Unified description of the low lying states of the ground bands of even-even nuclei

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(Received 30 May 1995)

A unified description of the low lying excited states of the ground bands of even-even nuclei throughout the Periodic Table is obtained. A new variable in the geometrical part of the interaction characterizing the different types of collective modes is evaluated and analyzed. A phenomenological formula for the dynamical coefficient of the interaction, depending on the nuclear shell quantum numbers, is derived. A good agreement between the so-obtained theoretical ground band energies and the experimental ones is obtained for 927 low lying states of 271 nuclei from five major shells.

PACS number(s): 21.10.Re, 21.60.Fw

The low lying parts of the spectra of the even-even nuclei is the richest and best known part of experimental data in the nuclear structure physics. Nevertheless, the interest in its theoretical investigation [1–7] does not diminish due to the possibility to investigate some of the general features of atomic nuclei on the basis of an appropriate systematics [8–11].

The purpose of this work is to investigate the general features of the low lying collective states of as large number of even-even nuclei as possible. Such a problem requires the use of a systematics, in which the physical observables of the collective modes will show a smooth and unified behavior. The development of a systematics of this kind is closely related to the choice of the specific nuclear characteristics, which mainly condition the collective phenomena under consideration. This choice could be motivated by empirical evidence that suggests that it is the valence nucleons in a given nuclear shell that are primarily responsible for nuclear collective motion. This is also the basis behind the highly successful $N_\pi N_\nu$ scheme of Casten [9] and the well known proton-neutron version of IBM [12,13].

In order to be fully consistent in our considerations we developed a classification scheme [11] for the even-even nuclei in a way similar to the classifications of elementary particles. This is a new approach to the theoretical nuclear physics, but it can be combined with the very practical and successful notion of a group of dynamical symmetry (DG). Traditionally, the DG is chosen in such a way that one appropriate irreducible representation of the DG contains the entire spectrum of the collective states of a given nucleus [1].

In order to clarify the classification problem applied to the nuclear chart we introduced the concept of the generalized dynamical group (GDG) [11] as a group, one of which irreducible representations gives the spectrum of collective states not of one, but of a sequence of nuclei. Hence, for the description of collective states we assume that a GDG contains some kind of a direct or semidirect product of a classification group (CG) and a DG:

$$\mathrm{GDG} \supset \mathrm{CG} \otimes \mathrm{DG}.$$

In this way the introduction of the GDG leads to the description of the energy spectra of a series of nuclei in a unified way [14], i.e., by means of a common Hamiltonian, whose coefficients are the same functions of the quantum numbers of the representations of the CG. The type of the interaction is given by the DG.

For the moment, the question of the proper choice of the DG and respectively of the GDG is left open and we motivate and use further the group Sp(4,R) as a CG for the even-even nuclei.

Our classification scheme is based on the reduction of the boson representation of the algebra of the noncompact group Sp($4k,R$),$k \geqslant 1$ to its subgroup SU(k) [14]. Sp(4,R) is the simplest and least ambiguous case for its application and will be presented briefly. The standard boson representation of sp(4,R) algebra can be simply constructed with the help of two one-dimensional creation (π^+,ν^+) and annihilation (π,ν) operators. The corresponding generators of the algebra are:

$$\begin{aligned} &\pi^+\pi^+,\nu^+\nu^+,\pi^+\nu^+, \quad \pi\pi,\nu\nu,\pi\nu, \\ &N_\pi=\pi^+\pi, N_\nu=\nu^+\nu, \quad F_+=\pi^+\nu, F_-=\nu^+\pi. \end{aligned} \tag{1}$$

The space of the boson representation of sp(4,R) denoted by $\mathscr{H}$ is reducible and splits into two irreducible subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$. The operator

$$N=N_\pi+N_\nu \tag{2}$$

is the first order Casimir invariant of the maximal compact subgroup U(2) of Sp(4,R). Acting on the spaces $\mathscr{H}_+$and $\mathscr{H}_-$, N reduces them into a direct sum of totally symmetric irreducible unitary representations [IUR] of su(2) labeled by $N=0,2,4,\ldots$.(N even) for $\mathscr{H}_+$ and $N=1,3,5,\ldots$(N odd) for $\mathscr{H}_-$. The boson representation of sp(4,R) also contains the components $F_\pm$ (1) and

$$F_0=\tfrac{1}{2}(N_\pi-N_\nu) \tag{3}$$

of the operator of the F spin, whose third component F_0 does not differ essentially from the first order Casimir invariant of the noncompact subgroup U(1,1):$F_0=\frac{1}{2}(C_{(1,1)}+1)$.

By means of this operator a ladder representation of U(1,1) defined by the fixed values of F_0 is realized in the space $\mathscr{H}$. The same operator F_0 (3) reduces each U(2) representation (fixed value of N) to the representations of the direct product $U_\pi(1)\otimes U_\nu(1)$ defined by the values of N_π and N_ν. The same is obtained by reducing the U(1,1) representations with the operator N. The splitting of the boson spaces $\mathscr{H}_\pm$ corresponding to the reduction

$$\begin{array}{ccc} & \mathrm{sp}(4,R) & \\ {}^{F_0}\swarrow & & \searrow^{N} \\ \mathrm{u}(1,1) & & \mathrm{su}(2) \\ {}_{N}\searrow & & \swarrow_{F_0} \\ & \mathrm{u}_\pi(1)\oplus\mathrm{u}_\nu(1) & \end{array}$$

is presented on schemes I and II, respectively:

Scheme I: $\mathscr{H}_+$

N \ F_0	...	+2	+1	0	-1	-2	...
0				0,0			
2			2,+1	2,0	2,−1		
4		4,+2	4,+1	4,0	4,−1	4,−2	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Scheme II: $\mathscr{H}_-$

N \ F_0	...	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	...
1				$1,\frac{1}{2}$	$1,-\frac{1}{2}$			
3			$3,\frac{3}{2}$	$3,\frac{1}{2}$	$3,-\frac{1}{2}$	$3,-\frac{3}{2}$		
5		$5,\frac{5}{2}$	$5,\frac{3}{2}$	$5,\frac{1}{2}$	$5,-\frac{1}{2}$	$5,-\frac{3}{2}$	$5,-\frac{5}{2}$	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The columns represent the ladders defined by F_0 and the rows represent the IUR's of su(2) defined by N. Each cell corresponds to a given IUR of $\mathrm{u}_\pi(1)\oplus\mathrm{u}_\nu(1)$, defined by the pair of numbers (N_π,N_ν) or which is the same by (N,F_0), given on the schemes.

In the above schemes the even-even nuclei are classified by introducing the appropriate physical meaning of the eigenvalues of the reduction operators. It is straightforward to interpret

$$N_\pi=\tfrac{1}{2}(N_p-N_p^{\mathrm{mag}}) \quad N_\nu=\tfrac{1}{2}(N_n-N_n^{\mathrm{mag}}) \tag{4}$$

as the numbers of proton and neutron valence pairs of the nucleus from a given shell, where N_p and N_n are the total numbers of protons and neutrons of the nucleus and N_p^{mag} and N_n^{mag} are the corresponding magic numbers of the shell to which it belongs. Then N (2) and F_0(3) are obviously, as in the IBM-2 [1], the total number of valence bosons and the third projection of the F spin, related to the nuclear characteristics in the following way:

$$N=\tfrac{1}{2}(A-A^{\mathrm{mag}}), \quad F_0=\tfrac{1}{2}(M_T-M_T^{\mathrm{mag}}), \tag{5}$$

where $A=N_p+N_n$ is the nucleus' mass number and $M_T=\frac{1}{2}(N_p-N_n)$ is the third projection of the isospin.

Further the nuclei from a given major shell are arranged in the symplectic multiplets in the spaces $\mathscr{H}_\pm$ by mapping the shell onto the schemes I and II. It is convenient to define the shells by two pairs of double magic nuclei $(N_p^{(1)}\ N_n^{(1)})$ and $(N_p^{(2)}\ N_n^{(2)})$ which denote the proton and neutron magic numbers of the nucleus at the beginning and at the end of the shell, respectively; $(N_p^{(2)}>N_p^{(1)}\ N_n^{(2)}>N_n^{(1)})$. The double magic number $(N_p^{(1)}\ N_n^{(1)})$ corresponds to the vacuum state $(N=0,F_0=0)$ in $\mathscr{H}$. Using formulas (4) and (5) each nucleus is placed in a definite cell in the space $\mathscr{H}_+$ or $\mathscr{H}_-$, which represents a given IUR of $\mathrm{u}_\pi(1)\oplus\mathrm{u}_\nu(1)$. The IUR's of $\mathrm{u}_\pi(1)\oplus\mathrm{u}_\nu(1)$ are one dimensional, which means that each nucleus is uniquely represented in the boson representation of the classification group Sp(4,R). The symplectic multiplets obtained in this way are denoted by $(N_p^{(1)},N_n^{(1)}|N_p^{(2)}\ N_n^{(2)})_+$ if N is even and $(N_p^{(1)},N_n^{(1)}|N_p^{(2)},N_n^{(2)})_-$ if N is odd. Each row of a given sp(4,R) multiplet contains nuclei belonging to a u(2) submultiplet (isobar), and each column contains nuclei from a given u(1,1) submultiplet (isofer).

The expediency of this classifications scheme [11] is further motivated by the qualitative analysis of the energies of the first excited 2^+ levels of the nuclei ordered in the above way in symplectic multiplets. The energies lay on smooth noncrossing curves when plotted as functions of N for each fixed value of F_0. Even more, in each shell and from shell to shell these curves demonstrate a periodic behavior. This lead us to the idea to unify the description of the ground band energies of the three main types of nuclear collectivity:

vibrational [15]

$$E_v(L)=\alpha_1 L, \tag{6}$$

rotational [15]

$$E_r(L)=\alpha_2 L(L+1), \tag{7}$$

and transitional [16]

$$E_{\mathrm{tr}}(L)=\alpha_3 L(L+\omega) \tag{8}$$

by evaluating their models' parameters $\alpha_i, i=1,2,3$:

$$\alpha_i\{h_k\}=\alpha_i\{A_p,A_n,N,F_0,\bar{N},\bar{F}_0\}, \tag{9}$$

TABLE I. The table contains the nuclei from the shell $(28,28|50,50)_-$ mapped on the odd H_- (N-odd) subspace of Sp(4,R). Each nucleus is defined by the total number of valence boson particles N or boson holes $\bar{N}$, which enumerate the rows from left and right, respectively, and the third projections of the F ($\bar{F}$) spin, which enumerate the columns at the top and the bottom, respectively. Next to the label of the each nucleus is given the value of (ω), in parentheses and the number of states with good theoretical description of the energies.

	F_0					
N	$-1/2$	$-3/2$	$-5/2$	$-7/2$	$-9/2$	$\bar{N}$
1	^{58}Ni					21
3	^{62}Zn(28), 2	^{62}Ni				19
5	^{66}Ge(26), 2	^{66}Zn(31), 2	^{66}Ni			17
7	^{70}Se(22), 2	^{70}Ge(26), 2	^{70}Zn(21), 2			15
9	^{74}Kr(8), 5	^{74}Se(13), 3	^{74}Ge(13), 2	^{74}Zn		13
11	^{78}Sr(4), 3	^{78}Kr(9), 5	^{78}Se(13), 3	^{78}Ge(13), 2	^{78}Zn	11
13	^{82}Zr(6), 4	^{82}Sr(10), 2	^{82}Kr(15), 3	^{82}Se(14), 2	^{82}Ge	9
15		^{86}Zr(13), 2	^{86}Sr(19), 2	^{86}Kr		7
17		^{90}Mo(15), 2	^{90}Zr			5
19		^{94}Ru				3
21	^{98}Cd					1
N	1/2	3/2	5/2	7/2	9/2	$\bar{N}$
	$\bar{F}_0$					

as functions of the set $\{h_k\}$ of the classification quantum numbers

$$A_p=N_p^{(1)}+N_p^{(2)}, \quad A_n=N_n^{(1)}+N_n^{(2)},$$
$$N=N_\pi+N_\nu, \quad F_0=\tfrac{1}{2}(N_\pi-N_\nu), \tag{10}$$
$$\bar{N}=\bar{N}_\pi+\bar{N}_\nu, \quad \bar{F}_0=\tfrac{1}{2}(\bar{N}_\pi-\bar{N}_\nu),$$

related to the shell model quantum numbers through the above systematics [11]. N_π and N_ν are defined in (4), while $\bar{N}_\pi$ and $\bar{N}_\nu$ are numbers of the corresponding hole pairs. N,F_0 from (10) coincide with the ones defined in (2,3), while $\bar{N}$ is the total number of valence boson holes and $\bar{F}_0$ is the third projection of the $\bar{F}$ spin, respectively (see [15,16]).

It is clear even from only empirical experience that in order to describe collective modes we need just six variables, which are the numbers of protons N_p and neutrons N_n of the nucleus and the four boundary numbers $(N_p^{(1)},N_n^{(1)})$ and $(N_p^{(2)},N_n^{(2)})$ of the shell to which it belongs. In our case it is convenient to use the set of six variables (10) which are obtained from these initial quantum numbers, but are naturally linked (2,3) to the quantum number of the classification algebra sp(4,R) and to the very popular model language of IBM [15,16].

The symplectic multiplets $(28,28|50,50)_\pm$, $(28,50|50,82)_\pm$, $(50,50|82,82)_\pm$, $(50,82|82,126)_\pm$, and $(82,126|126,184)_\pm$ formed in this way are presented in Tables I–X, respectively. The values of the nuclear characteristics (10) can be taken from these tables as the quantum numbers $N(\bar{N})$ enumerate their rows and $F_0(\bar{F}_0)$ their columns.

We have obtained simple and easy to apply phenomenological formulas for the dynamical coefficients of the interactions (6), (7), and (8) by evaluating their particular functional dependence on the above quantum numbers (10). Now the problem of the unification of the interaction, related to the choice of the DG, should be addressed in order to obtain the universal description of the ground state bands of all even-even nuclei already classified in symplectic multiplets.

An empirical argumentation for the unification of the interactions of the three main collective modes, namely, rotational, vibrational, and transitional is provided by the investigation of the energy ratio $R_2=E(4^+)/E(2^+)$. This ratio has been widely used as an indicator of collectivity [17]. The theoretical value of the energy ratio for the interaction (8) can be presented in the form:

$$R_2(\omega)=2+\frac{4}{(2+\omega)} \tag{11}$$

TABLE II. As the caption of Table I for the shell $(28,28|50,50)_+$.

	F_0				
N	0	-1	-2	-3	$\bar{N}$
0	^{56}Ni				22
2	^{60}Zn(30), 2	^{60}Ni			20
4		^{64}Zn(30), 2	^{64}Ni		18
6		^{68}Ge(27), 2	^{68}Zn(30), 2	^{68}Ni	16
8		^{72}Se(18), 2	^{72}Ge(18), 2	^{72}Zn	14
10		^{76}Kr(8), 4	^{76}Se(12), 4	^{76}Ge(12), 2	12
12		^{80}Sr(6), 5	^{80}Kr(11), 3	^{80}Se(19), 2	10
14		^{84}Zr(9), 3	^{84}Sr(14), 3	^{84}Kr(17), 3	8
16		^{88}Mo(15), 2	^{88}Zr(18), 2	^{88}Sr	6
18			^{92}Mo		4
N	0	1	2	3	$\bar{N}$
		$\bar{F}_0$			

TABLE III. As the caption of Table I for the shell $(28,50|50,82)_+$.

	F_0					
N	0	1	2	3	4	$\bar{N}$
6		^{90}Kr(22), 2	^{90}Sr(24), 2			21
8		^{94}Sr(25), 2	^{94}Zr			19
10	^{98}Sr(2), 2	^{98}Zr	^{98}Mo(19), 2	^{98}Ru(17), 3	^{98}Pd	17
12	^{102}Zr(1), 2	^{102}Mo(5), 2	^{102}Ru(11), 3	^{102}Pd(13), 3	^{102}Cd(20), 2	15
14	^{106}Mo(2), 3	^{106}Ru(5), 2	^{106}Pd(12),3	^{106}Cd(16), 3		13
16	^{110}Ru(3), 4	^{110}Pd(7), 3	^{110}Cd(15), 3			11
18	^{114}Pd(5), 3	^{114}Cd(11), 2				9
20	^{118}Cd(9), 2					7
N	−5/2	−7/2	−9/2	−11/2	−13/2	$\bar{N}$
	$\bar{F}_0$					

which makes explicit its dependence on the geometrical parameter ω. Obviously its limiting values $R_2=2$ and $R_2=3.33$ are obtained at $\omega\to\infty$ for (6) and $\omega=1$ for (7), respectively. So the value of the parameter ω accounts for the changes of the nuclear structure through the nuclear chart, incorporating as limiting cases the interactions in (6) and (7).

The first introduction of a similar idea is the Ejiri formulas

$$E_L=aL(L+1)+bL$$

according to which the parameter $\omega=(a+b)/a$ scales the different degrees of collectivity by fixing the arbitrary ratio of the terms in L and L^2 [3].

Interactions of the type (8) can be found in reviewing the models, dealing successfully with the description of the low-lying part of different types of collective spectra. In the Hamiltonians of the exactly solvable limits for the vibrational [SU(5)] and gamma soft [O(6)] nuclei of IBM-1 [1], terms with $\omega\neq 1$ appear through the eigenvalues of the corresponding Casimir operators with the labels of their respective representations. We remark that the energy expression (8) looks like the eigenvalue $C_2=2f(f+2w)$ of the second order Casimir invariant of the most symmetric representation $m=(f,0,\ldots,0)$ of the algebras of B_n, C_n, and D_n types of the groups $O(2n+1)$, $Sp(2n)$ and $O(2n)$ with $w=n-1/2$, $w=n$, and $w=n-1$, respectively.

As mentioned above we have tested the interaction (8) in the description of the low lying states of the ground bands of transitional nuclei [16]. In order to unify their description we have evaluated at the same time the parameter ω in the geometric part of the interaction and also the dynamical coefficient in front of it as a function of the nuclear shell quantum numbers. We have compared our results with the recently observed [18], for all nuclei with $2.05\leq R_2\leq 3.15$, universal behavior of an anharmonic vibrator

$$E(4_1^+)=2E(2_1^+)+\epsilon_4 \tag{12}$$

TABLE IV. As the caption of Table I for the shell $(28,50|50,82)_-$.

	F_0								
N	9/2	7/2	5/2	3/2	1/2	−1/2	−3/2	−5/2	$\bar{N}$
3				^{84}Se	^{84}Ge				24
5			^{88}Sr	^{88}Kr(23), 2	^{88}Se				22
7		^{92}Mo	^{92}Zr	^{92}Sr	^{92}Kr				20
9	^{96}Pd	^{96}Ru	^{96}Mo(21), 2	^{96}Zr	^{96}Sr(21), 2				18
11	^{100}Cd	^{100}Pd(16), 3	^{100}Ru(14), 3	^{100}Mo(12), 2	^{100}Zr(3), 2	^{100}Sr(1), 2			16
13	^{104}Sn	^{104}Cd(16), 3	^{104}Pd(14), 4	^{104}Ru(7), 4	^{104}Mo(2), 2				14
15		^{108}Sn	^{108}Cd(15), 2	^{108}Pd(9), 3	^{108}Ru(4), 2	^{108}Mo			12
17			^{112}Sn	^{112}Cd(13), 2	^{112}Pd(6), 2	^{112}Ru(3), 2			10
19				^{116}Sn	^{116}Cd(10), 2	^{116}Pd(5), 2			8
21					^{120}Sn	^{120}Cd(8), 2			6
23						^{124}Sn	^{124}Cd		4
25							^{128}Sn		2
27								^{132}Sn	0
N	−7	−6	−5	−4	−3	−2	−1	0	$\bar{N}$
	$\bar{F}_0$								

TABLE V. As the caption of Table I for the shell $(50,50|82,82)_-$.

	F_0								
N	$-1/2$	$-3/2$	$-5/2$	$-7/2$	$-9/2$	$-11/2$	$-13/2$	$-15/2$	$\bar{N}$
3	^{106}Te	^{106}Sn							29
5	^{110}Xe	^{110}Te	^{110}Sn						27
7		^{114}Xe	^{114}Te(18), 2	^{114}Sn					25
9			^{118}Xe(6), 4	^{118}Te(15)	^{118}Sn				23
11			^{122}Ba(3), 4	^{122}Xe(7), 4	^{122}Te(13), 2	^{122}Sn			21
13			^{126}Ce	^{126}Ba(5), 5	^{126}Xe(8), 4	^{126}Te(16), 2	^{126}Sn		19
15			^{130}Nd(1), 3	^{130}Ce(4), 4	^{130}Ba(7), 4	^{130}Xe(11), 3	^{130}Te(19), 2	^{130}Sn	17
17			^{134}Sm	^{134}Nd(5), 3	^{134}Ce(8), 3	^{134}Ba(12), 2	^{134}Xe(19), 2	^{134}Te	15
19			^{138}Gd	^{138}Sm(5), 3	^{138}Nd(9), 4	^{138}Ce(16), 2	^{138}Ba		13
21				^{142}Gd(8), 3	^{142}Sm(14), 2	^{142}Nd			11
23				^{146}Dy(11), 2	^{146}Gd				9
25				^{150}Er					7
27				^{154}Hf					5
N	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	$\bar{N}$
				$\bar{F}_0$					

with nearly constant anharmonicity ϵ_4. The physical meaning of the phenomenological parameter α_3 can be interpreted as the anharmonicity of the two phonon level $\epsilon_4(\alpha_3=\frac{1}{8}\epsilon_4)$. Although the anharmonicity is rather constant the changes in the underlying structure are accounted for through the dependence of the value of ω

$$\omega=\frac{4E(2)}{\epsilon_4}-2=\frac{E(2)}{2\alpha_3}-2 \tag{13}$$

on the energies of the first exited 2^+ levels, which in these regions decrease with N (2) from nearly closed shell values of >1 MeV towards the very small rotational values of the deformed rotor (100–200 keV). This justifies further the introduction of the variable ω in the geometric part of the interaction, which actually shows how the investigated nuclei pass through transitional regions. But the effect of unification of the description of the nuclei in the transitional region was obtained by the evaluation of the coefficient α_3 of the interaction (8), related to the dynamical aspects of the problem, as a function of the nuclear characteristics expressed in terms of the shell model quantum numbers. The detailed analysis of the phenomenological formulas for this coefficient although consistent with the constancy of the anharmonicity proved its applicability also to some rotational-like nuclei with $\omega=1$ and some nuclei with vibrational nature with rather large $\omega\geq 20$. The good agreement between the experimental and theoretically calculated with the use of

TABLE VI. As the caption of Table I for the shell $(50,50|82,82)_+$.

	F_0								
N	-1	-2	-3	-4	-5	-6	-7	-8	$\bar{N}$
2	^{104}Sn								30
4		^{108}Sn							28
6		^{112}Te(17), 2	^{112}Sn						26
8		^{116}Xe(8), 2	^{116}Te(16), 2	^{116}Sn					24
10		^{120}Ba(2), 2	^{120}Xe(6), 2	^{120}Te(13), 2	^{120}Sn				22
12		^{124}Ce	^{124}Ba(4), 4	^{124}Xe(7), 5	^{124}Te(14), 2	^{124}Sn			20
14		^{128}Nd(1), 3	^{128}Ce(3), 3	^{128}Ba(5), 5	^{128}Xe(9), 3	^{128}Te(18), 2	^{128}Sn		18
16		^{132}Sm	^{132}Nd(4), 2	^{132}Ce(6), 4	^{132}Ba(9), 3	^{132}Xe(14), 2	^{132}Te	^{132}Sn	16
18			^{136}Sm	^{136}Nd(6), 4	^{136}Ce(11), 3	^{136}Ba(18), 2	^{136}Xe		14
20			^{140}Gd	^{140}Sm(9)	^{140}Nd(15), 2	^{140}Ce			12
22			^{144}Dy	^{144}Gd(13)	^{144}Sm				10
24			^{148}Er(10), 2	^{148}Dy					8
26			^{152}Yb						6
N	1	2	3	4	5	6	7	-8	$\bar{N}$
				$\bar{F}_0$					

TABLE VII. As the caption of Table I for the shell $(50,82|82,126)_+$.

	F_0									
N	5	4	3	2	1	0	−1	−2	−3	$\bar{N}$
0						^{132}Sn				38
2					^{136}Xe	^{136}Te				36
4				^{140}Ce	^{140}Ba(26), 2	^{140}Xe(17), 2				34
6			^{144}Sm	^{144}Nd(27), 2	^{144}Ce(19), 3	^{144}Ba(8), 4	^{144}Xe			32
8		^{148}Dy	^{148}Gd(28), 2	^{148}Sm(24), 2	^{148}Nd(13), 3	^{148}Ce(6), 3	^{148}Ba			30
10	^{152}Yb	^{152}Er(27), 2	^{152}Dy(24), 2	^{152}Gd(13), 3	^{152}Sm(4), 7	^{152}Nd(1), 4	^{152}Ce(1), 2			28
12	^{156}Hf	^{156}Yb(19), 2	^{156}Er(12), 5	^{156}Dy(4), 7	^{156}Gd(2), 7	^{156}Sm(1), 4				26
14	^{160}W	^{160}Hf	^{160}Yb(8), 4	^{160}Er(4), 4	^{160}Dy(1), 5	^{160}Gd(1), 4				24
16	^{164}Os	^{164}W	^{164}Hf(7), 4	^{164}Yb(4), 4	^{164}Er(1), 5	^{164}Dy(1), 5				22
18	^{168}Pt	^{168}Os	^{168}W(7), 4	^{168}Hf(3), 5	^{168}Yb(1), 6	^{168}Er(1), 5	^{168}Dy			20
20		^{172}Pt	^{172}Os(7), 4	^{172}W(3), 5	^{172}Hf(2), 6	^{172}Yb(1), 6	^{172}Er			18
22			^{176}Pt	^{176}Os(3), 5	^{176}W(2), 6	^{176}Hf(1), 5	^{176}Yb(1), 5			16
24			^{180}Hg	^{180}Pt	^{180}Os(3), 5	^{180}W(2), 5	^{180}Hf(1), 4			14
26			^{184}Pb	^{184}Hg	^{184}Pt(4), 4	^{184}Os(3), 6	^{184}W(3), 5	^{184}Hf		12
28				^{188}Pb	^{188}Hg	^{188}Pt(8), 5	^{188}Os(4), 3	^{188}W		10
30					^{192}Pb	^{192}Hg	^{192}Pt(9), 2	^{192}Os(5), 4		8
32						^{196}Pb	^{196}Hg	^{196}Pt(9), 3	^{196}Os	6
34							^{200}Pb	^{200}Hg	^{200}Pt(12), 2	4
36								^{204}Pb	^{204}Hg	2
38									^{208}Pb	0
N	−8	−7	−6	−5	−4	−3	−2	−1	0	$\bar{N}$
	$\bar{F}_0$									

TABLE VIII. As the caption of Table I for the shell $(50,82|82,126)_-$.

	F_0										
N	11/2	9/2	7/2	5/2	3/2	1/2	−1/2	−3/2	−5/2	−7/2	$\bar{N}$
1						^{134}Te	^{134}Sn				37
3					^{138}Ba	^{138}Xe(27), 2	^{138}Te				35
5				^{142}Nd	^{142}Ce(30), 2	^{142}Ba(16), 2	^{142}Xe				33
7			^{146}Gd	^{146}Sm(28), 2	^{146}Nd(20), 2	^{146}Ce(11), 2	^{146}Ba(7), 3				31
9		^{150}Er	^{150}Dy(27), 2	^{150}Gd(26), 2	^{150}Sm(13), 5	^{150}Nd(4), 4	^{150}Ce(3), 4				29
11	^{154}Hf	^{154}Yb	^{154}Er(21), 2	^{154}Dy(12), 4	^{154}Gd(4), 7	^{154}Sm(1), 6	^{154}Nd(1), 4				27
13	^{158}W	^{158}Hf	^{158}Yb(12), 4	^{158}Er(7), 4	^{158}Dy(2), 5	^{158}Gd(1), 6	^{158}Sm(1), 4				25
15		^{162}W	^{162}Hf(10), 4	^{162}Yb(6), 4	^{162}Er(3), 5	^{162}Dy(1), 7	^{162}Gd				23
17		^{166}Os	^{166}W	^{166}Hf(5), 5	^{166}Yb(2), 6	^{166}Er(1), 5	^{166}Dy(1), 2				21
19		^{170}Pt	^{170}Os	^{170}W(4), 4	^{170}Hf(2), 6	^{170}Yb(1), 7	^{170}Er(1), 5				19
21			^{174}Pt	^{174}Os(4), 3	^{174}W(3), 5	^{174}Hf(1), 6	^{174}Yb(1), 4				17
23			^{178}Hg	^{178}Pt	^{178}Os(3), 5	^{178}W(2), 6	^{178}Hf(1), 6	^{178}Yb			15
25				^{182}Hg	^{182}Pt(4), 4	^{182}Os(3), 4	^{182}W(2), 4	^{182}Hf(1), 4			13
27				^{186}Pb	^{186}Hg	^{186}Pt(5), 4	^{186}Os(3), 3	^{186}W(3), 4			11
29					^{190}Pb	^{190}Hg	^{190}Pt(9), 3	^{190}Os(5), 3	^{190}W		9
31						^{194}Pb	^{194}Hg	^{194}Pt(9), 3	^{194}Os(5), 2		7
33							^{198}Pb	^{198}Hg	^{198}Pt(11), 3		5
35								^{202}Pb	^{202}Hg		3
37									^{206}Pb	^{206}Hg	1
N	−17/2	−15/2	−13/22	−9/2	−7/2	−5/2	−3/2	−1/2	1/2	$\bar{N}$	
	$\bar{F}_0$										

TABLE IX. As the caption of Table I for the shell $(82,126|126,184)_-$.

	F_0						
N	3/2	1/2	−1/2	−3/2	−5/2	−7/2	$\bar{N}$
1		^{210}Po	^{210}Pb				50
3	^{214}Ra	^{214}Rn	^{214}Po	^{214}Pb			48
5	^{218}Th	^{218}Ra(19), 2	^{218}Rn(18), 2	^{218}Po			46
7	^{222}U	^{222}Th(9), 4	^{222}Ra(5), 2	^{222}Rn(10), 2			44
9		^{226}U	^{226}Th(3), 3	^{226}Ra(3), 2	^{226}Rn		42
11			^{230}U(1), 8	^{230}Th(2), 5	^{230}Ra		40
13			^{234}Pu	^{234}U(1), 6	^{234}Th(2), 4		38
15			^{238}Cm	^{238}Pu(1), 6	^{238}U(1), 6		36
17		^{242}Fm	^{242}Cf	^{242}Cm(1), 3	^{242}Pu(1), 6	^{242}U(2)	34
19			^{246}Fm	^{246}Cf	^{246}Cm(1), 4	^{246}Pu(1), 2	32
21			^{250}No	^{250}Fm	^{250}Cf(1), 4	^{250}Cm	30
23				^{254}No	^{254}Fm(1), 2	^{254}Cf	28
25					^{258}No	^{258}Fm	26
N	−5	−4	−3	−2	−1	0	$\bar{N}$
				$\bar{F}_0$			

(8) energies from the spectra of some rotational and vibrational nuclei clearly suggests a further generalization of our approach. The different types of collective motion could be integrated by means of the empirical parameter $\omega=1,2,\ldots$ in the expression of the low lying yrast energies for all the even-even nuclei:

$$E_L=\alpha L(L+\omega). \tag{14}$$

The analysis of these results not only provides a deeper understanding of the onset and development of collectivity and deformation in nuclei, but even more, it leads to a possibility of a further generalization even of the rather complex and diverse interactions in a vast region of nuclei. The investigation of such a generalization, which will unify the description of the low lying energies of all even-even nuclei with a collective spectra is a subject of this work. Such a unification, in our opinion, is the bridge that will relate the observed widespread phenomena in nuclear collective spectra to the underlying shell structure and interactions in atomic nuclei.

In order to evaluate the values of ω we compare the theoretical ratios

$$R_L=\frac{E_{L+2}}{E_L}=\frac{(L+2)(L+2+\omega)}{L(L+\omega)}, \tag{15}$$

for $L=2,4,6,\ldots$ and $\omega=1,2,\ldots$ with the experimental ones for each nuclei. This gives us from one side the starting value ω_0 of the parameter ω at which the interaction coefficient α in (14) will be evaluated and also the number of the

TABLE X. As the caption of Table I for the shell $(82,126|126,184)_+$.

	F_0					
N	1	0	−1	−2	−3	$\bar{N}$
2	^{212}Rn	^{212}Po				49
4	^{216}Ra	^{216}Rn				47
6		^{220}Ra(9), 2	^{220}Rn(14), 2			45
8		^{224}Th(5), 2	^{224}Ra(4), 3			43
10			^{228}Th(2), 5	^{228}Ra(2), 3		41
12			^{232}U(1), 7	^{232}Th(1), 8		39
14			^{236}Pu(1), 7	^{236}U(1), 6		37
16				^{240}Pu(1), 7	^{240}U(1), 2	35
18			^{244}Cf	^{244}Cm(1), 4	^{244}Pu(1), 6	33
20				^{248}Cf	^{248}Cm(1), 9	31
22				^{252}Fm	^{252}Cf(1), 2	29
24					^{256}Fm(2), 4	27
N	−9/2	−7/2	−5/2	−3/2	−1/2	$\bar{N}$
				$\bar{F}_0$		

ground band states which will be included in the fit. We cut the band at the values of L, where a sudden change in the experimental ω appears, revealing a phase transition, like the backbending.

The good agreement between the theoretical and experimental results in the rotational, vibrational, and transitional cases, which we would like to incorporate in this paper as limiting cases of a general description of the low-lying energies of all the even-even nuclei clearly imposes the use of the same set of quantum numbers [15, 16] as variables of a universal function , which we consider to be the dynamical coefficient $\alpha\{h_k\}$ of the general interaction (14) with $\omega \geqslant 1$.

Like in [15] and [16] we assume the parameter $\alpha(h_k)$ to be a polynomial up to a second degree in the variables (10), because we obviously take into account only one- and two-body interactions. Thus we deduce a formula for the dynamical parameter of the interaction by evaluating the linearly independent coefficients in the second order polynomials in the nuclear characteristics. We determine the polynomial constants as solutions of the overdetermined system of the linear equations

$$E^i_{\rm th}=E^i_{\rm exp}, \quad i=1,2,\ldots,k, \tag{16}$$

where $k=927$ is the number of the considered yrast states and

$$E^i_{\rm th}=\alpha(A_p,A_n,N,F_0,\bar{N},\bar{F}_0)L(L+\omega_0). \tag{17}$$

We apply our unified formula (17) to 927 levels from the ground bands of 271 rotational, vibrational, and transitional nuclei belonging to the multiplets: $(28,28|50,50)_\pm$, $(28,50|50,82)_\pm$, $(50,50|82,82)_\pm$, $(50,82|82,126)_\pm$, and $(82,126|126,184)_\pm$. Most of the nuclei which can not be described in the framework of this simple unification are at the borders of the shells mapped on the symplectic multiplets. These nuclei are either with proton or neutron closed shell, or with a minimal value of N or $\bar{N}$, or a maximal value of $|F_0|$ for the multiplet under consideration. The interpretation of this fact is that our phenomenological generalization of interactions includes only purely collective degrees of freedom, but not the influence of the single particle or quasiparticle movements.

The values of $E^i_{\rm exp}$ are taken from [19] and [20] and for $E^i_{\rm th}$ we take (17) with α a second order polynomial in the nuclear characteristics containing 16 coefficients, which are the same for all 271 nuclei, since the value of the function α depends on the quantum characteristics of each nucleus. The problem is solved by means of the minimization of the function:

$$\chi^2=\sum_{i=1}^{k}\frac{W^2(E^i_{\rm th}-E^i_{\rm exp})^2}{k-s}, \tag{18}$$

where $k=927$ is the number of the considered nuclear states, s is the number of the independent coefficients in the second order polynomial, and W is a diagonal weighting matrix of order k. This fit is carried out by means of an autoregularized iterational method of the Gauss-Newton type [21]. The minimization procedure is realized in two steps: (i) with a unit weighting matrix $W=\mathbf{E}$; (ii)with $W=(E^i_{\rm th}-E^i_{\rm exp})^{-1}$, where

TABLE XI. Parameters D_i.

i	D_i[MeV]	$\pm\Delta D_i$[MeV]
1	+0.0261526	0.0001740
2	−0.0009279	0.0000284
3	−0.0021273	0.0000232
4	−0.0323361	0.0001986
5	−0.0302709	0.0001770
6	−0.0000098	0.0000002
7	+0.0000059	0.0000001
8	+0.0009395	0.0000064
9	−0.000730	0.0000049
10	+0.0001599	0.0000013
11	−0.000153	0.0000011
12	+0.001724	0.0000107
13	−0.001318	0.0000064

$E^i_{\rm th}$ are calculated with the values of the parameters D_i, $i=1,2\ldots,13$, obtained in step (i). The second step (ii) consists of the application of the procedure called LCH [22]. This allows the estimation of the inherited errors of the parameters $D_i\rightarrow\pm\Delta D_i$. The final result is a simple formula for the energies of all the ground band states from the vast region of the even-even nuclei from five major nuclear shells, with

$$\begin{aligned}\alpha(A_p,A_n,N,F_0,\bar{N},\bar{F}_0)=&D_1+D_2N+D_3\bar{N}+D_4F_0+D_5\bar{F}_0\\&+D_6N^2+D_7\bar{N}^2+D_8F_0^2+D_9\bar{F}_0^2\\&+D_{10}NF_0+D_{11}\overline{NF}_0+D_{12}A_p\\&+D_{13}A_n.\end{aligned} \tag{19}$$

The values of the parameters $D_i, i=1,\ldots,13$ and their uncertainties $\pm\Delta D_i$, $i=1,\ldots,13$ are given in Table XI.

It should be mentioned that for the unknown values of $N_p^{(2)}$ and $N_n^{(2)}$ in the multiplets $(82,126|\ldots,\ldots)_\pm$ we had to use the values $N_p^{(2)}=126$ and $N_n^{(2)}=184$, predicted by the shell model. The total rms deviation $\sigma=\sqrt{\Sigma_{i=1}^k(E^i_{\rm th}-E^i_{\rm exp})^2/k}$ ($k=927$ is the number of the nuclear states) in the fit to all k states of 271 nuclei is 45 keV. The value of the statistical criterion $\chi^2=1.0001$ corresponds to the solution of the problem at the second step. Its closeness to unity shows that the obtained solution is a good one in a statistical sense also.

As a result of the used computational method the initial 16 parameters are reduced to 13. It is interesting to compare formula (19) for the dynamical coefficient of the general interaction for the collective yrast bands of the even-even nuclei with the respective formulas for the vibrational (6):

$$\begin{aligned}\alpha_1=&B_1+B_2(N+\bar{N})+B_3(F_0+\bar{F}_0)+B_4(4F_0^2-3\bar{F}_0^2)\\&+B_5\overline{NF}_0+B_6A_p+B_7A_n,\end{aligned} \tag{20}$$

rotational (7):

$$\alpha_2 = C_1 A_n + C_2(N+\bar{N}) + C_3(F_0 + 0.5\bar{F}_0) + C_4(N^2 + 0.5\bar{N}^2) + C_5(F_0^2 - \bar{F}_0^2), \quad (21)$$

and transitional nuclei (8):

$$\alpha_3 = K_1 + K_2(N+\bar{N}) + K_3(F_0+\bar{F}_0) + K_4(N^2 - 3\bar{N}^2) + K_5(F_0^2 - 0.7\bar{F}_0^2) + K_6(NF_0 - 2\overline{NF}_0) + K_7 A_p + K_8 A_n, \quad (22)$$

first considered independently. The values of the parameters B_i $(i=1,\ldots,7)$, C_i $(i=1,\ldots,5)$, and K_i $(i=1,\ldots,8)$ are given in [15] and [16].

Formula (19), which provides a unified description of the ground-band energies of all nuclei, involves thirteen independent coefficients in the dynamical parameter of the interaction, while the corresponding formula for the ground bands of rotational nuclei (21) involves only five coefficients, the one for vibrational nuclei (20)—seven [15] and in the transitional case (22) they are eight [16]. In all these subcases we have investigated very carefully the correlations between the obtained parameters. In the present very general consideration we work over a very large amount of data on a widely dispersed region of nuclei and in order to keep the accuracy we have left each nonzero parameter as independent. The greater number of parameters in this case may be attributed to this fact, which also made the above analysis of the obtained formula and its comparison to our previous results easier and more transparent.

In all cases (19), (20), (22), except the rotational (21), there is a free parameter, which for the vibrational energies is rather big and actually determines the high values of the ground-band energies in each shell.

The second degree terms of the numbers A_p and A_n, defining the shell, do not take part in either of them. In the rotational case (21) even the first degree of the characteristics of the proton shell A_p does not exist, because the well deformed nuclei belong to multiplets with valence neutrons in the higher nuclear shell.

The first order terms of the proton and neutron boson (hole) quantum numbers $N,(\bar{N})$ and the F-spin third projections $F_0,(\bar{F}_0)$ are obtained in the three particular cases (20), (21), (22) and in this general one (19). In the cases with defined collectivity (20), (21), and (22) the boson and hole total numbers have equal coefficients, thus indicating only the proton and neutron length of the shells $1/2[N_p^{(2)} - N_p^{(1)} + (N_n^{(2)} - N_n^{(1)})]$ and as a result giving the same contribution within each of them. The same is obtained for the term depending on $(F_0 + \bar{F}_0)$ in the transitional and vibrational cases, but with a dependence on the F spin of the shell: $F = 1/4[N_p^{(2)} - N_p^{(1)} - (N_n^{(2)} - N_n^{(1)})]$.

The analysis of the second order terms is of importance, because through them an $N_\pi N_\nu$ dependence of the energies can be obtained, whose importance was revealed by Casten [9]. A dependence on the square of the total number of bosons (boson holes) $N^2(\bar{N}^2)$ is obtained in all cases except the vibrational one. The square terms of the third projection of the F spin F_0^2 and $\bar{F}_0^2$ also take part in all separate cases and in this general one, although with very small parameters.

Mixed terms of the type NF_0 and $\overline{NF}_0$ do not appear in the description of the well deformed nuclei (21); the hole version of it is obtained in the nearly spherical ones (20) and both terms with differing parameters take part in the dynamical coefficients of the transitional (22) and general (19) cases. In these cases—vibrational, transitional, and general—these terms contribute to the correct description of the asymmetry of the transition from vibrational to rotational and from rotational to nearly vibrational nuclei through the different values of the coefficients in front of the boson and hole numbers, like the first order terms F_0 and $\bar{F}_0$, when their parameters differ.

We consider α as a fundamental nuclear characteristics because

through its evaluated dependence on the shell model quantum numbers of each nucleus the description of the ground state bands of all the even-even nuclei is unified and related to the microscopic characteristics of the nuclear system;

through the so-obtained unification the universal behavior of all collective types of nuclear spectra can be investigated.

Through an evaluation of $\epsilon_4 = 8\alpha$ by means of (19) we obtain the following average values of the constant of anharmonicity (12) [18] for the considered in this general case 271 nuclei from five major shells:

$(28,28|50,50)_\pm$ – $\epsilon_4 = 172 \pm 46$ keV for 38 nuclei;
$(28,50|50,82)_\pm$ – $\epsilon_4 = 156 \pm 47$ keV for 43 nuclei;
$(50,50|82,82)_\pm$ – $\epsilon_4 = 167 \pm 59$ keV for 49 nuclei;
$(50,82|82,126)_\pm$ – $\epsilon_4 = 101 \pm 36$ keV for 105 nuclei;
$(82,126|126,184)_\pm$ – $\epsilon_4 = 57 \pm 9$ keV for 36 nuclei.

These values are about 20% smaller than the ones obtained only for the transitional nuclei, although we have included here the rotational and vibrational nuclei. They are still in good agreement with the ones obtained by linear least square fits in [18]. We would like once again to underline that by applying formula (17) we exactly evaluate the constant of anharmonicity for each nucleus, depending on the 13 parameters D_i from Table XI and its shell characteristics.

From the analysis of the values of the parameters α it can be seen (Fig. 1) that though very close to each other they gradually increase with N in a given F_0 multiplet. In a fixed N multiplet the deviations of α for the different F_0 values are much smaller, or this is the case when α can be considered as constant.

It is important to observe that after the addition of the rotational and vibrational nuclei in the generalized consideration of the transitional ones, which is actually the present case, there is no dramatic change in the behavior of the constants of interaction α, for the different types of collectivity—rotational, vibrational, and transitional.

After the evaluation of α we improve our fit by further adjusting the values of the initial parameter ω_0 by means of (13) for $L=2$, as in the case of the transitional nuclei [16]. Without a deterioration of the accuracy of the fit we take for ω the nearest integer value to the one obtained by means of (13) with the already obtained values of α for each nucleus. We stick to the initial idea of ω integer because of the resemblance of this kind of interactions to the eigenvalues of the Casimir invariants of the algebras of some groups, as mentioned in the Introduction, which can be further investigated as groups of dynamical symmetry. The values of the

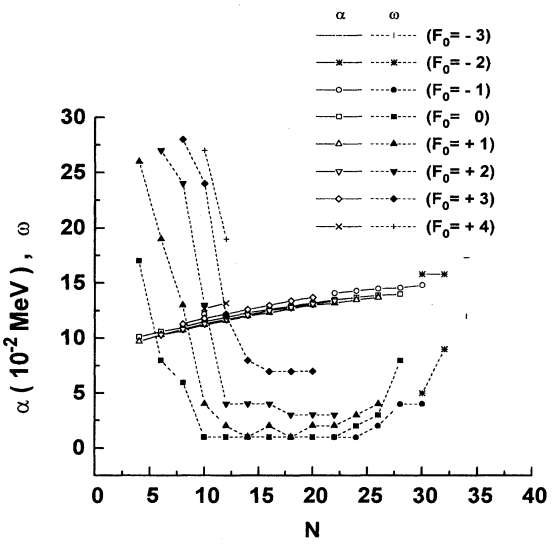

FIG. 1. Values of the dynamical coefficient α (solid lines) and the geometrical parameter ω (dashed lines) as functions of N at each fixed value of F_0 for the multiplet $(50,82|82,126)_+$.

parameter ω for each nucleus are given in parentheses next to their labels in Tables I–X.

It is interesting to investigate the changes in the values of ω in this general case in comparison with its values in the transitional case, when the rotational and vibrational nuclei were not considered. In the present case ω has its highest values for all the 35 nuclei described by means of (6) as vibrational. For most of the transitional nuclei ω is bigger than in the case when they were considered separately [16]. So the largest values of ω are obtained for the nearly spherical nuclei which were considered as vibrational [15] or the transitional ones near them. They are placed mainly at the upper borders of the multiplets and the diagonals next to them. These are the nuclei with smallest values of N and highest values of $\bar{N}$. This increase in the values of ω in the general case is the reason for the observed decrease of the mean values of α in each shell, but it should be underlined that for the nuclei which have preserved their values of ω in respect to their consideration in the separate cases, like most of the rotational nuclei and some of the transitional ones, we obtain the same value of α with formula (17) and (21) or (22), respectively.

In general in each of the F_0 – multiplets (the columns in the tables) ω starts with its highest value gradually decreases with the increase of N and then more smoothly increases again, but not to such high values as at the beginning (see Fig. 1). Clearly the values of ω follow the trend of the E_2 values [11] and reflect the asymmetry between the two halves of each F_0 multiplet.

In the lighter shells, where rotational nuclei are not observed, the multiplets are shorter and the changes in ω are steeper. In the heavier shells $(50,82|82,126)_\pm$ and $(82,126|126,182)_\pm$ most of the rotational nuclei are present. Now, when included in these general considerations, some of them in the rare-earth region are described better with a slightly bigger $\omega=2,3$. In [15] we have described as rotational 70 nuclei, now we obtain a good description with $\omega=1$ only for 49 of them.

In this shell, in the middle of the F_0 multiplets, series of nuclei with equal or close values of ω are observed. These are the so called regions of saturated collectivity ($\omega\sim3$) or deformation ($\omega=1$). The nuclei from these parts of the F_0 multiplets obviously have very similar yrast bands, which can be considered in some cases as identical. Depending on the values of ω these bands can be identical rotational ($\omega=1$) or with a "saturated" collectivity ($\omega=3,4$). Following the discovery of identical bands [23] in the region of superdeformations and their recognition among "normal" bands [24] we have located especially the most interesting newly discovered ones [25] in the symplectic multiplets in our classification scheme. The even-even nuclei exhibiting ground bands with very close transition energies belong mainly to the same F_0 multiplets and have equal or very close values of ω. See, for example, the sequence of nuclei ^{156}Dy ($\omega=4$), ^{160}Er ($\omega=4$), ^{164}Yb ($\omega=4$), ^{168}Hf ($\omega=3$), ^{172}W ($\omega=3$) and ^{180}Os ($\omega=3$) presented in [25]. The first five of them belong to the $F_0=2$ multiplet of the shell $(50,82|82,126)$ with $N=12,14,16,18,20$, respectively. The last one ^{180}Os is with $F_0=1$ and $N=24$ and can be replaced with the nucleus ^{176}Os ($\omega=3, F_0=2, N=22$), without a deterioration in the quantities qualifying the identical band. The comparison between the theoretically evaluated by

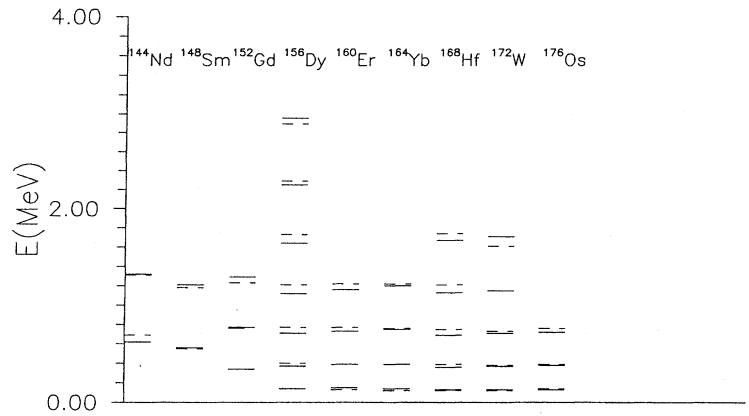

FIG. 2. Comparison of the theoretical E_{th} (solid lines) and experimental E_{exp} (dashed lines) energy values of the nuclei belonging to the $F_0=2$ column of the multiplet $(50,82|82,126)_+$.

means of (17) and the experimental energies for the $F_0=22$ column of the $(50,82|82,126)_+$ multiplet (Table VII) is presented on Fig. 2. The agreement of our results with the experimental data is very good for all the nuclei and it is very easy to observe the ones which form identical bands. As such can be considered all sequences of nuclei belonging to the same F_0 multiplet (column) in the Tables I–X with equal values of ω.

By means of these values of ω and the interaction coefficient α in (17) we obtain a good theoretical description for different number of low-lying states for each nucleus. The number of states is given next to the value of ω in Tables I–X. The highest values of L are obtained in the description of the rotational nuclei and the ones close to them in the last two shells.

As an effect of the generalization of the interaction by means of the parameter ω we were able to introduce in our considerations and to describe correctly 12 more nuclei than in the three separate cases together—vibrational (35), rotational (70), and transitional (159). They are placed mainly at the borders of the light multiplets and in this way we are able to enlarge the limits of the described nuclear region.

This proves once again that the basis of this consideration, namely, the symplectic classification scheme is very rich in the opportunities to investigate the existing general features as well as the singularities in the collective spectra of the even-even nuclei.

In conclusion, as a result of the symplectic classification scheme, a smooth and periodic behavior of the experimental energies of the ground-state bands of the even-even nuclei is observed. This permits the investigation of the collective interaction without including any details about the involved nuclear forces. But the classification numbers which are used suggest the importance of the nuclear shell characteristics in the description of collective phenomena. This leads to a possibility to unify the description of nuclear spectra by evaluating the dependence of the interaction coefficient on the shell quantum numbers, defined in (10). The results obtained, although very simple and easy for application, are general enough to cover vast nuclear regions or types of collectivity. Analyzing them an insight on the physics behind the nuclear structure can be reached. This approach can lead in future to establishing a connection between the phenomenological model parameters of the collective modes and their microscopic dependence on the shell model quantum numbers. A more fundamental understanding of the different types of collectivity is intended by the investigation of the appropriate DG's through the parameter ω in the geometrical part of the interaction.

The authors are grateful to Professor S. Pittel and Professor D. Karadjov for their permanent interest and useful discussion. This paper is supported in part by Contracts No. F415 and No. F310 with the Ministry of Science and Education, Bulgaria.

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