lines to baryons, and dashed lines to the two-quark interaction. Summations are implied.

B. Mapping of the model Hamiltonian

In order to apply the above mapping to the three-color delta model, we first rewrite the Hamiltonian (4) in terms of the colorless one- and two-body operators of Eqs. (9) and (15), respectively,

$$\hat{H}=\hat{T}+\hat{V}=\sum_{k}\frac{k^2}{2M}\,A_{kk}-\frac{G}{4}\sum_{k_1k_2k_3} B_{k_2,k_1-k_2,k_1-k_3,k_3}. \tag{17}$$

Applying (10) to the one-body operator $\hat{T}$ and symmetrizing, we obtain

$$\hat{T}\rightarrow\sum_{k_1k_2k_3}\frac{k_1^2+k_2^2+k_3^2}{2M}\,\Lambda^{\dagger}_{k_1k_2k_3}\Lambda_{k_1k_2k_3}. \tag{18}$$

Likewise, by means of (16), we obtain the baryon image of the two-body term

$$\hat{V}\rightarrow -3G\sum_{k_1\rightarrow k_4}\Lambda^{\dagger}_{k_2,k_1-k_2,k_4}\Lambda_{k_1-k_3,k_3,k_4} -\frac{9G}{2}\sum_{k_1\rightarrow k_7}\Lambda^{\dagger}_{k_2,k_4,k_5}\Lambda^{\dagger}_{k_1-k_2,k_6,k_7}\Lambda_{k_1-k_3,k_3,k_4}\Lambda_{k_5,k_6,k_7}. \tag{19}$$

From (19), we see that the two-quark interaction maps onto an operator in the baryon space with both one- and two-body parts. Thus the complete baryon Hamiltonian can be represented as a sum of two parts, $\hat{H}_B=\hat{H}_0+\hat{V}_B$. The first part, the one-body operator $\hat{H}_0$, collects the baryon image of the kinetic energy operator (18) with the one-baryon interaction term arising from the two-quark interaction (19),

$$\hat{H}_0=\sum_{k_1k_2k_3}\frac{k_1^2+k_2^2+k_3^2}{2M}\Lambda^\dagger_{k_1k_2k_3}\Lambda_{k_1k_2k_3}$$
$$-3G\sum_{k_1k_2k_3k_4}\Lambda^\dagger_{k_2,k_1-k_2,k_4}\Lambda_{k_1-k_3,k_3,k_4}. \quad (20)$$

The two-body operator V_B arises solely from the two-quark interaction and is given from (19) by

$$\hat{V}_B=-\frac{9G}{2}\sum_{k_1\to k_7}\Lambda^\dagger_{k_2,k_4,k_5}\Lambda^\dagger_{k_1-k_2,k_6,k_7}\Lambda_{k_1-k_3,k_3,k_4}\Lambda_{k_5,k_6,k_7}. \quad (21)$$

C. Transformation to Jacobi coordinates

We expect that the structure of the colorless baryons that dominate in the ground state of the system should be governed by the relative motion of the three-quark clusters. Thus we next perform a transformation from the single-quark linear momenta k_1,k_2,k_3 entering the colorless baryon operators $\Lambda^\dagger_{k_1k_2k_3}$ and $\Lambda_{k_1k_2k_3}$ to the Jacobi momenta

$$p_1=\tfrac{1}{2}k_1-\tfrac{1}{2}k_2,$$
$$p_2=\tfrac{1}{3}k_1+\tfrac{1}{3}k_2-\tfrac{2}{3}k_3, \quad (22)$$
$$p\equiv p_3=k_1+k_2+k_3,$$

with p being equal to the total momentum of the three-quark cluster. The inverse transformation is

$$k_1=p_1+\tfrac{1}{2}p_2+\tfrac{1}{3}p_3,$$
$$k_2=-p_1+\tfrac{1}{2}p_2+\tfrac{1}{3}p_3, \quad (23)$$
$$k_3=-p_2+\tfrac{1}{3}p_3.$$

We denote the creation and annihilation operators for a colorless baryon with total momentum p and internal momenta p_1 and p_2 by $\Gamma^\dagger_{p_1p_2p}$ and $\Gamma_{p_1p_2p}$, respectively. The anticommutation relations for these operators follow from the Jacobi transformation (22) of Eq. (12):

$$\{\Gamma_{p_1p_2p},\Gamma^\dagger_{p_1'p_2'p'}\}=\frac{1}{6}\delta_{pp'}Q(p_1p_2,p_1'p_2'), \quad (24)$$

where

$$Q(p_1p_2,p_1'p_2')=\delta_{p_2,p_2'}(\delta_{p_1,p_1'}+\delta_{p_1,-p_1'})$$
$$+\delta_{p_2,-p_1'-\frac{1}{2}p_2'}(\delta_{p_1,-\frac{1}{2}p_1'+\frac{3}{4}p_2'}$$
$$+\delta_{p_1,\frac{1}{2}p_1'-\frac{3}{4}p_2'})$$
$$+\delta_{p_2,p_1'-\frac{1}{2}p_2'}(\delta_{p_1,-\frac{1}{2}p_1'-\frac{3}{4}p_2'}$$
$$+\delta_{p_1,\frac{1}{2}p_1'+\frac{3}{4}p_2'}). \quad (25)$$

In the same way, applying the Jacobi transformation (22) to Eq. (11), we obtain the symmetry relations

$$\Gamma_{p_1,p_2,p}=\Gamma_{-\frac{1}{2}p_1-\frac{3}{4}p_2,p_1-\frac{1}{2}p_2,p}=\Gamma_{-\frac{1}{2}p_1+\frac{3}{4}p_2,-p_1-\frac{1}{2}p_2,p}$$
$$=\Gamma_{-p_1,p_2,p}=\Gamma_{\frac{1}{2}p_1-\frac{3}{4}p_2,-p_1-\frac{1}{2}p_2,p}$$
$$=\Gamma_{\frac{1}{2}p_1+\frac{3}{4}p_2,p_1-\frac{1}{2}p_2,p}, \quad (26)$$

and likewise for the $\Gamma^\dagger$ operators.

In terms of the operators $\Gamma^\dagger_{p_1p_2p}$ and $\Gamma_{p_1p_2p}$, the baryon Hamiltonian (20) and (21) reads

$$\hat{H}_B=\hat{H}_0+\hat{V}_B, \quad (27)$$

where

$$\hat{H}_0=\sum_{p_1p_2p_1'p}\left[\frac{2p_1^2+\frac{3}{2}p_2^2+\frac{1}{3}p^2}{2M}\delta_{p_1p_1'}-3G\right]\Gamma^\dagger_{p_1p_2p}\Gamma_{p_1'p_2p} \quad (28)$$

and

$$\hat{V}_B=-\frac{9G}{2}\sum_{PQ}\sum_{p_1p_2q_1q_2p}$$
$$\times\Gamma^\dagger_{[p_2+\frac{1}{2}q_2+\frac{1}{3}(Q-P)+\frac{1}{2}p],(q_2+\frac{1}{3}p),(Q+p)}$$
$$\times\Gamma^\dagger_{p_1,(p_2+\frac{2}{3}p),(P-p)}\Gamma_{q_1q_2Q}\Gamma_{p_1p_2P}. \quad (29)$$

IV. HARTREE-FOCK APPROXIMATION

The baryon Hamiltonian derived in the previous section cannot be diagonalized exactly. In the infinite-matter limit and at low densities, a natural variational approximation to consider is that of Hartree-Fock (HF). Since the baryon Hamiltonian is non-Hermitian, however, it is necessary to use a nonunitary HF approximation [2], for which we now develop the relevant formalism.

A. Introduction of a nonunitary collective transformation

The Hartree-Fock approximation is most readily derived by introducing an appropriate collective transformation. For a non-Hermitian Hamiltonian, this transformation is nonunitary and can be written as

$$\chi^\dagger_{\alpha p}=\sum_{p_1p_2}X^\alpha_{p_1p_2}\Gamma^\dagger_{p_1p_2p},$$
$$\tilde{\chi}_{\alpha p}=\sum_{p_1p_2}Y^\alpha_{p_1p_2}\Gamma_{p_1p_2p}. \quad (30)$$

Note that the operators $\chi^\dagger_{\alpha p}$ and $\tilde{\chi}_{\alpha p}$ of the collective basis are not necessarily Hermitian adjoints of one another. It is easy to show that the transformation coefficients $X^\alpha_{p_1p_2}$ and $Y^\alpha_{p_1p_2}$ satisfy analogous symmetry properties (26) to the operators $\Gamma^\dagger_{p_1p_1p}$ and $\Gamma_{p_1p_2p}$. Finally, by imposing the condition

$$\{\tilde{\chi}_{\alpha p},\chi^\dagger_{\alpha' p'}\}=\delta_{\alpha\alpha'}\delta_{pp'}, \quad (31)$$

we guarantee that the transformation (30) is canonical.

With the aid of Eqs. (24), (30), and (31), we obtain the following properties of the collective transformation coefficients:

$$\sum_{p_1p_2} Y^{\alpha}_{p_1p_2}X^{\alpha'}_{p_1p_2}=\delta_{\alpha\alpha'},$$

$$\sum_{\alpha} Y^{\alpha}_{p_1p_2}X^{\alpha}_{p_1'p_2'}=\tfrac{1}{6}Q(p_1p_2,p_1'p_2'). \tag{32}$$

They can be used to invert the collective transformation (30), yielding

$$\Gamma^{\dagger}_{p_1p_2p}=\sum_{\alpha} Y^{\alpha}_{p_1p_2}\chi^{\dagger}_{\alpha p},$$

$$\Gamma_{p_1p_2p}=\sum_{\alpha} X^{\alpha}_{p_1p_2}\tilde{\chi}_{\alpha p}, \tag{33}$$

where we have assumed that the transformation (or structure) coefficients are real.

The basic idea will be to substitute (33) into (27)–(29) and then carry out a nonunitary Hartree-Fock treatment of the problem. The lowest ($\alpha=1$) solution will then define the collective baryons that dominate in the ground state of the system. Since we will only be interested in the $\alpha=1$ solution, we simplify the notation by replacing

$$\chi^{\dagger}_{\alpha=1p}\rightarrow\chi^{\dagger}_{p},$$

$$\tilde{\chi}_{\alpha=1p}\rightarrow\tilde{\chi}_{p},$$

$$X^{\alpha=1}_{p_1p_2}\rightarrow X_{p_1p_2}, \tag{34}$$

$$Y^{\alpha=1}_{p_1p_2}\rightarrow Y_{p_1p_2}.$$

The part of the baryon Hamiltonian that only involves the dominant collective baryons and their structure coefficients is

$$\hat{H}_B=\hat{H}_0+\hat{V}_B, \tag{35}$$

where

$$\hat{H}_0=\sum_{p}\sum_{p_1p_1'p_2}\left[\frac{2p_1^2+\frac{3}{2}p_2^2+\frac{1}{3}p^2}{2M}\delta_{p_1p_1'}-3G\right]Y_{p_1'p_2}X_{p_1p_2}\chi^{\dagger}_{p}\tilde{\chi}_{p}, \tag{36}$$

and

$$\hat{V}_B=-\frac{9G}{2}\sum_{p}\sum_{PQ} f(p,Q-P)\chi^{\dagger}_{(Q+p)}\chi^{\dagger}_{(P-p)}\tilde{\chi}_{Q}\tilde{\chi}_{P}, \tag{37}$$

with

$$f(p,Q-P)=\sum_{p_1p_2q_1q_2} Y_{[p_2+\frac{1}{2}q_2+\frac{1}{3}(Q-P)+\frac{1}{2}p],(q_2+\frac{1}{3}p)}\times Y_{p_1,(p_2+\frac{2}{3}p)}X_{p_1p_2}X_{q_1q_2}. \tag{38}$$

B. The nonunitary Hartree-Fock method

In the Hartree-Fock method, the ground state is described by a Slater determinant built up from single-particle states with definite momentum p. For a non-Hermitian Hamiltonian, the bra and ket determinantal state vectors need not be adjoints of one another. Thus we introduce separate bra and ket states

$$|\psi\rangle=\prod_{p=-p_f\rightarrow p_f}\chi^{\dagger}_{p}|0\rangle,$$

$$\langle\phi|=\langle 0|\prod_{p=-p_f\rightarrow p_f}\tilde{\chi}_{p}, \tag{39}$$

where p_f is the Fermi momentum and the Fermi sea contains all baryon states with momenta $|p|\leq p_f$ occupied.

The variational condition that defines the structure coefficients $X_{p_1p_2}$ and $Y_{p_1p_2}$ involves minimizing the quantity

$$E_B^{(\lambda)}(X,Y)\equiv E_B(X,Y)-\lambda N_B(X,Y)=\frac{\langle\phi|\hat{H}_B|\psi\rangle}{\langle\phi|\psi\rangle}-\lambda\frac{\langle\phi|\hat{N}_B|\psi\rangle}{\langle\phi|\psi\rangle}, \tag{40}$$

subject to the constraint that the number of baryons equals one-third the total number of quarks N,

$$N_B(X,Y)\equiv\frac{\langle\phi|\hat{N}_B|\psi\rangle}{\langle\phi|\psi\rangle}=\frac{1}{3}N. \tag{41}$$

The expectation value of the baryon number operator $\hat{N}_B$ taken between the trial states (39) is

$$N_B(X,Y)=\left(\sum_{p=-p_f}^{p_f}\right)\sum_{p_1p_2} X_{p_1p_2}Y_{p_1p_2}. \tag{42}$$

On the other hand, the structure coefficients $X_{p_1p_2}$ and $Y_{p_1p_2}$ must be normalized according to Eq. (32), which for $\alpha=1$ becomes

$$\sum_{p_1p_2} X_{p_1p_2}Y_{p_1p_2}=1. \tag{43}$$

Taking the infinite-matter limit (5) leads to a simple relation between the quark density

$$\rho\equiv\lim_{N,L\rightarrow\infty}\frac{N}{L}=\lim_{N,L\rightarrow\infty}\frac{1}{L}\frac{\langle\phi|\hat{N}|\psi\rangle}{\langle\phi|\psi\rangle}=\lim_{N,L\rightarrow\infty}\frac{3}{L}\frac{\langle\phi|\hat{N}_B|\psi\rangle}{\langle\phi|\psi\rangle} \tag{44}$$

and the Fermi momentum p_f, namely,

$$\rho=\frac{3p_f}{\pi}. \tag{45}$$

With this relation between ρ and p_f, the constraint condition (41) reduces to the normalization condition (43).

Summarizing, to find the collective structure coefficients $X_{p_1p_2}$ and $Y_{p_1p_2}$, we must solve the system of equations

$$\frac{\partial}{\partial X_{p_1p_2}}[E_B(X,Y)-\lambda N_B(X,Y)]=0,$$
$$\frac{\partial}{\partial Y_{p_1p_2}}[E_B(X,Y)-\lambda N_B(X,Y)]=0, \tag{46}$$

such that the solution is normalized according to Eq. (43).

C. Application of the Hartree-Fock approximation to the three-color delta model

1. Complete equations

A straightforward calculation, using the particular form of the Hamiltonian (35)–(38) and the Wick theorem, which is valid for the trial states (39), gives the energy functional

$$E_B(X,Y)\equiv\frac{\langle\phi|\hat{H}_B|\psi\rangle}{\langle\phi|\psi\rangle}=E_0(X,Y)+E_V(X,Y), \tag{47}$$

where

$$E_0(X,Y)\equiv\frac{\langle\phi|\hat{H}_0|\psi\rangle}{\langle\phi|\psi\rangle}=\sum_{p=-p_f}^{p_f}\sum_{p_1p_1'p_2}\left[\frac{2p_1^2+\frac{3}{2}p_2^2+\frac{1}{3}p^2}{2M}\delta_{p_1p_1'}-3G\right]X_{p_1'p_2}Y_{p_1p_2} \tag{48}$$

and

$$E_V(X,Y)\equiv\frac{\langle\phi|\hat{V}_B|\psi\rangle}{\langle\phi|\psi\rangle}=\frac{9G}{2}\sum_{P,Q=-p_f}^{p_f}\sum_{p_1q_1p_2q_2}[Y_{[-\frac{1}{2}p_1-\frac{1}{4}p_2+q_2+\frac{1}{6}(Q-P)],[-p_1-\frac{1}{2}p_2+\frac{1}{3}(Q-P)]}Y_{p_1p_2}$$
$$-Y_{[-\frac{1}{2}p_1-\frac{1}{4}p_2+q_2-\frac{1}{3}(Q-P)],(-p_1-\frac{1}{2}p_2)}Y_{[p_1+\frac{1}{2}(Q-P)],(p_2+\frac{1}{3}(Q-P))}]X_{p_1p_2}X_{q_1q_2}. \tag{49}$$

In order to obtain the explicit form of the variational equations (46), we first substitute Eqs. (47)–(49) and (42) in Eq. (40) and then minimize the resulting expression for $E_B^{(\lambda)}(X,Y)$ with respect to $X_{p_1p_2}$ and $Y_{p_1p_2}$. Due to the symmetry properties of the structure coefficients [see Eq. (26) and the comment after (30)], we use the derivatives

$$\frac{\partial X_{p_1p_2}}{\partial X_{q_1q_2}}\equiv\frac{\partial Y_{p_1p_2}}{\partial Y_{q_1q_2}}=\frac{1}{6}Q(p_1p_2,q_1q_2), \tag{50}$$

with $Q(p_1p_2,q_1q_2)$ defined by Eq. (25). The resulting system of nonlinear equations for the structure coefficients then reads

$$\left[\frac{(2p_1^2+\frac{3}{2}p_2^2)+a}{2M}-\lambda\right]X_{p_1p_2}=G\sum_{q=-\infty}^{\infty}(X_{p_1+q,p_2}+X_{p_1-\frac{1}{2}q,p_2-q}+X_{p_1-\frac{1}{2}q,p_2+q})+\frac{3}{\rho L}\frac{\partial E_V}{\partial X_{p_1p_2}},$$

$$\left[\frac{(2p_1^2+\frac{3}{2}p_2^2)+a}{2M}-\lambda\right]Y_{p_1p_2}=G\sum_{q=-\infty}^{\infty}(Y_{p_1+q,p_2}+Y_{p_1-\frac{1}{2}q,p_2-q}+Y_{p_1-\frac{1}{2}q,p_2+q})+\frac{3}{\rho L}\frac{\partial E_V}{\partial Y_{p_1p_2}}, \tag{51}$$

where the derivatives of Eq. (49) are implicitly given and

$$a=\frac{\sum_{p=-p_f}^{p_f}\frac{p^2}{3}}{\sum_{p=-p_f}^{p_f}1}\to\frac{p_f^2}{9}. \tag{52}$$

Due to the complex structure of the derivative terms $\partial E_V(X,Y)/\partial Y_{p_1p_2}$ and $\partial E_V(X,Y)/\partial Y_{p_1p_2}$, Eq.(51) is not in general analytically solvable. We will see, however, that it is analytically solvable for the energy per quark through third order in ρ. Higher order contributions to the energy per particle require numerical analysis. For our purposes, however, it suffices to study the solutions through $O(\rho^3)$.

2. Expansion in powers of the density

To study the solution at low densities, it is useful to expand the various quantities of interest in powers of the density ρ:

$$X_{p_1p_2}=X^{(0)}_{p_1p_2}+\rho X^{(1)}_{p_1p_2}+\rho^2X^{(2)}_{p_1p_2}+\rho^3X^{(3)}_{p_1p_2}+O(\rho^4),$$

$$Y_{p_1p_2}=Y^{(0)}_{p_1p_2}+\rho Y^{(1)}_{p_1p_2}+\rho^2Y^{(2)}_{p_1p_2}+\rho^3Y^{(3)}_{p_1p_2}+O(\rho^4),$$

$$\lambda=\lambda^{(0)}+\rho\lambda^{(1)}+\rho^2\lambda^{(2)}+\rho^3\lambda^{(3)}+O(\rho^4). \tag{53}$$

Inserting (53) into the normalization condition (43) and equating powers of ρ leads to the relations

$$\sum_{p_1p_2} X^{(0)}_{p_1p_2}Y^{(0)}_{p_1p_2}=1,$$

$$\sum_{p_1p_2} (X^{(0)}_{p_1p_2}Y^{(1)}_{p_1p_2}+X^{(1)}_{p_1p_2}Y^{(0)}_{p_1p_2})=0,$$

$$\sum_{p_1p_2} (X^{(0)}_{p_1p_2}Y^{(2)}_{p_1p_2}+X^{(1)}_{p_1p_2}Y^{(1)}_{p_1p_2}+X^{(2)}_{p_1p_2}Y^{(0)}_{p_1p_2})=0,$$

$$\sum_{p_1p_2} (X^{(0)}_{p_1p_2}Y^{(3)}_{p_1p_2}+X^{(1)}_{p_1p_2}Y^{(2)}_{p_1p_2}+X^{(2)}_{p_1p_2}Y^{(1)}_{p_1p_2}+X^{(3)}_{p_1p_2}Y^{(0)}_{p_1p_2})$$

$$=0. \tag{54}$$

An analogous expansion of the nonlinear equations (51) is more difficult, since it is not *a priori* obvious what density dependence is implicitly contained in the partial derivatives of E_V. As a reminder, E_V involves integrations up to the Fermi momentum p_f, which is related to the density. As we now show, however, the first nonzero contribution from these terms is of $O(\rho^3)$.

To see this, we first rewrite Eq. (49) schematically as

$$E_V(X,Y)=\sum_{P,Q=-p_f}^{p_f}\sum_{p_1q_1p_2q_2}[B(p_1,p_2,q_1,q_2,Q-P) - C(p_1,p_2,q_1,q_2,Q-P)]. \tag{55}$$

Both Q and P are limited in magnitude by p_f. Thus, at low densities, we can perform a Taylor series expansion of $B(p_1,p_2,q_1,q_2,Q-P)$ and $C(p_1,p_2,q_1,q_2,Q-P)$ in powers of $Q-P$, viz.,

$$B(p_1,p_2,q_1,q_2,Q-P)=\sum_{n=0}^{\infty} b_n(p_1,p_2,q_1,q_2)(Q-P)^n,$$

$$C(p_1,p_2,q_1,q_2,Q-P)=\sum_{n=0}^{\infty} c_n(p_1,p_2,q_1,q_2)(Q-P)^n. \tag{56}$$

By looking at the explicit forms for $B(p_1,p_2,q_1,q_2,Q-P)$ and $C(p_1,p_2,q_1,q_2,Q-P)$, we see that they are identically equal at $Q-P=0$. The reason for this can be directly traced to the Pauli principle *between baryons*. Two baryons with the same internal momenta cannot have relative momentum 0. Thus the power series expansions for the *full integrand* begins with the linear term in $Q-P$.

It is straightforward to demonstrate that all odd terms from the power series expansion (56) give a zero contribution to $E_V(X,Y)$ when integrated over Q and P. Thus the lowest nonzero contribution to the total energy comes from the term $(Q-P)^2$, and it is of $O(p_f^4)$. From this, we conclude that the lowest nonzero contribution to the energy per particle arising from the two-baryon interaction is of $O(\rho^3)$. Furthermore, to calculate the energy per particle up to this order in ρ, we need to know the zeroth order wave functions $X^{(0)}_{p_1p_2}$ and $Y^{(0)}_{p_1p_2}$ only.

Exactly the same analysis can be applied (albeit with a bit more difficulty) to the two partial derivatives entering the system of equations (51) for the structure functions X and Y. The partial derivatives get their first nonzero contributions at $O(\rho^4)$. Thus the two-baryon interaction first contributes to $X_{p_1p_2}$ and $Y_{p_1p_2}$ at third order in ρ. To make this evident, we rewrite Eq. (51) schematically as

$$\left[\frac{(2p_1^2+\frac{3}{2}p_2^2)+a}{2M}-\lambda\right]X_{p_1p_2}=G\sum_{q=-\infty}^{\infty}(X_{p_1+q,p_2}+X_{p_1-\frac{1}{2}q,p_2-q}+X_{p_1-\frac{1}{2}q,p_2+q})+\rho^3D_{p_1p_2},$$

$$\left[\frac{(2p_1^2+\frac{3}{2}p_2^2)+a}{2M}-\lambda\right]Y_{p_1p_2}=G\sum_{q=-\infty}^{\infty}(Y_{p_1+q,p_2}+Y_{p_1-\frac{1}{2}q,p_2-q}+Y_{p_1-\frac{1}{2}q,p_2+q})+\rho^3F_{p_1p_2}. \tag{57}$$

The functions $D_{p_1p_2}$ and $F_{p_1p_2}$ depend of course on X and Y. When the zeroth-order approximations $X^{(0)}$ and $Y^{(0)}$ are used, they have no density dependence. Higher order approximations yield higher powers in the density.

3. Zeroth-order wave functions and chemical potential

Inserting the expansions (53) into (51), isolating the term that varies as ρ^0, and taking the infinite-matter limit, we see that the zeroth-order wave functions $X^{(0)}_{p_1p_2}$ and $Y^{(0)}_{p_1p_2}$ both satisfy the same integral equation:

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)X^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(X^{(0)}_{p_1+q,p_2}+X^{(0)}_{p_1-\frac{1}{2}q,p_2-q}+X^{(0)}_{p_1-\frac{1}{2}q,p_2+q})dq,$$

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)Y^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(Y^{(0)}_{p_1+q,p_2}+Y^{(0)}_{p_1-\frac{1}{2}q,p_2-q}+Y^{(0)}_{p_1-\frac{1}{2}q,p_2+q})dq. \tag{58}$$

Equation (58) is the integral equation that arises in Fadeev theory for three particles interacting via a delta-function potential. It is straightforward to show that its eigenvalue is

$$\lambda^{(0)}=-Mg^2, \tag{59}$$

and that the corresponding eigenvectors are

$$X^{(0)}_{p_1p_2}=\mathcal{N}^{(0)}_X Z_{p_1p_2} \tag{60}$$

and

$$Y^{(0)}_{p_1p_2}=\mathcal{N}^{(0)}_Y Z_{p_1p_2}, \tag{61}$$

where

$$Z_{p_1p_2}=\frac{c^2(6c^2+2p_1^2+\frac{3}{2}p_2^2)}{(c^2+p_2^2)[c^2+(p_1-\frac{1}{2}p_2)^2][c^2+(p_1+\frac{1}{2}p_2)^2]}, \tag{62}$$

and $c=Mg$.

Note that there are two independent norms that appear. The reason is that it is only the product of the two functions that must be normalized. From the zeroth order normalization condition of (54), we find that

$$\mathcal{N}^{(0)}_X\mathcal{N}^{(0)}_Y=\frac{2}{3}\frac{c^6}{L^2}. \tag{63}$$

We discuss shortly how we fix the one remaining undetermined quantity, the ratio of the two normalization constants $\mathcal{N}^{(0)}_X$ and $\mathcal{N}^{(0)}_Y$.

4. The first- and second-order wave functions and chemical potentials

Next, we turn to the first- and second-order variational equations. At first order, the equations that result are

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)X^{(1)}_{p_1p_2}-\lambda^{(1)}X^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(X^{(1)}_{p_1+q,p_2}+X^{(1)}_{p_1-\frac{1}{2}q,p_2-q}+X^{(0)}_{p_1-\frac{1}{2}q,p_2+q})dq,$$

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)Y^{(1)}_{p_1p_2}-\lambda^{(1)}Y^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(Y^{(1)}_{p_1+q,p_2}+Y^{(1)}_{p_1-\frac{1}{2}q,p_2-q}+Y^{(1)}_{p_1-\frac{1}{2}q,p_2+q})dq. \tag{64}$$

Multiplying the first equation by $X^{(0)}_{p_1p_2}$ (or the second by $Y^{(0)}_{p_1p_2}$) and then integrating over p_1 and p_2 yields (after some changes of variables)

$$\lambda^{(1)}=0. \tag{65}$$

Inserting (65) back into (64), we find that $X^{(1)}_{p_1p_2}$ and $Y^{(1)}_{p_1p_2}$ satisfy *exactly* the same integral equations (58) as the corresponding zeroth-order wave functions. Thus they are both proportional to the same wave functions as given in (60) and (61), viz.,

$$X^{(1)}_{p_1p_2}=\mathcal{N}^{(1)}_X Z_{p_1p_2},$$

$$Y^{(1)}_{p_1p_2}=\mathcal{N}^{(1)}_Y Z_{p_1p_2}. \tag{66}$$

A similar analysis can be carried out at second order in ρ. The equations for the structure coefficients are

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)X^{(2)}_{p_1p_2}-\left(\lambda^{(2)}-\frac{\pi^2}{162M}\right)X^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(X^{(2)}_{p_1+q,p_2}+X^{(2)}_{p_1-\frac{1}{2}q,p_2-q}+X^{(2)}_{p_1-\frac{1}{2}q,p_2+q})dq,$$

$$\left(\frac{2p_1^2+\frac{3}{2}p_2^2}{2M}-\lambda^{(0)}\right)Y^{(2)}_{p_1p_2}-\left(\lambda^{(2)}-\frac{\pi^2}{162M}\right)Y^{(0)}_{p_1p_2}=\frac{g}{\pi}\int_{-\infty}^{\infty}(Y^{(2)}_{p_1+q,p_2}+Y^{(2)}_{p_1-\frac{1}{2}q,p_2-q}+Y^{(2)}_{p_1-\frac{1}{2}q,p_2+q})dq, \tag{67}$$

where we have already used the fact that $\lambda^{(1)}=0$.

Precisely the same strategy as was used in first order shows that

$$\lambda^{(2)}=\frac{\pi^2}{162M} \tag{68}$$

and that $X^{(2)}_{p_1p_2}$ and $Y^{(2)}_{p_1p_2}$ satisfy exactly the same integral equations as the corresponding zeroth- and first-order wave functions. Thus they can be written as

$$X^{(2)}_{p_1p_2}=\mathcal{N}^{(2)}_X Z_{p_1p_2},$$

$$Y^{(2)}_{p_1p_2}=\mathcal{N}^{(2)}_Y Z_{p_1p_2}. \tag{69}$$

As discussed earlier, the first- and second-order contributions to $X_{p_1p_2}$ and $Y_{p_1p_2}$ have no effect on any properties of the system. Thus their normalization coefficients are indeed completely arbitrary. We now show, however, that in the process of removing the arbitrariness in the zeroth-order normalization coefficients, we also remove the arbitrariness in those that arise in first and second order.

5. Imposition of a physical condition

Arbitrary relative normalizations are an unavoidable consequence of using a nonunitary variational principle, with distinct right and left vectors. In analogous work carried out on the use of the nonunitary Hartree-Bose approximation following a boson mapping [10,9], it was shown how to remove this arbitrariness by introducing a physical condition. Here, too, we follow a similar strategy.

The Hartree-Fock solution represented by the bra and ket vectors $|\psi\rangle$ and $\langle\phi|$, respectively, refers to a system with N_B baryons. For the discussion to follow, it is useful to introduce the following state vectors for a system of N_B-1 baryons:

$$|\tilde{\psi}\rangle=\tilde{\chi}_0|\psi\rangle,$$
$$\langle\tilde{\phi}|=\langle\phi|\chi_0^+. \tag{70}$$

These state vectors are obtained from the HF solution for N_B baryons by simply removing the collective $P=0$ baryon.

Consider now the colorless three-quark creation and annihilation operators

$$Q^\dagger_{k_1k_2k_3}=\frac{1}{6}\sum_{123}\epsilon_{123}q^\dagger_{1k_1}q^\dagger_{2k_2}q^\dagger_{3k_3},$$
$$Q_{k_1k_2k_3}=\frac{1}{6}\sum_{123}\epsilon_{123}q_{3k_3}q_{2k_2}q_{1k_1}. \tag{71}$$

Using the mapping equations of Ref. [5], we obtain for their baryon images

$$Q^\dagger_{k_1k_2k_3}\to\tilde{Q}^\dagger_{k_1k_2k_3}=\Lambda^\dagger_{k_1k_2k_3}+\sum_{k_4k_5k_6}(\Lambda^\dagger_{k_4k_5k_1}\Lambda^\dagger_{k_2k_3k_6}+\Lambda^\dagger_{k_4k_5k_2}\Lambda^\dagger_{k_3k_1k_6}+\Lambda^\dagger_{k_4k_5k_3}\Lambda^\dagger_{k_1k_2k_6})\Lambda_{k_4k_5k_6},$$
$$Q_{k_1k_2k_3}\to\tilde{Q}_{k_1k_2k_3}=\Lambda_{k_1k_2k_3}. \tag{72}$$

The physical condition that we impose is that the matrix elements of these two baryon operators (transformed first to Jacobi coordinates), taken between the above states of N_B and N_B-1 baryons, are the same; viz.,

$$\langle\tilde{\phi}|\tilde{Q}_{p_1p_20}|\psi\rangle=\langle\phi|\tilde{Q}^\dagger_{p_1p_20}|\tilde{\psi}\rangle. \tag{73}$$

The relation between the X and Y structure coefficients that results from imposing this condition is

$$X_{p_1p_2}=Y_{p_1p_2}+\sum_{r=-p_f}^{p_f}\sum_s[Y_{s,-p_1-\frac{1}{2}p_2}Y_{-\frac{1}{2}p_1+\frac{3}{4}p_2,\frac{2}{3}r-p_1-\frac{1}{2}p_2}X_{s,\frac{2}{3}r-p_1-\frac{1}{2}p_2}-Y_{s,\frac{1}{3}r-p_1-\frac{1}{2}p_2}Y_{p_1,p_2}X_{s,\frac{1}{3}r-p_1-\frac{1}{2}p_2}$$
$$-Y_{s,\frac{1}{3}r+p_1-\frac{1}{2}p_2}Y_{p_1,p_2}X_{s,\frac{1}{3}r+p_1-\frac{1}{2}p_2}+Y_{s,p_1-\frac{1}{2}p_2}Y_{\frac{1}{2}p_1+\frac{3}{4}p_2,\frac{2}{3}r+p_1-\frac{1}{2}p_2}X_{s,\frac{2}{3}r+p_1-\frac{1}{2}p_2}+Y_{s,p_2}Y_{p_1,\frac{2}{3}r+p_2}X_{s,\frac{2}{3}r+p_2}$$
$$-Y_{s,\frac{1}{3}r+p_2}Y_{p_1,p_2}X_{s,\frac{1}{3}r+p_2}]. \tag{74}$$

We can analyze this equation in much the same way as we analyzed Eq. (55). The conclusion is that the sum of six terms of the form YYX does not contribute until third order in ρ. Thus

$$X^{(i)}_{p_1p_2}=Y^{(i)}_{p_1p_2},\quad \text{for } i=0,\ 1, \text{ and } 2. \tag{75}$$

From Eqs. (75) and (54), we see that

$$X^{(0)}_{p_1p_2}=Y^{(0)}_{p_1p_2} \tag{76}$$

and

$$X^{(1)}_{p_1p_2}=X^{(2)}_{p_1p_2}=Y^{(1)}_{p_1p_2}=Y^{(2)}_{p_1p_2}=0. \tag{77}$$

Thus the physical condition removes the ambiguity in the relative normalization of $X^{(0)}$ and $Y^{(0)}$ and fixes the first- and second-order contributions to X and Y to be identically zero.

6. The energy per particle through $O(\rho^3)$

We have shown that to determine the energy per quark up to third order in ρ we only need the zeroth-order approximations to X and Y. The total energy through that order can thus be obtained from

$$E=E_0(X^{(0)},Y^{(0)})+E_V(X^{(0)},Y^{(0)}), \tag{78}$$

with the functions E_0 and E_V given by Eqs. (48) and (49), respectively.

Using the explicit results for $X^{(0)}$ and $Y^{(0)}$ given in Eqs. (60), (61), and (76), we find that

$$\frac{E}{N}\frac{1}{Mg^2}=-\frac{1}{3}+\frac{\pi^2\tilde{\rho}^2}{486}+\frac{47\,249\pi^2\tilde{\rho}^3}{5\,668\,704}+O(\tilde{\rho}^4). \tag{79}$$

These results are plotted in Fig. 2 along with the exact results (8) for the model.

Comparing Eq. (79) with Eq. (8), we see that Hartree-Fock approximation carried out in the baryon space (with the assumption that the internal structure of all collective baryons is the same) reproduces the exact results through second order in ρ, but begins to deviate at third order. The HF analysis yields too much repulsion at third order, by roughly a factor of 6.

It is not surprising that deviations begin to show up at third order. As we have seen, it is not until third order that effects due to the residual interaction between baryons contribute. The baryon-baryon interaction that derives from the model Hamiltonian is repulsive, reflecting quark exchange effects. Most likely, a Brueckner-like treatment is needed to better incorporate the short-range correlations that result from this repulsive interaction. A similar finding emerged in the study of boson mappings applied to the two-color delta model [9], although there the repulsive boson-boson interaction contributed to the energy per particle already at first order in the density.

It is gratifying that, even at the level of the Hartree-Fock approximation, we are able to reproduce the results through second order in ρ and obtain some (though not all) of the correlations that arise in higher orders. To the best of our knowledge, no approximate treatment has yet been carried out for this problem that extends beyond second order in the density. The method reported by Tosa [8], for example, was only able to extract information on E/N through second order in ρ. Most importantly, as we show shortly, the baryon mapping procedure permits us to readily determine the residual interaction between collective baryons, as required to go beyond the Hartree-Fock approximation.

7. The structure of the baryon-baryon interaction

Once we have defined the structure of the dominant collective baryons, we can readily determine from Eqs. (37) and (38) the interaction between them. In what follows, we assume that the structure of the collective baryons is as given by our zeroth-order approximation. Then

$$\hat{V}_B=-\frac{9G}{2}\sum_p \sum_{PQ} f^{(0)}(p,Q-P)\chi^\dagger_{(Q-p)}\chi^\dagger_{(P+p)}\tilde{\chi}_Q\tilde{\chi}_P\,, \tag{80}$$

where, in terms of the scaled variables $z=p/Mg$ and $\Delta=(Q-P)/Mg$, the interaction amplitudes are

$$f^{(0)}(z,\Delta)=11\,664\frac{A(z,\Delta)}{B(z,\Delta)}\,, \tag{81}$$

with

$$\begin{aligned}A(z,\Delta)=&\,1\,534\,329\,216+46\,912\,608\Delta^2+500\,904\Delta^4+2718\Delta^6+6\Delta^8-[64\,746\,864\Delta+1\,308\,960\Delta^3+10\,809\Delta^5+33\Delta^7]z\\&+[119\,497\,680+3\,182\,328\Delta^2+23\,733\Delta^4+77\Delta^6]z^2-[3\,930\,768\Delta+36\,450\Delta^3+106\Delta^5]z^3+[2\,904\,336\\&+52\,272\Delta^2+128\Delta^4]z^4-[50\,517\Delta+157\Delta^3]z^5+[23\,085+173\Delta^2]z^6+136\Delta z^7+48z^8,\end{aligned} \tag{82}$$

and

$$B(z,\Delta)=(81+\Delta^2)[81+(\Delta-2z)^2][36+(\Delta-z)^2][144+(\Delta-z)^2](9+z^2)(36+z^2)^2[36+(\Delta+z)^2]^2. \tag{83}$$

The above two-baryon interaction, limited to momenta less than p_f, was used in the calculation of the Hartree-Fock energy (79). The same interaction can be used in many-body approximations that go beyond Hartree-Fock, but which still work within the subspace of collective baryons only. With it, for example, we should be able to incorporate the effects of short-range baryon-baryon correlations via Brueckner theory.

8. Quark occupation numbers

The most troublesome feature of mapping techniques, whether to baryons or bosons, is the existence of unphysical states after the mapping. Only when the mapped Hamiltonian is diagonalized exactly, as can never be done for real problems, is there a complete separation of these unphysical states (which do not preserve the original quark Pauli principle) from the physical states of interest. In the Hartree-Fock approximation, for example, there is invariably some admixture of unphysical (or spurious) components in the wave function(s) describing the ground state.

The mapping developed in Ref. [5] has the feature that unphysical states are pushed up in energy, providing some hope that their admixtures in the HF ground state (and approximations to other low-lying states) may be small. Nevertheless, in some cases, this is not enough. In Hartree-Bose studies following a Dyson boson mapping [10], for example, it was shown that the variational solutions provide a good approximation to the exact results only if they do not overpopulate any of the original fermion states. Similar conclusions were reached in an early application of baryon mappings [2] to the Bonn quark shell model [11].

To address this issue in the context of the current analysis, we calculated the number of quarks with a given momentum k implied by our HF solutions. Only if this number remains less than 3 for all k can we have confidence in our solutions.

The number operator for quarks with momentum k can be expressed as

$$\hat{N}(k)=\sum_1 q^\dagger_{1k}q_{1k}\,. \tag{84}$$

Mapping this operator to the space of colorless baryons, transforming to Jacobi coordinates, and truncating to the dominant collective baryons yields

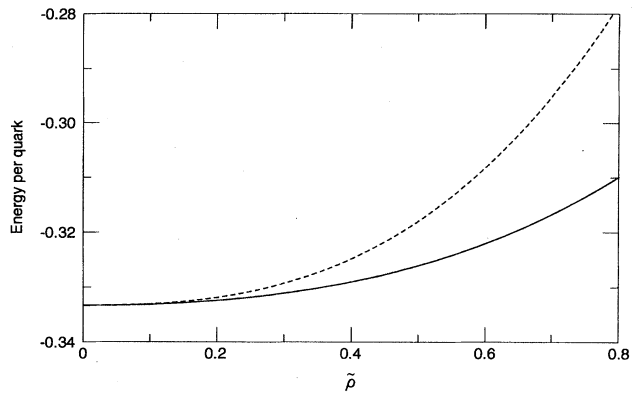

FIG. 2. The energy per quark $(E/N)(1/Mg^2)$ for the three-color delta model as a function of the scaled density $\tilde{\rho}=\rho/Mg$. The solid line gives the exact results; the dotted line gives the results obtained in the Hartree-Fock calculations described in the text.

$$\hat{N}(k)\rightarrow 3\sum_{sp} X_{(1/2)(k-s),k+s-(2/3)p} \times Y_{(1/2)(k-s),k+s-(2/3)p}\chi_p^\dagger\tilde{\chi}_p \,. \tag{85}$$

Evaluating the expectation value $N(k)$ of this operator in the Hartree-Fock ground state gives

$$N(k)=\sum_{s=-\infty}^{\infty}\sum_{p=-p_F}^{p_F} X_{(1/2)(k-s),k+s-(2/3)p} \times Y_{(1/2)(k-s),k+s-(2/3)p} \,. \tag{86}$$

We have evaluated $N(k)$ from this expression using the zeroth-order HF solutions, viz.,

$$N(k)=\sum_{s=-\infty}^{\infty}\sum_{p=-p_F}^{p_F}[X^{(0)}_{(1/2)(k-s),k+s-(2/3)p}]^2. \tag{87}$$

The results are plotted in Fig. 3 for $\tilde{\rho}$ up to 0.7. For larger values of $\tilde{\rho}$, $X^{(0)}$ and $Y^{(0)}$ are not good approximations to the self-consistent HF solution, since terms of $O(\rho^3)$ no doubt contribute significantly.

For all values of ρ that are considered, the quark occupations never exceed 3. Thus the zeroth-order HF solution never overpopulates any quark state at those densities for which it is a good approximation to the full self-consistent solution. Most likely, the Hartree-Fock solutions continue not to violate the quark Pauli principle up to somewhat higher densities, as long as their higher-order contributions are properly taken into account.

V. THE HIGH-DENSITY LIMIT

We have focused our analysis on the low-density limit of the model, in part because real nuclei exist in a relatively low-density world. Here, we would like to briefly discuss the high-density limit for the sake of completeness.

The three-color delta model can also be solved analytically in the high-density limit. The energy per quark in this limit is given by

$$\frac{E}{N}\frac{1}{Mg^2}=\frac{\pi^2\tilde{\rho}^2}{54} \,. \tag{88}$$

This is precisely the energy per quark for a free quark gas. That the system behaves in this way at high densities follows from the fact that the model scales as ρ/Mg, which tells us that the limit of very high density $(\rho\rightarrow 0)$ is equivalent to the limit of very weak interaction $(g\rightarrow 0)$.

The baryon mapping written down in Sec. III applies equally well at all densities. As we now show, however, it is not particularly useful at high densities.

At very high densities, where only the quark kinetic energy term needs to be considered, it is simpler to carry out the mapping to the original one-baryon creation operators $\Lambda^\dagger_{k_1k_2k_3}$ and not to implement the Jacobi transformation. In terms of these operators, the *physical* ground state of the system can be represented in the baryon space as

$$|\psi_1\rangle=\prod_{k=-p_f\rightarrow p_f}\Lambda^\dagger_{kkk}|0\rangle. \tag{89}$$

In this state, there are precisely three quarks with all momenta up to p_f, and thus no violation of the quark Pauli principle. Furthermore, it is straightforward to show that this baryon state has an energy per quark in exact agreement with (88).

It is important to realize, however, that this is not the ground state of *the baryon system*, nor could it be realized in a variational treatment. To illustrate this point, consider another baryon state

$$|\psi_2\rangle=\prod_{k=-p_f\rightarrow p_f}\Lambda^\dagger_{00k}|0\rangle. \tag{90}$$

The energy per quark associated with this state is

$$\frac{E}{N}\frac{1}{Mg^2}=\frac{\pi^2\tilde{\rho}^2}{162} \,, \tag{91}$$

a factor of 3 lower than for the physical ground state (89).

The state (90) is itself not the HF solution of the system. There are independent-baryon states even lower in energy. For the purposes of this discussion, however, it is unnecessary to construct the HF solution. What is already clear is that the Hartree-Fock approximation carried out in the high-

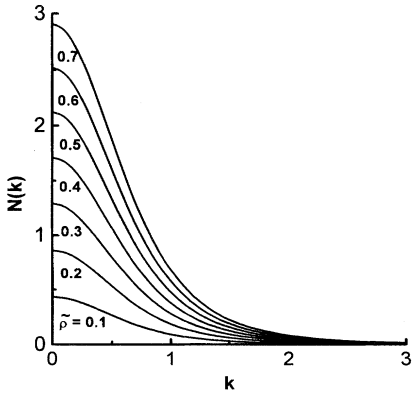

FIG. 3. The number of quarks with momentum k contained in the zeroth-order Hartree-Fock solutions discussed in the text. Below each curve is given the scaled density $\tilde{\rho}=\rho/Mg$ to which it refers.

density limit will of necessity lead to a solution that is well below the physical ground state. As such, the Hartree-Fock approximation is useless in this regime.

The fact that the Hartree-Fock approximation gives a result significantly lower than the exact energy in the high-density limit is easy to understand. It too can be traced to the fact that the high-density limit is equivalent to the weak-interaction limit. In the absence of an interaction between quarks, the baryon system that results from a mapping of necessity has an unphysical ground state, with the Pauli principle violated at the quark level. This is clearly the case for the state (90), and it is likewise true for the HF ground state. It is the non-Hermitian mapping of the two-body interaction that pushes up unphysical states, through implicit inclusion of quark Pauli effects. When there is no active two-body interaction, there is no mechanism to push up unphysical states. Thus, if we carry out a Hartree-Fock analysis subsequent to the mapping of this model in the high-density limit, we should expect to obtain an energy much lower than that of the real ground state. And we do.

The message to be learned from this is that baryon mappings are only *useful* in a regime in which the interactions between the constituents are sufficiently strong so as to produce meaningful cluster correlations. When this is the case, as in the low-density limit of our model, the non-Hermiticity of the effective baryon-baryon interaction will serve to push up unphysical states that violate the Pauli principle at the quark level, so that variational methods (such as HF) can yield meaningful approximations to the physical ground state of the system.

VI. SUMMARY AND CONCLUDING REMARKS

In this work, we applied a recently developed method for mapping constituent quark models onto colorless baryons to a three-color delta model of quarks in the infinite-matter limit. The three-color delta model has several features that make it particularly attractive as a testing ground for baryon mapping methods: (1) it is exactly solvable, and (2) at low densities it admits strong spatial three-quark correlations. All earlier tests were carried out on models that do not admit spatial three-quark correlations, although in real nuclei they prevail.

One of the more interesting conclusions of this work is that effects due to the two-baryon interaction, arising from quark exchange, are suppressed at low densities. More precisely, we find that such effects do not begin to influence either the wave functions or the energy per quark until third order in the density. This result, which was traced to the important role of the Pauli principle between baryons, may have important consequences in efforts to build real nuclei from quarks, since for such systems the densities are relatively low. The same conclusion was also noted in Ref. [8].

Baryon mappings, by themselves, are not especially useful. They only become useful when they can be reliably combined with variational many-body methods. Here, we focused on the low-density limit of the model and considered the use of the Hartree-Fock approximation on the baryon Hamiltonian resulting from the mapping. Several promising features emerged.

(i) We were able to reproduce the energy per quark of the model through second order in the density exactly.

(ii) Because of the strong spatial correlations within the baryon, correlations that are preserved at low densities, our variational Hartree-Fock solution did not overpopulate any quark state, an important criterion for it to provide a reliable reproduction of the exact quark dynamics.

On the other hand, deviations between the exact results and our HF results did begin to show up at third order in the density. This is not terribly surprising. The repulsive baryon-baryon interaction should at some level produce short-range correlations between baryons, which the HF approximation is unable to treat. Most likely, a Brueckner treatment is required to incorporate these correlations, and efforts along these lines are currently under way. As discussed in the paper, our baryon mapping provides a natural prescription for building the two-baryon matrix elements required for such extended many-body approximations.

Many of the conclusions reached here are similar to those that emerged from an earlier application of boson mapping methods to a two-color version of the same model. There, however, deviations began to show up at first order in the density, precisely because there is no Pauli principle active between bosons to suppress these very low-density deviations.

Assuming that we can improve our description of the low-density behavior through a many-body treatment that incorporates short-range correlations, we will (in our view) have gone a long way towards demonstrating the utility of baryon mapping methods to derive *real* nuclei from constituent quark models. Preliminary investigations [12] suggest that these methods can be extended to three-dimensional systems, albeit with major computational effort.

Of course, all of this is predicated on the assumption that constituent quark models are a proper starting point for a quark description of nuclei. This is by no means firmly established yet. Constituent quark models, by their very nature, are limited in their inclusion of several important ingredients of QCD, e.g., confinement (approximate) chiral symmetry, and relativistic effects. An important question still to be addressed is: Is it possible to establish more direct contact between QCD and constituent quark models? Considering the successes of constituent quark models in describing one- and two-baryon systems and the now-present possibility of using baryon mapping methods to build many-nucleon systems from them, we believe that this question needs to be seriously addressed.

It is important to stress, however, that modern constituent quark models do incorporate at some level many of these key features of QCD. The physics of confinement, for example, is often modeled through a quadratic term of the form $-\lambda_i \cdot \lambda_j a r_{ij}^2$ [1], or sometimes through a linear or logarithmic term. Likewise, partial restoration of chiral symmetry can be achieved through the introduction of one-pion and one-sigma exchange [13]. These are the models that we envisage using in more realistic calculations of the quark structure of nuclei.

Another key issue still to be addressed is: What new physics would be present in a quark description of finite nuclei that is not already contained in the traditional picture of nucleons interacting through the exchange of mesons? The

usual view is that explicit quark effects will only show up in processes that probe the high-momentum structure of nuclei. Our view is that a theory that is able to incorporate on the same footing both the quark structure of the nucleon and the intimately related interactions between nucleons, while at the same time accurately describing traditional nuclear properties, can provide a unique handle on where precisely to search for explicit quark effects in nuclear physics. Perhaps they will indeed be limited to high-momentum effects and perhaps not. It is our hope and in fact the ultimate goal of this work to shed light on this fundamental issue in nuclear physics.

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