Towards a model independent analysis of single particle spectra: Application to hypernuclei

R. J. Lombard and J. Mareš*
Division de Physique Théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex, France
(Received 3 February 1995)

Considering one particle motion in a local potential, we derive a set of bounds relating moments of the ground state density to successive excitation energies. We check the validity of these bounds for potentials currently used in nuclear physics. The method is then applied to hypernuclear spectra. We investigate the possibility of determining the local potential experienced by the Λ particle inside the nucleus in a model independent way.

PACS number(s): 21.80.+a, 21.10.Pc, 21.60.−n

I. INTRODUCTION

Up to now the analysis of hypernuclear spectra has been performed in a model dependent way, by guessing or deriving the radial shape of the average potential and fitting parameters to the measured single particle energies. Although this approach is legitimate, giving at least a first insight into the physical situation, it is always desirable to establish or to complete the scheme with a model independent analysis. This has been the case, for instance, of the nuclear charge density, allowing then for valuable comparisons between models and experiments.

The situation is somewhat different for a discrete spectrum, in comparison to a scattering amplitude. A continuous variable, namely the momentum transfer, which can be measured up to large values, is replaced by a set of data usually considerably smaller and extending not so high above the ground state. The standard approach relies on the inverse problem method, which is mathematically involved [1]. Consequently, determining a potential from a bunch of bound levels could be a difficult task.

It is the purpose of the present work to investigate the possibility of extracting information from a single particle spectrum by using a method which is simpler than the standard inverse problem. The technique is based on bounds which are established from sum rules. It generalizes the approach of Bertlmann and Martin [2], linking the rms radius of the ground state orbital to the $1s$-$1p$ energy splitting, and which has already been applied to hypernuclei [3]. This method is specially well adapted to one-body problems, and the hypernuclei offer a unique possibility of application in the context of strong interactions.

A somewhat similar approach has been proposed long ago for the analysis of muonic x-ray energies [4,5], and later on applied to pionic atoms [6]. From transition energies, moments of the nuclear charge or matter density were extracted. The method was established empirically and shown to be roughly model independent [7]. The problem of mesic atoms is simpler than the one we are treating here, because it is dominated at large distances by the $1/r$ tail of the Coulomb potential.

It is important to recall that the derivation of the bounds presented here is only valid in the case of the Schrödinger equation with a local potential. This is an obvious lack of generality. From this point of view, it would be very interesting to have internal consistency relationships, telling us how good a local approximation is. If the nonlocality can be expressed in the form of an effective mass depending on the radial coordinate, the techniques used in Ref. [3] can be applied. However, we restrict our discussion to local potentials and postpone more detailed analysis of nonlocal effects to future investigation.

The paper is organized as follows. In Secs. II and III we derive the bounds in the one- and three-dimensional space, respectively, and we check their validity in a number of simple potentials. In Sec. IV, the technique is applied to hypernuclei. Conclusions are drawn in Sec. V.

II. BOUNDS ON $\langle x^n \rangle$ IN THE ONE-DIMENSIONAL CASE

The model independent method we are proposing is based on a set of bounds for the ground state average values of $\langle 0|x^n|0\rangle$. First, we sketch the derivation of the bounds and test their degree of confidence, i.e., how far they are saturated in the case of some typical potentials. Although limited in its applications, the one-dimensional case is useful to establish and illustrate the method. In this respect, the Pöschl-Teller potential [8] provides us with an interesting possibility of analytic tests.

We start from the Schrödinger equation for a particle of mass m in a local potential $V(x)$:

$$H\Psi_j = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi_j = E_j\Psi_j \quad , \qquad (1)$$

*Permanent address: Nuclear Physics Institute, 250 68 Řež, Czech Republic.

0556-2813/95/52(1)/170(7)/$06.00

where j labels the single-particle states, and the basic commutation relations:

$$[H, x^n] = -\frac{ni\hbar}{m}x^{n-1}p - \frac{n(n-1)}{2m}\hbar^2 x^{n-2} \quad . \tag{2}$$

From (2), it is easy to obtain the set of sum rules corresponding to x^n:

$$\sum_j (E_j - E_0)\langle 0|x^n|j\rangle^2 = \frac{n^2\hbar^2}{2m}\langle 0|x^{2n-2}|0\rangle \ . \tag{3}$$

Here, $|0\rangle$ and $|j\rangle$ denote the ground and jth excited state, respectively. For $n = 1$, we get the usual Thomas-Reiche-Kuhn sum rule.

In order to get inequalities we can apply to (3) the procedure used by Bertlmann and Martin (BM) [2]. It consists in minoring the sum by retaining only the lowest transition energy E_j. In the dipole case ($n = 1$), we get

$$(E_1 - E_0)\sum_j \langle 0|x|j\rangle^2 \ \leq \ \frac{\hbar^2}{2m} \quad , \tag{4}$$

and using the closure rule to sum over j gives the first BM inequality:

$$\langle x^2\rangle \ \leq \ \frac{\hbar^2}{2m(E_1 - E_0)} \quad . \tag{5}$$

For $n = 2$, the technique is similar, but we have to take care of the fact that the matrix element $\langle 0|x^2|0\rangle$ is not zero, and therefore should be substracted. To be explicit, consider

$$\sum_{j=0}^{\infty}(E_j - E_0)\langle 0|x^2|j\rangle^2$$

$$= \sum_{j\geq 2}(E_j - E_0)\langle 0|x^2|j\rangle^2 + (E_1 - E_0)\langle 0|x^2|1\rangle^2$$

$$+(E_0 - E_0)\langle 0|x^2|0\rangle^2 \ . \tag{6}$$

The last two terms on the right-hand side do not contribute, the matrix element or the energy difference being zero. However, in order to use the closure while factorizing $(E_2 - E_0)$, we shall add and substract $(E_2 - E_0)\langle 0|x^2|0\rangle^2$ to the whole sum. Consequently we are left with

$$(E_2 - E_0)(\langle x^4\rangle - \langle x^2\rangle^2) \ \leq \ \frac{2\hbar^2}{m}\langle x^2\rangle \quad . \tag{7}$$

Here, $\langle x^n\rangle$ stands for the ground state average value of x^n. In principle we can proceed further and obtain the whole set of inequalities relating $\langle x^{2n}\rangle$ to $\langle x^{2n-2}\rangle$ in a similar way. However, with $n \geq 3$, the quantities to be substracted are transition matrix elements, and thus the great advantage of the two first cases ($n = 1, 2$) of dealing only with ground state expectation values is lost. As we shall see, the situation is different in three dimensions due to orthogonality of the wave functions with respect to the angular momentum. In one dimension, this could be simulated by considering polynomial operators $O_n(x)$ rather than x^n, in such a way that the lowest nonzero matrix element of $O_n(x)$ connects the ground to the nth excited state. However this procedure leads to complicated relations, and we discard this possibility in the present work.

In general, inequalities are not so useful, unless they are close to being equalities. Indeed, it is easy to check that in the case of the harmonic oscillator both bounds (5) and (7) are saturated, i.e. are strict equalities.

The infinite square well potential, with particle in the interval $[-\frac{a}{2}, \frac{a}{2}]$, is providing us with a second trivial example. From

$$\langle x^2\rangle = \frac{(2\pi^2 - 12)}{24}\left(\frac{a}{\pi}\right)^2 \ ,$$

$$\langle x^4\rangle \ = \left[\frac{1}{5}\left(\frac{\pi}{2}\right)^4 - \left(\frac{\pi}{2}\right)^2 + \frac{3}{2}\right]\left(\frac{a}{\pi}\right)^4 \ , \tag{8}$$

and

$$E_2 - E_0 = \frac{4\hbar^2\pi^2}{ma^2}$$

it is easy to verify that inequality (7) yields $0.1463 \ \leq \ 0.1612$. In other words it is saturated within 10 %.

As a third example, we chose the modified Pöschl-Teller potential, a case we have used in a previous work devoted to the BM inequalities [3]. We recall that this potential is defined by [8]

$$V(x) = -\frac{\hbar^2}{2m}\alpha^2\frac{\lambda(\lambda-1)}{\cosh^2(\alpha x)} \quad . \tag{9}$$

The two parameters to be fixed are the strength λ and the length scale α. The energy eigenvalues are determined by

$$E_j = -\frac{\hbar^2\alpha^2}{2m}(\lambda - 1 - j)^2 \quad . \tag{10}$$

The ground state averages of x^2 and x^4 are given by

$$\langle x^2\rangle_\lambda = \frac{1}{2\alpha^2(\lambda-2)}\left[2(\lambda-2)\alpha^2\langle x^2\rangle_{\lambda-1} - \frac{1}{(\lambda-2)}\right] \tag{11}$$

and

$$\langle x^4\rangle_\lambda = \frac{1}{2\alpha^2(\lambda-2)}\left[2(\lambda-2)\alpha^2\langle x^4\rangle_{\lambda-1} - \frac{6}{(\lambda-2)}\langle x^2\rangle_{\lambda-1}\right] \ , \tag{12}$$

respectively.

The last two equations have to be taken for $\lambda > 2$. The index j is the order of the level: $j = 0, 1, 2$, etc. For $\lambda = 2$, we have

$$\langle x^2\rangle_{\lambda=2} = \frac{\pi^2}{12\alpha^2} \quad \text{and} \quad \langle x^4\rangle_{\lambda=2} = \frac{7\pi^4}{240\alpha^4} \quad .$$

TABLE I. Test of the inequality (7) in the case of the Pöschl-Teller potential for few values of λ.

λ	$\langle x^4\rangle - \langle x^2\rangle^2$	$\frac{2}{\alpha^2(4\lambda-8)}\langle x^2\rangle$
3	0.2695	0.3225
4	0.0926	0.098
5	0.0454	0.0476
6	0.0266	0.0278

Taking into account

$$E_2 - E_0 = \frac{\hbar^2\alpha^2}{2m}(4\lambda - 8) \ ,$$

the inequality (7) becomes

$$\langle x^4\rangle - \langle x^2\rangle^2 \ \leq \ \frac{2}{\alpha^2(4\lambda-8)}\langle x^2\rangle \ . \qquad (13)$$

Note that the inequality is independent of α, as it can be checked easily. It is also independent of the mass of the particle. The comparison of the two sides of the inequality (13) is displayed for a few values of λ in Table I. Being off by about 20% for $\lambda = 3$, the discrepancy is still of the order of 4% for $\lambda = 6$. Even if the difference continues to diminish at a rate of about 0.5% as λ increases by 1 unit, the inequality is not expected to reach the 1% level before $\lambda \simeq 15$. Thus the situation is not as favorable as for inequality (5). In this case, the 1% level is reached already with $\lambda = 5$.

III. BOUNDS ON $\langle r^n\rangle$ IN THE THREE-DIMENSIONAL SPACE

As stated before, the three-dimensional case allows us to extend further the method sketched in the one-dimensional case and to derive a series of inequalities for the moments of the ground state density $\langle r^n\rangle$. This is due to the orthogonality of states with different angular momentum. Note that this is strictly true under the assumption of spherical symmetry. In the case of a nonspherical potential, corrections are expected to play a role and to become increasingly important as the departure from sphericity is increasing. In the present work we restrict the discussion to the spherical case.

In order to derive the inequalities, we start from the well-known sum rule for the multipole operator $Q_{\nu,0} = r^\nu Y_{\nu,0}$, namely

$$\sum_j (E_j - E_0)|\langle 0|Q_{\nu,0}|j\rangle|^2 = \frac{\hbar^2}{2m}\nu(2\nu+1)\langle r^{2\nu-2}\rangle \ . \qquad (14)$$

(This sum rule has been derived in many papers. Here we refer to lecture notes given by Fallieros at Orsay [9].) By using the same method as before, the sum rule is bound from below by

$$(E_1 - E_0)\sum_j |\langle 0|Q_{\nu,0}|j\rangle|^2 \leq \sum_j (E_j - E_0)|\langle 0|Q_{\nu,0}|j\rangle|^2 \ .$$

Noticing that

$$\langle 0|Q^*_{\nu,0}Q_{\nu,0}|0\rangle = \langle r^{2\nu}\rangle \ = \langle r^n\rangle \ ,$$

we easily get

$$\langle r^{2\nu}\rangle \ \leq \ \frac{\hbar^2}{2m}\nu(2\nu+1)\frac{\langle r^{2\nu-2}\rangle}{(E_1(\nu) - E_0)} \ , \qquad (15)$$

or equivalently

$$\frac{2m}{\hbar^2}(E_1(\nu) - E_0)\frac{\langle r^{2\nu}\rangle}{\nu(2\nu+1)\langle r^{2\nu-2}\rangle} \ \leq \ 1 \ . \qquad (16)$$

Here, $E_1(\nu)$ is the energy of the lowest state reachable with the operator $Q_{\nu,0}$ from the ground state. Because of the orthogonality of the single-particle wave function with respect to the angular momentum, states with energy lower than $E_1(\nu)$ do not contribute to $\sum_j \langle 0|Q^*_{\nu,0}|j\rangle\langle j|Q_{\nu,0}|0\rangle$. Thus, these inequalities have a simpler form than in the one-dimensional case.

The monopole transition operator, r^2, does not belong to the class of $r^\nu Y_{\nu,0}$. Thus, it allows us to establish an independent inequality from

$$\sum_j (E_j - E_0)|\langle 0|r^2|j\rangle|^2 \ = \ \frac{2\hbar^2}{m}\langle r^2\rangle \ .$$

Note, however, that in this case, a correction occurs, which is similar to the one encountered in one dimension, due to the fact that the operator r^2 has a nonzero average value in the ground state. In this case, as it can be verified, the inequality reads

$$\langle r^4\rangle \ \leq \ \frac{2\hbar^2}{m}\frac{\langle r^2\rangle}{(E_{2s} - E_0)} + \langle r^2\rangle^2 \ . \qquad (17)$$

Contrary to the one-dimensional case, in three dimensions we have to face corrections arising from the spin-orbit splitting. This problem was already discussed in our previous work [3] dealing with the TRK sum rule. The same argument applies here: since we are considering sum rules which are linear in energy, $E_1(\nu)$ should be replaced by the average

$$E_1(\nu) = \frac{2j+1}{2(2\ell+1)}E_{j=\ell+1/2} \ + \ \frac{2j+1}{2(2\ell+1)}E_{j=\ell-1/2} \ .$$

Since the spin-orbit effects are negligible in Λ hypernuclei, we shall not care about this average and pursue our reasoning as if the two spin-orbit partners were degenerated.

As stated in the preceeding section, the key point is to check how far the derived inequalities can be used to

extract information on the moments of the ground state density. This has to be investigated by considering a few typical potentials. Whereas the method is not general, it nevertheless fixes limits of confidence.

As a first example, it is easy to show that the inequalities (15)–(17) are saturated for the harmonic oscillator potential, i.e. they become equalities in this particular case.

The $-1/r$ potential constitutes another interesting analytical case. A simple relationship exists, for this potential, among the moments of the ground state density, namely

$$\langle r^n \rangle = \frac{(n+1)(n+2)}{4} a_0^2 \langle r^{n-2} \rangle \ , \tag{18}$$

where a_0 is related to the decay of the wave function and to the radius ($\langle r^2 \rangle = 3a_0^2$). The energy difference is given by

$$E_1(\nu) - E(0) = \frac{\hbar^2}{2ma_0^2} \frac{n(n+4)}{(n+2)^2} \ , \quad \nu = n/2 \ . \tag{19}$$

The inequality (15) takes the following form:

$$\frac{(n+1)(n+2)}{4} \leq \frac{n}{2}(n+1)\frac{(n+2)^2}{n(n+4)} \ , \tag{20}$$

$$1 \leq \frac{2(n+2)}{(n+4)} \ .$$

Consequently we see that (15) is never close to an equality. It is already off by 33% for $n = 2$.

At this point, it is opportune to remark that in the analysis of mesic atoms [4–7], it was found useful to introduce the equivalent radius $R(n)$ instead of the moment $M(n)$. The former is defined by

$$R(n) \ = \ M(n)^{1/n} \ = \ \langle r^n \rangle^{1/n} \ . \tag{21}$$

By taking the nth root on both sides of (15) and (17), we get automatically bounds for $R(n)$. The equivalent radius does not contain more information than the moment; however, its spreading is much smaller, in particular as n becomes large. This can be shown easily by expanding $M(n)$ and $R(n)$, respectively, around some average value. As a consequence, the inequalities for $R(n)$ are much better saturated than for $M(n)$. Taking the nth root on both sides of (20) shows that even for the $1/r$ potential, the inequality for $R(n)$ is saturated as $n \to \infty$, the worst case being around $n = 4$. This result underlines the specificity of the moment $\langle r^n \rangle$ at large n in discriminating among various wave functions.

The next question to ask is to which extent the inequalities (15)–(17) are saturated for potentials typical for nuclear or hypernuclear physics. The tests are achieved numerically. To this aim we consider a four parameter Woods-Saxon potential:

$$V(r) = \left[1 + b\frac{r^2}{R^2}\right] V(0) \left[1 + \exp\frac{r-R}{a}\right]^{-1} \ . \tag{22}$$

The presence of a positive or negative hump allows us to investigate the inequalities under slightly different geometrical conditions. The parameters have been fixed to the following values in model WSI (WSII, WSIII): $V(0) = -28.82$ MeV ($-27.6, -29.8$), $r_0 = 1.057$ fm (0.98, 1.18), $a = 0.4$ fm (0.4, 0.4), $b = 0.0$ ($+0.2, -0.2$). The radius parameter R from (22) is defined as $R = r_0 A^{1/3}$, where $A = 209$. For comparison, we used also the harmonic oscillator (HO) potential $V(r) = V(0) + \frac{1}{2}m\omega^2 r^2$, with $V(0) = -30.75$ MeV, and $\hbar\omega = 3.9$ MeV. The above parametrizations describe more or less a Λ hyperon embedded in Pb. The nucleus is large enough to allow bound states up to $l = 5$, which means the $1h$ state in the spectroscopic notation. The bounds on $R(n)$ can thus be tested up to $n = 10$. The results are displayed in Table II. For the three Woods-Saxon potentials the bounds are saturated within 2%. We have verified on a number of similar examples that these results are typical for a wide range of potentials. It fixes the confidence degree of the method within a couple of percent.

Note that the few analytical cases we have worked out in one and three dimensions show that the inequalities are independent of the mass of the particle. In the numerical examples, however, the mass has to be specified in order to solve the Schrödinger equation. In view of application to hypernuclear spectra, it has been fixed to the free Λ mass, namely 1115.6 MeV.

The monopole inequality (17) provides us with a second constraint on $\langle r^4 \rangle$, and consequently a consistency check. To illustrate the situation, the two inequalities for $R(4)$ obtained from (15) and (17), respectively, are displayed in Table III, for the three potentials quoted above.

The results of Table II show that the inequalities for $R(n)$ are saturated within 1–3 % even up to $n = 10$. This is fixing the confidence level to which they can be used as equalities. The two bounds for $R(4)$, on the other hand, agree to better than 1% (see Table III).

We shall end up this section by a remark concerning nonlocal effects. The present work is exploratory in nature, therefore we keep the difficulty at its lowest level and consider local potential. On the other hand, if the

TABLE II. Test of the inequalities (16) for the equivalent radius $R(n)$ divided by the sum rule ($\frac{\langle r^n \rangle^{1/n}}{\text{sum rule}}$) is listed up to $n = 10$. The three potentials WSI, WSII, and WSIII correspond to those given in the text. Harmonic oscillator values are recalled for comparison.

	HO	WSI	WSII	WSIII
$n = 2$	1.000	0.991	0.985	0.996
$n = 4$	1.000	0.988	0.982	0.994
$n = 6$	1.000	0.985	0.979	0.992
$n = 8$	1.000	0.984	0.979	0.990
$n = 10$	1.000	0.984	0.977	0.990

TABLE III. Comparison of the equivalent radius of order 4 $\langle r^4\rangle^{1/4}$ obtained from the quadrupole sum rule (15, ν=2) and from the value deduced from the monopole sum rule (17). The potentials are the same as in Table II.

$\langle r^4\rangle^{1/4}$	HO	WSI	WSII	WSIII
Quadrupole sum rule	4.163	4.039	3.997	4.103
Monopole sum rule	4.163	4.005	3.939	4.088

nonlocality can be put under the form of an effective mass depending on the radial coordinate, the problem can be tackled without tears, as was shown for the TRK sum rule in a preceeding work [3]. The hint is to recall that for Hamiltonians of the type

$$H = \frac{p^2}{2m(r)} + V(r) ,$$

the commutation relation with the multipole operator $Q_{\nu,0}$ reads

$$[H, Q_{\nu,0}] = \frac{1}{2m(r)}[p^2, Q_{\nu,0}] .$$

In such a case, the right-hand side of the inequality (15) undergoes the following transformation

$$\frac{1}{m}\langle r^{2\nu-2}\rangle \rightarrow \left\langle \frac{r^{2\nu-2}}{m(r)} \right\rangle .$$

To the extent that

$$\left\langle \frac{r^{2\nu-2}}{m(r)} \right\rangle \simeq \frac{\langle r^{2\nu-2}\rangle}{\langle m(r)\rangle} ,$$

is an acceptable approximation, we conclude that the ratio $R(n+2)/R(n)$ is nearly independent of nonlocal effects (within the class of nonlocality considered here). Obviously such a conjecture needs to be confirmed by a careful numerical analysis, which we postpone to future work.

IV. APPLICATION TO HYPERNUCLEAR SPECTRA

Hypernuclei constitute an almost ideal case to apply the technique presented in the preceeding sections and to try a model independent analysis of the single particle spectra. Indeed, the hypernucleus is essentially a one-body problem, due to the absence of Pauli correlations between the hyperon and the nucleons, small nonlocality effects and almost negligible spin-orbit interaction [10]. Consequently, in this section, we shall investigate how the inequalities we have derived are working in practice. Two hypernuclei will be considered for which experimental data exist: ${}^{89}_{\Lambda}$Y and ${}^{209}_{\Lambda}$Pb. In the latter case, the spectrum has been measured recently at KEK [11], however, the results of the analysis are still to be published [12]. Thus we take the predictions of the scalar derivative coupling model, relying on its good fit to known hypernuclear spectra [13] to expect reasonable extrapolated values for ${}^{209}_{\Lambda}$Pb. We shall refer to these values as experimental data for simplicity.

Equivalent radii $R(n)$ obtained from the inequalities can be first compared to values calculated from a few simple models for the ground state density. Assuming spherical symmetry, three typical forms are used. They are listed below together with their expression for $\langle r^n\rangle^{1/n}$.

(a) The homogenous sphere with sharp boundary,

$$\rho_0(r) = \frac{3}{4\pi R_0^3}\Theta(R_0 - r) \quad , \quad \langle r^n\rangle^{1/n} = R_0\left(\frac{3}{n+3}\right)^{1/n} . \tag{23}$$

(b) The Gaussian,

$$\rho_0(r) = \frac{\beta^3}{\pi\sqrt{\pi}} e^{-\beta^2 r^2} ,$$
$$\langle r^n\rangle^{1/n} = \frac{[3\cdot 5\cdot 7 ... (n+1)]^{1/n}}{\sqrt{2}\beta} , (n \text{ even!}) . \tag{24}$$

(c) The sum of two exponentials,

$$\rho_0(r) = \frac{2\alpha^3}{15\pi}\left(e^{-\alpha r} - \frac{1}{2}e^{-2\alpha r}\right) ,$$
$$\langle r^n\rangle^{1/n} = \left[\frac{8}{15}(n+2)!\right]^{1/n}\frac{1}{\alpha} . \tag{25}$$

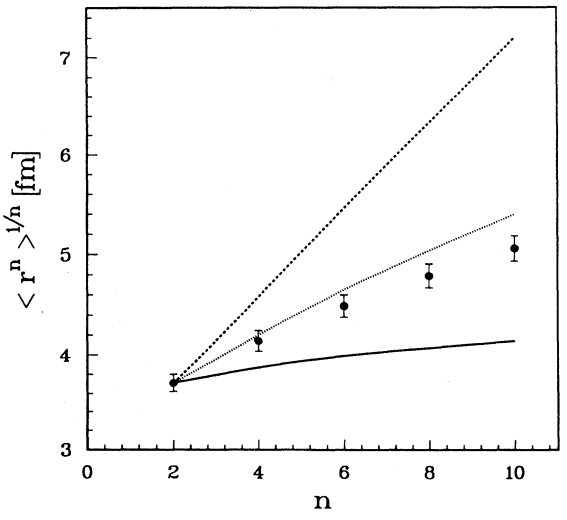

FIG. 1. The equivalent radii $R(n)$ obtained from the ${}^{209}_{\Lambda}$Pb spectrum (crosses) are compared to three simple models of the ground state density of the Λ. The solid line is the homogenous sphere with sharp boundary (22). The dotted line is the Gaussian (23) and the dashed line is a sum of two exponentials (24). An experimental error of 2.5% is plotted for the sake of comparison.

Taking $^{209}_{\Lambda}$Pb as an illustrative example, we fix the rms radius to 3.7 fm, according to the values obtained from the BM inequality applied to the Λ in the $1s$ state. It determines in each case the single free parameter, R_0, β, and α, respectively. The calculated values of $\langle r^n \rangle^{1/n}$ are plotted against n in Fig. 1, and compared to experiments. The quoted error of 2.5% is essentially reflecting the uncertainty in the inequalities, and is displayed in order to judge the uniqueness of the solution. It is quite clear from Fig. 1 that the two extreme ground state densities are excluded by experimental data.

The three ground state densities we have used in Fig. 1 have simply been adjusted to the rms radius, without any reference to the measured energy levels. Ignoring the spectrum is justified as long as one can deal with a sufficiently large functional basis for the trial functional space, together with a sufficient amount of measured moments. This would constitute an ideal situation leading to a properly model independent determination of the ground state density. In practice, however, the analysis cannot be achieved in this way, the number of values to be fitted being too small. Consequently, it is more efficient to take into account constraints brought by the spectrum itself. This is done by choosing an ensemble of potentials which fit the spectrum, in particular the $1s$-$1p$ splitting, and by comparing their predictions for the moments with the estimates obtained from inequalities.

The procedure is obviously not unique and could be biased by peculiar choices. On the other hand, one can wonder if fitting the spectrum is not sufficient by itself to fix the potential. Actually, apart from the very few lower levels, the spectrum is never fitted with such a great accuracy. It leaves some undetermination which can be completed by the bounds on moments. Moreover, testing the moments of the ground state density yields a rather direct way of appreciating the geometrical shape of the potential and its pertinence. Thus, even if an excellent fit is achieved, the bounds on moments are checking the uniqueness of the potential.

For illustrative purpose, both cases of the $^{89}_{\Lambda}$Y and $^{209}_{\Lambda}$Pb spectra have been analyzed. We used Woods-Saxon (22) as well as harmonic oscillator potentials. For

TABLE IV. Λ single particle spectra for $^{89}_{\Lambda}$Y and $^{209}_{\Lambda}$Pb.

		$1s$	$1p$	$1d$	$1f$	$1g$	$1h$
	HO	-22.7	-16.0	-9.3	-2.6		
	WSI	-22.7	-16.0	-8.4	-0.3		
$^{89}_{\Lambda}$Y	WSII	-22.7	-16.05	-8.0	+0.6		
	WSIII	-22.7	-16.0	-8.5	-0.8		
	ZM_6	-22.7	-16.6	-9.9	-2.9		
	exp.	-22.7	-16.	-9.9	-2.9		
	HO	-24.9	-21.0	-17.1	-13.2	-9.3	-5.4
	WSI	-24.9	-21.0	-16.2	-10.6	-4.5	+2.1
$^{208}_{\Lambda}$Pb	WSII	-24.9	-21.0	-15.9	-9.9	-3.2	+3.9
	WSIII	-24.9	-21.0	-16.6	-11.6	-6.2	-0.4
	ZM_6	-24.9	-21.0	-16.7	-11.7	-6.4	-0.8

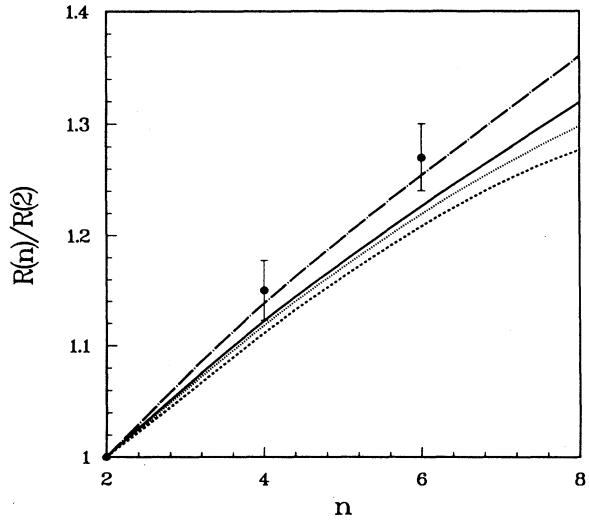

FIG. 2. The equivalent radii, relative to the rms radius, obtained from the $^{89}_{\Lambda}$Y spectrum (crosses) compared to the values calculated by using potentials fitting the lowest energy levels. The dotted, dashed, and solid lines correspond to the potential WSI, WSII, and WSIII, respectively. The dot-dashed line is the harmonic oscillator case. An experimental error of 2.5% has been plotted for the sake of comparison.

$^{89}_{\Lambda}$Y, the parameters of WSI(WSII,WSIII) were the following: $V(0) = -29.95$ MeV $(-28.4, -30.98)$, $r_0 = 1.037$ fm (0.968, 1.1095), $a = 0.5$ fm (0.4, 0.5), $b = 0.0$ $(+0.2, -0.2)$; for HO: $V(0)$=-32.75 MeV, $\hbar\omega$=6.7 MeV. For $^{209}_{\Lambda}$Pb the parameters were those presented already in the previous section. The spectra are listed in Table IV, and the equivalent radii $R(n)$ are displayed in Figs. 2 and 3. Here, $\langle r^n \rangle^{1/n}$ relative to $\langle r^2 \rangle^{1/2}$ are plotted against n. The continuous curves show the results

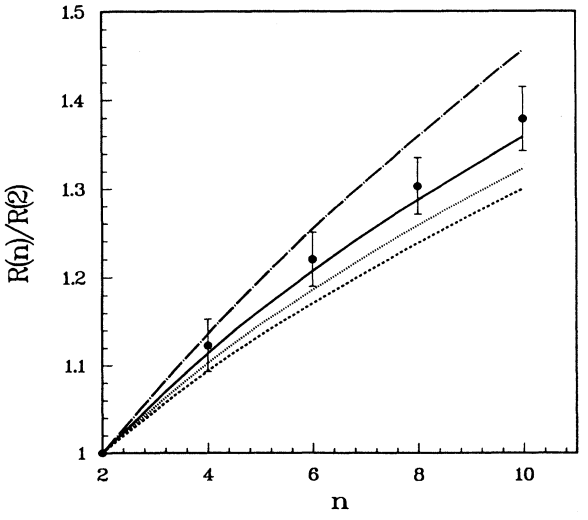

FIG. 3. Same legend as Fig 2 for the case of $^{209}_{\Lambda}$Pb.

obtained from the fitted potentials, whereas the discrete values are obtained from the experimental energies. The free mass of the Λ was used, which is consistent with our assumption of locality.

As in the preceeding case of Fig. 1, we have attributed a 2.5% error to the moments deduced from experiment, which merely accounts for the uncertainty due to the nonsaturation of the inequalites. Although it does not give the actual error bars, it indicates the limits of the analysis assuming accurate measurements of the energy levels, which is not reached at present (the quoted errors on experimental levels is of the order of 1–2 MeV). The main salient feature is the lack of clear discrimination between the harmonic oscillator and the Woods-Saxon shapes, specially for ${}^{89}_{\Lambda}$Y. In this respect the highest moments are of key importance, as shown by the last value obtained in ${}^{209}_{\Lambda}$Pb. It means that besides the radius and the strength of the potential, the energy spectrum is not terribly sensitive to details of the radial shapes.

V. CONCLUSIONS

In the present work, following the approach of Bertlmann and Martin [2], we have derived a set of inequalities for moments of the ground state density in one and three dimensions. In the three-dimensional case, they appear as recurrence relations among the even moments $\langle r^n \rangle$, n even. The derivation is restricted to local Schrödinger equations.

The bounds obtained for the moments $M(n)$ are readily extended to the equivalent radii $R(n)$, defined as the nth roots of $M(n)$.

These bounds become equalities in the case of the harmonic oscillator potential. We have verified on some examples that the inequalities for $R(n)$ are saturated within a few percent for potentials currently used in nuclear physics. Thus, in principle, the bounds can be used to detemine the ground state density (or the ground state wave function) from the measured energy levels. The potential is then obtained by inverting the Schrödinger equation. This method provides an alternative to the standard inverse method [1], simpler mathematically but of course less general.

In practice, however, the very finite number of moments available from experiment limits the possibility of the method. This is what we have learned from the application to hypernuclear spectra. In ${}^{89}_{\Lambda}$Y, with $n \leq 6$, it is not possible to disentangle between a harmonic oscillator or a Woods-Saxon shape for the potential. Actually it is surprising that the experimental results favor the harmonic oscillator potential instead of the Woods-Saxon potential, as expected from the shape of nuclear densities. The situation is somewhat different in ${}^{209}_{\Lambda}$Pb, with $n \leq 8$ and a Woods-Saxon shape prefered by the experimental moments. These two examples show that the method appears merely complementary to the usual fit to energy levels. It basically yields a possibility of testing the uniqueness of the potential in a model independent way.

Division de Physique Théorique is Unité de Recherche des Universités Paris 11 et Paris 6 Associée au CNRS.

[1] L. Kay and H.E. Moses, J. Appl. Phys. **27**, 1503 (1956); H.B. Thacker, C. Quigg, and J.L. Rosner, Phys. Rev. D **18**, 274 (1978); **18**, 278 (1978).
[2] R.A. Bertlmann and A. Martin, Nucl. Phys. **B168**, 111 (1980); **182**, 35 (1981).
[3] R.J. Lombard, S. Marcos, and J. Mares, Phys. Rev. C **50**, 2900 (1994).
[4] K.W. Ford and J.G. Wills, Phys. Rev. **185**, 1429 (1969).
[5] R.C. Barrett and D.F. Jackson, *Nuclear Sizes and Structure* (Clarendon, Oxford, 1977).
[6] F. Scheck, *Leptons, Hadrons and Nuclei* (North-Holland, Amsterdam, 1983).
[7] Y.N. Kim, Sheldon Wald, and Asok Ray, in *Radial Shape of Nuclei,* Second Nuclear Physics Divisional Conference of the European Physical Society, Cracow, 1976, edited by A. Budzanowski and A. Kapuscik (Jagellonian University, IPN, Cracow, 1976).
[8] See S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1974), p. 94.
[9] S. Fallieros, "Electromagnetic interactions and nuclear excitations," Lecture notes, Orsay, 1975 (unpublished).
[10] D.J. Millener, C.B. Dover, and A. Gal, Phys. Rev. C **38** 2700 (1988).
[11] O. Hashimoto, "The Λ Hypernuclear Spectroscopy with the SKS Spectrometer at KEK 12 GeV PS," Invitational talk at the International Conference on Hypernuclear and Strange Particle Physics, Vancouver, 1994 (unpublished).
[12] T. Motoba, private communication.
[13] R. J. Lombard, S. Marcos, and J. Mareš, Phys. Rev. C **50**, 2900 (1994).