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Analytic interpretation of universal anharmonic vibrator behavior in the interacting boson approximation model

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The recently discovered nearly universal anharmonic vibrator behavior (with *constant* anharmonicity) of nuclei with $E(4_1^+)/E(2_1^+)$ between 2.0 and 3.15 has been shown to be a natural and nearly automatic outcome of numerical interacting boson approximation (IBA) calculations. Here, we present an approximate analytic derivation and discussion of this based on the idea of the Q phonon.

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Recently, it was shown [1,2] that a nearly universal empirical behavior characterizes nuclei between the vibrator and rotor limits. Specifically, for all nuclei from $Z=38-82$ with

$$2.05 \leqslant R_{4/2} \equiv E(4_1^+)/E(2_1^+) \leqslant 3.15, \tag{1}$$

$E(4_1^+)$ is empirically linear in $E(2_1^+)$, with a slope of 2.0. Such behavior is described by the equation

$$E(4_1^+) = 2.0E(2_1^+) + \varepsilon_4, \tag{2}$$

where ε_4 is the (constant) intercept. This equation is that of an anharmonic vibrator (AHV) and can be generalized for higher spin yrast states as

$$E(n) = nE(2_1^+) + \frac{n(n-1)}{2}\varepsilon_4 + \frac{n(n-1)(n-2)}{6}\varepsilon_6, \tag{3}$$

where $n = I/2$ is viewed as the phonon number and where ε_6 is another parameter. Such an expression has been discussed, for example, by Das, Dreizler, and Klein [3]. For $I \gtrsim 8$, it is sometimes useful to include the ε_6 term. ε_6 is found to be much smaller than ε_4.

The remarkable feature is that the anharmonicity, ε_4, is *constant* for nuclei of such varying underlying structure. The AHV behavior is also reflected [2] in yrast $B(E2)$ values as well, raising again the question of whether the success of the AHV equations actually reflects an underlying phonon structure of nearly all collective nonrotational nuclei.

An equally surprising theoretical result has also recently been obtained [2]. The interacting boson approximation (IBA) model [4] naturally reproduces the linear AHV behavior of both yrast energies and $B(E2)$ values, for a *very wide range* of Hamiltonian parameters.

We show in Fig. 1 (reproduced from Ref. [2]) the empirical and IBA results for ε_4 to highlight both the remarkable linearity of the data (i.e., the adequacy of an AHV interpretation with constant anharmonicity) and the excellent reproduction of this behavior with the IBA. The only significant constraint on the IBA calculations of this AHV behavior is that, with the IBA Hamiltonian

$$H = \varepsilon n_d - \kappa Q \cdot Q, \tag{4}$$

where

$$Q = (s^\dagger d + d^\dagger s) + \chi (d^\dagger d)^{(2)}, \tag{5}$$

the calculated $R_{4/2}$ value must be less than 3.15 (i.e., nonrotational nuclei) and the quantity

$$\varepsilon/4\kappa N > \frac{1}{2}\left[1 - \frac{3}{N} + \frac{\chi^2}{2N}\right], \tag{6}$$

where N is the boson number. Hence $\varepsilon/4\kappa N > 0.1$ to 0.4. A practical upper limit on $\varepsilon/4\kappa N$ is ~ 2.

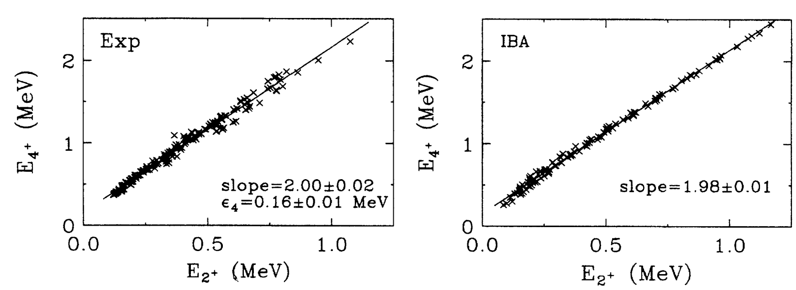

FIG. 1. Correlation of $E(4_1^+)$ with $E(2_1^+)$. Left, empirical values for all collective, nonrotational even-even nuclei (i.e., $2.05 \leqslant R_{4/2} \leqslant 3.15$) for $Z=38-82$. Right, IBA calculations for a broad range of parameters that give this same range of $R_{4/2}$ values. Based on Ref. [2].

This theoretical result is so striking both in the quality of reproduction of the data and in the breadth of parameters that accommodate such a linear behavior that it suggests a simple origin reflecting basic features of the model. It is therefore interesting, and the purpose of this Rapid Communication, to show that the numerical IBA results can indeed be derived in a simple and analytic way by using the concept of global multi-quadrupole-phonon structure of the low-lying even-spin yrast states [5,6]. With this idea, both the wave functions and the Hamiltonian are known, and it is therefore possible to calculate the energies in an approximate way. This derivation therefore explicitly supports the phonon origins of the empirically universal correlations of Ref. [1].

To approach this problem, we consider an (yrast or stretched) state (with spin $I=2n$) in the Q-phonon basis [5]

$$|I,M=I\rangle \equiv |n\rangle = \mathcal{N}_n^{-1/2} Q_{22}^n |\text{g.s.}\rangle, \tag{7}$$

where $\mathcal{N}_n^{-1/2}$ is a normalization factor. For simplicity, we drop the subscripts on Q_{22} in the following. We consider an IBA Hamiltonian H and evaluate $H|n\rangle \equiv \mathcal{N}_n^{-1/2} H Q^n |\text{g.s.}\rangle$. The simple relation

$$HQ = QH + [H,Q] \tag{8}$$

is an example of a general theorem that

$$HQ^n = Q^n H + nQ^{n-1}[H,Q] + \frac{n(n-1)}{2} Q^{n-2}[[H,Q],Q] + \frac{n(n-1)(n-2)}{3!} Q^{n-3}[[[H,Q],Q],Q] + \cdots. \tag{9}$$

For example, applying Eq. (8),

$$HQ^2 = HQQ = (QH + [H,Q])Q = Q^2 H + 2Q[H,Q] + [[H,Q],Q], \tag{10}$$

which is Eq. (9) for $n=2$. Equation (10) also shows explicitly that Eq. (9) terminates after $n+1$ terms. In our case, this means that for the 4_1^+ state ($n=2$) Eq. (9) resembles the form of Eq. (3) with terms up through ε_4.[Of course, one has not yet shown that ε_4 in Eq. (3) is constant. That, in fact, is one of the aims of this paper.]

Applying the relation Eq. (9) to the $|\text{g.s.}\rangle$ (which we assign energy $E=0$) gives

$$H|n\rangle \equiv \mathcal{N}_n^{-1/2} H Q^n |\text{g.s.}\rangle = \mathcal{N}_n^{-1/2}\Big(nQ^{n-1}[H,Q]|\text{g.s.}\rangle + \frac{n(n-1)}{2} Q^{n-2}[[H,Q],Q]|\text{g.s.}\rangle + \frac{n(n-1)(n-2)}{3!} Q^{n-3}[[[H,Q],Q],Q]|\text{g.s.}\rangle + \cdots\Big). \tag{11}$$

Inserting the unit operator $\Sigma_{I,\alpha}|I_\alpha\rangle\langle I_\alpha| \equiv 1$, where α specifies additional quantum numbers, yields

$$H|n\rangle = \mathcal{N}_n^{-1/2}\Big(nQ^{n-1}\sum_\alpha |2_\alpha\rangle\langle 2_\alpha|[H,Q]|\text{g.s.}\rangle + \frac{n(n-1)}{2} Q^{n-2}\sum_\alpha |4_\alpha\rangle\langle 4_\alpha|[[H,Q],Q]|\text{g.s.}\rangle + \frac{n(n-1)(n-2)}{3!} Q^{n-3}\sum_\alpha |6_\alpha\rangle\langle 6_\alpha|[[[H,Q],Q]Q]|\text{g.s.}\rangle + \cdots\Big). \tag{12}$$

It has recently been shown [6] that the Q-phonon basis is an excellent approximation (at the 95% level) for even spin yrast wave functions both empirically and in the IBA. In effect, this means that

$$B(E2:0_1^+ \to 2_1^+) \gg \sum_{i>1} B(E2:0_1^+ \to 2_i^+). \tag{13}$$

Therefore, in the Q-phonon approximation, the dominant (by far) contributions to Eq. (12) are from the yrast states, so the Σ_α give contributions from one state only, namely that given by Eq. (7). Thus,

$$H|n\rangle = \mathcal{N}_n^{-1/2}\Big(nQ^{n-1}\mathcal{N}_1^{-1/2}Q|\text{g.s.}\rangle\langle 2_1^+|[H,Q]|\text{g.s.}\rangle + \frac{n(n-1)}{2} Q^{n-2}\mathcal{N}_2^{-1/2}Q^2|\text{g.s.}\rangle\langle 4_1^+|[[H,Q],Q]|\text{g.s.}\rangle + \frac{n(n-1)(n-2)}{3!} Q^{n-3}\mathcal{N}_3^{-1/2}Q^3|\text{g.s.}\rangle\langle 6_1^+|[[[H,Q],Q]Q]|\text{g.s.}\rangle + \cdots\Big). \tag{14}$$

Using Eq. (7) to replace the $\mathcal{N}_n^{-1/2}$ factors, we get

$$H|n\rangle = \left[n\frac{\langle 2_1^+|[H,Q]|\text{g.s.}\rangle}{\langle 2_1^+|Q|\text{g.s.}\rangle} + \frac{n(n-1)}{2}\frac{\langle 4_1^+|[[H,Q],Q]|\text{g.s.}\rangle}{\langle 4_1^+|QQ|\text{g.s.}\rangle} + \frac{n(n-1)(n-2)}{3!}\frac{\langle 6_1^+|[[[H,Q],Q]Q]|\text{g.s.}\rangle}{\langle 6_1^+|QQQ|\text{g.s.}\rangle} + \cdots\right]|n\rangle \tag{15}$$

or, finally, replacing $H|n\rangle$ by $E(n)|n\rangle$ and using Eq. (8)

$$E(n)=nE(2_1^+)+\frac{n(n-1)}{2}\frac{\langle 4_1^+|[[H,Q],Q]|\text{g.s.}\rangle}{\langle 4_1^+|QQ|\text{g.s.}\rangle}+\frac{n(n-1)(n-2)}{3!}\frac{\langle 6_1^+|[[[H,Q],Q]Q]|\text{g.s.}\rangle}{\langle 6_1^+|QQQ|\text{g.s.}\rangle}+\cdots. \quad (16)$$

This result, which arises, in essence, because of the goodness of the Q-phonon approximation for the ground band, is identical in form to Eq. (3): it reflects the phonon content of the yrast excitations. We note that the validity of this expression does not require any special type of phonon or phonon structure. It is quite generic and applies, for example, even to good rotor nuclei in which, certainly, Eq. (13) is well satisfied.

For simplicity, and since we are primarily interested in the behavior of $E(4_1^+)$, we calculate only the first two terms of Eq. (16). The treatment of the higher terms is analogous though more complicated. The next step is therefore to show that the second term in Eq. (16) does not vary significantly for broad ranges of IBA parameters. If so, then the empirical result, Eq. (2), with constant ε_4, is obtained. In order to evaluate the second term in Eq. (16), we need to specify an IBA Hamiltonian. Consistent with Ref. [2], we use Eq. (4). With Eq. (4), we obtain, after some straightforward algebra,

$$\begin{aligned}\langle 4_1^+|[[H,Q],Q]|\text{g.s.}\rangle=&2\varepsilon\langle 4_1^+|(d^\dagger\tilde{d})_{44}|\text{g.s.}\rangle+\kappa[4\sqrt{11/5}(1-\tfrac{4}{7}\chi^2)^2\langle 4_1^+|[(d^\dagger\tilde{d})_3(d^\dagger\tilde{d})_3]_{44}|\text{g.s.}\rangle+2(1+\tfrac{2}{7}\chi^2)\langle 4_1^+|QQ|\text{g.s.}\rangle\\&-\tfrac{3}{7}\chi(1-\tfrac{4}{7}\chi^2)\langle 4_1^+|Q(d^\dagger\tilde{d})_{22}+(d^\dagger\tilde{d})_{22}Q|\text{g.s.}\rangle+(\chi\sqrt{110}/7)(1-\tfrac{4}{7}\chi^2)\langle 4_1^+|(Q(d^\dagger\tilde{d})_4)_{44}\\&+((d^\dagger\tilde{d})_4Q)_{44}|\text{g.s.}\rangle]. \end{aligned}\quad (17)$$

To evaluate this expression approximately we require a technique which is applicable for the wide range of the Hamiltonian parameters. A suitable approach is the $1/N$ expansion developed for the IBM in [7]. In this formalism the states belonging to the ground band are given by the angular momentum projection of the intrinsic state [8] which is taken as a boson condensate

$$|\text{intrinsic g.s.}\rangle=\frac{1}{\sqrt{N!}}(b_{\text{g.s.}}^\dagger)^N|0\rangle, \quad (18)$$

where $b_{\text{g.s.}}^\dagger=x_0s^\dagger+x_2d_0^\dagger$, $|0\rangle$ is the boson vacuum, and $x_0^2+x_2^2=1$. This procedure has been shown to lead to a $1/N$ expansion for all matrix elements [7]. For many cases the results obtained in the leading order approximation are sufficient. Since we are interested here not in the exact matrix elements but in their range of variations, we calculate the matrix elements in Eq. (17) in leading order approximation in N.

The straightforward calculation of the matrix elements such as these in Eq. (17) produces the following expression for ε_4:

$$\begin{aligned}\varepsilon_4\equiv&\frac{\langle 4_1^+|[[H,Q],Q]|\text{g.s.}\rangle}{\langle 4_1^+|QQ|\text{g.s.}\rangle}\\=&2\kappa\left\{1+\frac{2}{7}\chi^2+\frac{1}{\sqrt{14}}\frac{|\chi|(1-\frac{4}{7}\chi^2)}{M}x_2+\frac{\varepsilon}{4\kappa N}\frac{1}{M^2}\right\},\end{aligned}\quad (19)$$

where

$$M=\sqrt{1-x_2^2}+\frac{|\chi|}{\sqrt{14}}x_2=x_0+\frac{|\chi|}{\sqrt{14}}x_2. \quad (20)$$

For any set of ε, κ, and χ, the parameters of the creation operator $b_{\text{g.s.}}^\dagger$, x_0 and x_2, can be found by minimizing the expectation value of H in the intrinsic state. We will present numerical results below, but first it is useful to analyze Eq. (19) approximately for $\varepsilon/4\kappa N$ values less than unity. First, we note that x_2 can be approximated by

$$x_2\cong\frac{1}{\sqrt{2}}\left[1-\frac{\varepsilon}{4\kappa N}\right]^{1/2},\quad 0\leqslant\frac{\varepsilon}{4\kappa N}<1,\quad |\chi|<\sqrt{7}/2. \quad (21)$$

This approximation is quite good for a wide range of Hamiltonian parameters. The overall dependence of solutions to the IBA Hamiltonian in Eq. (4) on the ratio of ε to κN is, in fact, a familiar property of the intrinsic state formalism [8].

From Eq. (21), we note that $x_2\to 0$ as $\varepsilon/4\kappa N\to 1$ and approaches 0.7 near SU(3). Hence x_0 varies *only* from 1.0 to ~0.7. Thus M does not vary significantly from unity. We therefore momentarily set $M=1$, obtaining from Eq. (19)

$$\frac{\varepsilon_4}{2\kappa}\cong 1+\frac{2}{7}\chi^2+\frac{1}{\sqrt{14}}|\chi|\left(1-\frac{4}{7}\chi^2\right)x_2+\frac{\varepsilon}{4\kappa N}. \quad (22)$$

The third term in Eq. (22) is always small. We therefore have approximately

$$\frac{\varepsilon_4}{2\kappa}\cong 1+\frac{2}{7}\chi^2+\frac{\varepsilon}{4\kappa N}. \quad (23)$$

For $0.2\leqslant\varepsilon/4\kappa N<1$ and $|\chi|<\sqrt{7}/2$, this gives a range of values

$$\frac{\varepsilon_4}{2\kappa}=1.85\pm 0.65. \quad (24)$$

Moreover, inserting the value of κ for IBA calculations that reproduce the empirical AHV results, namely $\kappa=0.032$ MeV, we get

$$\varepsilon_4\cong 0.12\pm 0.04\ \text{MeV}. \quad (25)$$

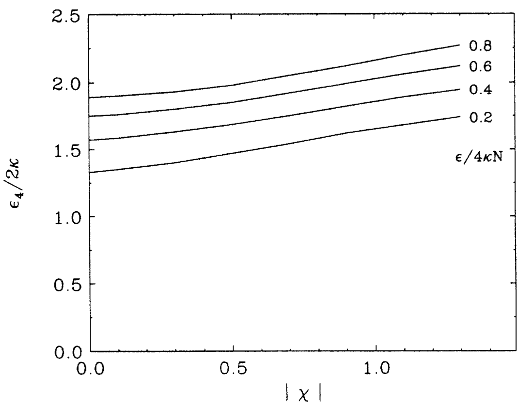

FIG. 2. Values for $\varepsilon_4/2\kappa$ as a function of χ for several values of $\varepsilon/4\kappa N<1$.

This approximate analysis therefore gives results close to the actual ε_4 values of 0.16 ± 0.05 MeV from the IBA calculations of Ref. [1].

It is useful, however, to show the actual numbers, and this is done in Fig. 2 for a range of values of $\varepsilon/4\kappa N$ and χ where it is seen that the results do in fact fall in a relatively narrow range.

To summarize, the data in Fig. 1 [complemented by those for higher-spin yrast states and for $B(E2)$ values [2]] suggest an anharmonic vibrator description, with nearly constant anharmonicity, for a large variety of nuclei which differ substantially in the energy spectra of their low-lying collective states. Numerical IBA calculations reproduce this behavior for the boson pairing plus quadrupole Hamiltonian of Eq. (4). Here, we have shown that this behavior can be approximately derived analytically and that the key ingredient is the fact that the IBA predicts [5,6] nearly pure Q-phonon purity over essentially the entire symmetry triangle. (The Q-phonon concept, for the nonrotational yrast states we are considering here, is essentially equivalent to the normal phonon picture.) Given the Q-phonon purity embedded in the IBA, the nearly constant anharmonicity of an AHV description in the IBA follows for a wide range of parameters. These results therefore suggest that the empirical success of the constant-ε_4 AHV expressions [e.g., Eq. (3)] is not accidental but indeed reflects the actual applicability of a phonon and multiphonon description of the yrast states of nearly all collective, nonrotational nuclei.

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