Effect of relativistic kinematics on the quark-quark interaction obtained from the proton form factor

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We have recently shown, assuming nonrelativistic kinematics, that the parametrized elastic charge form factor data for an N-particle system may be used to determine the underlying two-particle interaction. We now extend the validity of our formalism, using the prescription of Mitra and Kumari, to include relativistic kinematics, and apply it to a constituent quark model of the proton. We show analytically that, for a dipole fit to the form factor, the essential features of the quark-quark potential do not depend on the form of the kinematics. This provides an explanation of our earlier somewhat paradoxical finding that the small r singularity in the potential, which is predicted by first order QCD, is reproduced by a totally nonrelativistic inversion of the form factor data.

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Traditionally, the two-body potential assumed to govern the dynamics of a given many-body system has been fitted from two-body scattering and/or bound state data [1,2] mediated by theoretical considerations. Alternately, the interaction can be derived by inverting the scattering or spectral data [3]. However, for larger systems, for example the p-shell nuclei, an effective interaction is often obtained by a least squares fit to the binding energies of several similar systems [4,5]. Recently [6], we derived a new method by means of which the underlying two-body interaction governing the dynamics of particles in an N-body system can be deduced from the elastic electric form factor of that system. We applied this method to various parametrized fits (e.g., dipole and Gari-Krümpelmann [7]) to the form factor data of the proton, so obtaining quark-quark interactions consistent with these data in a consituent particle model.

Our method, which is described in detail in [6], assumes that the N-body wave function is well approximated by the first term in a hyperspherical harmonic (HH) expansion. This is known as the hypercentral approximation (HCA). The assumption that the leading term in the HH expansion of the wave function is much larger than any higher term implies the dominance of collective (monopole) dynamics over more complicated correlations. The presence of a hard core in the potential tends to generate higher correlations, whereas they are more or less absent in systems subject to interactions which do not exhibit short range repulsion.

In constituent quark models, the potential is generally assumed to consist of an attractive Coulomb-like term (predicted by first order QCD) and a linear confining term (consistent with a constant energy density in the flux tube connecting two interacting quarks) [2,8]. The three-quark system examined in the nonrelativistic quark potential model (NRQPM) would therefore appear to be ideally suited as an application of our form factor inversion formalism. However, relativistic corrections appear even in leading order in Q^2 [9], which places some doubt on the interpretation of results obtained by our method. Nevertheless, we were able to derive interactions whose features can be easily understood in terms of a constituent quark + meson cloud model of the nucleon. First, the potential so obtained is attractive and singular at small r, as predicted by QCD. This result is rather surprising, since first order QCD is applicable in the highly relativistic regime, whereas our original method is entirely nonrelativistic. Second, in a quark + meson cloud model, we would expect the imprint of the confining nature of the quark-quark interaction on the form factor to be washed out by the presence of the mesons, which are unconfined. Our results indeed display this feature.

For a full analysis of the form factor data, even within a constituent quark model, two types of relativistic correction need to be taken into account. First, the recoil kinematics of the electron-target system needs to be analyzed relativistically. In the current work we have used the transformation of Mitra and Kumari (MK) [10] to obtain the rest frame form factor $G_E^{(\mathrm{RF})}(Q^2)$ from the measured form factor $G_E(Q^2)$. The MK prescription, which is a modification of the earlier treatment of Licht and Pagnamenta (LP) [11], is derived by considering the Lorentz transformation of the coordinates of the wave function before Fourier transforming to obtain the form factor. The difference between the MK and LP approaches is that, in the former, the initial and final states are treated symmetrically. As a result, the MK prescription correctly predicts the $Q^{2(1-N)}$ asymptotic behavior of the form factor of an N-particle system, whereas the LP approach predicts a $Q^{(1-N)}$ falloff at high Q^2. Recently, both MK and LP approaches were used to investigate whether the discrepancy regarding the relation between the charge radius and the triplet scattering length a_t of the deuteron could be resolved by using relativistic kinematics [12]. Second, the quark dynamics needs to be made relativistic by using the proper kinetic energy operator and by treating the interaction covariantly. Although we are currently working on the problem of establishing a formalism for handling relativistic dynamics within the hypercentral approximation, the work is incomplete and we will therefore not present it in this paper.

In our original approach, we exploited the fact that the charge density $\rho(y)$ for an N-body system can be expressed in terms of the hyperradial component $\psi(r)$ of the wave function according to

$$\rho(y) \sim \int_{f_N y}^{\infty} |\psi(r)|^2 r (r^2 - f_N^2 y^2)^{\frac{D-5}{2}}\, dr, \tag{1}$$

where $f_N = \sqrt{\frac{2N}{N-1}}$ and $D = 3(N-1)$. (See Appendix C in [6].) For $N = 3$, Eq. (1) reduces to

$$\rho(y) \sim \int_{\sqrt{3}y}^{\infty} |\psi(r)|^2 r \sqrt{r^2 - 3y^2}\, dr. \tag{2}$$

Equation (2) can be expressed in the form

$$-\frac{1}{y}\frac{d\rho(\frac{y}{\sqrt{3}})}{dy} \sim \int_{y}^{\infty} \frac{r\,|\psi(r)|^2}{\sqrt{r^2 - y^2}}\, dr, \tag{3}$$

which is known as the Weyl fractional integral, and which has the solution [13]

$$|\psi(r)|^2 \sim \frac{1}{r}\frac{d}{dr}\int_{r}^{\infty} \frac{d\rho(\frac{y}{\sqrt{3}})}{dy}\frac{dy}{\sqrt{y^2 - r^2}}. \tag{4}$$

Alternately, the fact that the quantities $r^{\frac{D-2}{2}}|\psi(f_N r)|^2$ and $Q^{\frac{D-2}{2}} G_E^{(\mathrm{RF})}(Q^2)$ are related to each other by means of the Hankel transform of order $\frac{D-2}{2}$ [14], i.e.,

$$Q^{\frac{D-2}{2}} G_E^{(\mathrm{RF})}(Q^2) \sim \int_0^{\infty} r^{\frac{D-2}{2}} |\psi(f_N r)|^2 J_{\frac{D-2}{2}}(Qr) r\, dr \tag{5}$$

and

$$r^{\frac{D-2}{2}} |\psi(f_N r)|^2 \sim \int_0^{\infty} Q^{\frac{D-2}{2}} G_E^{(\mathrm{RF})}(Q^2) J_{\frac{D-2}{2}}(Qr) Q\, dQ, \tag{6}$$

can also be exploited to obtain $\psi(r)$ from $G_E^{(\mathrm{RF})}(Q^2)$.

Assuming the validity of the HCA, the hypercentral potential is then given by

$$V_0(r) = \frac{\hbar^2}{3m_q}\left[\frac{u''(r)}{u(r)} - \frac{15}{4r^2}\right] + \frac{E}{3}, \tag{7}$$

where the proton mass M_P is given by $M_P = 3m_q + E$ and where the reduced wave function is defined by

$$u(r) = r^{\frac{5}{2}}\psi(r). \tag{8}$$

The hypercentral potential is the first term in the expansion of the potential in terms of hyperspherical harmonics. It is defined as

$$V_0(r) = \frac{\int |Y_{[L]}(\Omega)|^2 V(r_{12})\, d\Omega}{\int |Y_{[L]}(\Omega)|^2\, d\Omega}, \tag{9}$$

where $Y_{[L]}(\Omega)$ is the lowest order hyperspherical harmonic basis function consistent with the Pauli principle, Ω is a set of angles and hyperangles, $V(r_{12})$ is the two-body potential, and r_{12} the relative coordinate between particles 1 and 2. In the case of three particles in an S state, Eq. (9) reduces to

$$V_0(r) = \frac{2}{\pi}\int_{-1}^{+1} V\left(r\sqrt{\frac{1+z}{2}}\right)(1 - z^2)^{1/2}\, dz. \tag{10}$$

As described in Ref. [15], the two-body interaction V can be obtained from the hypercentral potential V_0 for three bodies by using an Abel transform:

$$V(y) = \frac{1}{4y}\left[\int_0^{y^2} \frac{d^2}{dx^2}\left[x^2 V_0(\sqrt{x})\right]\frac{dx}{\sqrt{y^2 - x}} + \frac{1}{y}\lim_{x\to 0}\frac{d}{dx}\left[x^2 V_0(\sqrt{x})\right]\right]. \tag{11}$$

The MK application of the Lorentz transformation to the wave function leads to the following relationship between the measured and the rest frame form factors:

$$G_E(Q^2) = \left(1 + \frac{Q^2}{4M_P^2}\right)^{1-N} G_E^{(\mathrm{RF})}\left(\frac{Q^2}{1 + \frac{Q^2}{4M_P^2}}\right) \tag{12}$$

for an N-particle system. This relationship has a unique inverse, namely,

$$G_E^{(\mathrm{RF})}(k^2) = \left(1 - \frac{k^2}{4M_P^2}\right)^{1-N} G_E\left(\frac{k^2}{1 - \frac{k^2}{4M_P^2}}\right). \tag{13}$$

If we apply this inversion procedure to the dipole fit of the proton form factor data, i.e.,

$$G_E(Q^2) = \left(\frac{1}{1 + \frac{Q^2}{\mu^2}}\right)^2, \tag{14}$$

where $\mu^2 = 0.71$ GeV^2, we obtain

$$G_E^{(\mathrm{RF})}(k^2) = \left(\frac{1}{1 + \frac{k^2}{\mu'^2}}\right)^2, \tag{15}$$

where

$$\frac{1}{\mu'^2} = \frac{1}{\mu^2} - \frac{1}{4M_P^2}. \tag{16}$$

A mathematical comment is appropriate at this point. From Eqs. (12) and (13) it would at first appear that

$G_E^{(\mathrm{RF})}(k^2)$ is defined up to a finite value of k^2 only, namely, $k^2 = 4M_P^2$. However, it turns out that, for the dipole parametrization, the singularity in the argument of G_E in (13) is exactly canceled by the zero in the prefactor. More generally, if we assume that $G_E \to Q^{2-2N}$ as we approach the singularity from the left, then this cancellation also occurs.

Since the charge distribution associated with the form factor (15) is

$$\rho(y) = \rho_0 e^{-\mu' y}, \tag{17}$$

it is clear that the effect of using MK relativistic kinematics is simply to reduce the range of the charge distribution with respect to that obtained purely nonrelativistically. Obviously, this rather striking analytical result holds for a dipole form factor only. However, the dipole fit is so good (see, for example, Höhler *et al.* [16] for a comparison) that parametrizations with even better χ^2 fits to the data are unlikely to produce rest frame charge distributions which deviate much from it. The mean square radius of the dipole rest frame charge distribution is

$$-6\frac{\partial G_E^{(\mathrm{RF})}(k^2)}{\partial k^2}\bigg|_{k^2=0} = 12\left(\frac{1}{\mu^2} - \frac{1}{4M_P^2}\right). \tag{18}$$

This yields a value for the rms radius of 0.72 fm, as opposed to 0.81 fm for a purely nonrelativistic calculation. The Darwin-Foldy term therefore contributes an additional 0.09 fm to the proton radius. Since NRQPM's predict a much smaller proton size (~0.5 fm) for the mass spectrum to be consistent with experiment [8,17], we still need to invoke the presence of mesons or attempt to apply a relativistic dynamical theory to resolve the discrepancy between measured and calculated charge radii in a constituent particle model.

Applying Eq. (4) to the exponential charge density resulting from the dipole form factor, we obtain the following expression for the hyperradial component of the wave function (see [6]):

$$\psi(r) \sim \sqrt{\frac{K_1(\frac{\mu' r}{\sqrt{3}})}{r}}, \tag{19}$$

where $K_1(\frac{\mu' r}{\sqrt{3}})$ is a modified Bessel function.

The two-body potentials corresponding to both relativistic and nonrelativistic kinematics for the dipole form factor have been calculated numerically using (11), and the results are shown in Fig. 1. The singularity at $r = 0$, as well as the very flat behavior at large r, is evident for both forms of kinematics. However, it is instructive to examine the large r and the small r behavior of the potential analytically too. Using (7), we can show (see [6])

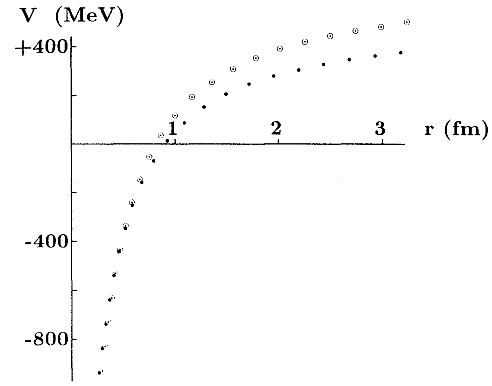

FIG. 1. Quark-quark potentials derived from the dipole form factor for nonrelativistic (•) and relativistic (⊙) kinetics.

that the long range behavior of $V_0(r)$ is given by

$$V_0(r) \to V_{\text{constant}} - \frac{\hbar^2}{3m_q}\left(\frac{7\sqrt{3}\mu'}{12r} + \frac{39}{16r^2}\right), \tag{20}$$

whereas at short distances we have

$$V_0(r) \to V_{\text{constant}} + \frac{\hbar^2}{3m_q}\left(-\frac{3}{r^2} + \frac{2}{3}\mu'^2 \ln \mu' r\right). \tag{21}$$

Using (11), we find that the two-body quark-quark interaction has the following form at short distances:

$$V_{q_i q_j}(r_{ij}) \to \frac{\hbar^2}{3m_q}\left(-\frac{3}{4r_{ij}^2} + \frac{2}{3}\mu'^2 \ln \mu' r_{ij}\right) + V_{\text{constant}}. \tag{22}$$

Equation (20) shows that the nonconfining nature of the potential derived from the proton form factor in a constituent quark model is not altered by the use of relativistic kinematics, since no confinement occurs whatever the value of μ'. In Appendix B in [6], where only nonrelativistic kinematics was applied, we showed that the leading term in Eq. (21) originates entirely in the Q^{-4} behavior of $G_E(Q^2)$. However, we have now shown that this term is independent of the form of kinematics. This independence therefore provides an explanation of why the singularity in the two-body potential at $r = 0$, which is predicted by first order QCD, survives the use of nonrelativistic kinematics, despite the fact that such kinematics are surely inappropriate at small r. It remains to be seen whether the use of a relativistic dynamics results in the correct order of the singularity, namely, $1/r$ rather than $1/r^2$.

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