Fine structure in the cluster decays of the translead nuclei

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Within the one level R-matrix approach several hindrance factors for the radioactive decays in which are emitted ^{4}He, ^{14}C, and ^{20}O atomic nuclei are calculated. The interior wave functions are supposed to be given by the recently proposed enlarged superfluid model, an extension of the JINR-Dubna quasiparticle phonon nuclear model. The spectroscopic factors are expanded in terms of products of cluster overlaps and intrinsic overlap integrals. The cluster overlaps are equivalents of the generalized coefficients of fractional parentage, while for the intrinsic overlap integrals we construct a model which is an extension of the usual models for simple particle decay such as deuteron, triton, and α decay. The exterior wave functions are calculated from a cluster-nucleus double-folding model potential obtained with the $M3Y$ interaction. As examples of the cluster decay fine structure we analyzed the particular cases of α decay of ^{255}Fm, ^{14}C decay of ^{223}Ra, and ^{20}O decay of ^{229}Th and ^{255}Fm. A relatively good agreement with the experimental data is obtained especially in the case of the α-decay fine structure.

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I. INTRODUCTION

The spontaneous emission of nuclear fragments heavier than α particles and lighter than the most probable fission fragments, termed exotic or cluster decays, has now become an experimentally confirmed reality (see Refs. [1–8] and references therein). Moreover, Hourani and his co-workers [9] experimentally discovered the fine structure in ^{14}C radioactivity, opening in this way a new area of research. All nuclei with $Z > 40$ are unstable with respect to radioactive decay into two nuclear fragments (A_1, Z_1 and A_2, Z_2 with $A = A_1 + A_2$ and $Z = Z_1 + Z_2$), i.e., the energy release $Q = M(A, Z) - M_1(A_1, Z_1) - M_2(A_2, Z_2)$ is positive; however, only for certain combinations, (A_1, Z_1) plus (A_2, Z_2), is the high value of the potential barrier (proportional to $Z_1 Z_2$) almost compensated by a high value of Q, and these decay modes may be detectable. By using a triple γ coincidence technique in the spontaneous fission of ^{252}Cf recently [6], neutronless fragmentations, such as ^{104}Mo + ^{148}Ba, ^{106}Mo + ^{146}Ba, and ^{104}Zr + ^{148}Ce, have been experimentally observed for the first time. In this way it was experimentally proved that the spontaneous decay with emission of light fragments, such as α, ^{14}C, ^{20}O, ^{23}F, ^{24}Ne, ^{28}Mg, and ^{32,34}Si (cluster radioactivity), and the neutronless spontaneous fission defined as a process where all the available energy goes into the total kinetic energy of the fragments (cold fission) may have an analogous decay mechanism. Also, for the first time [6], a double fine structure, i.e., decay to the excited states of both fragments of the final channel, was experimentally observed in analogy with the usual fine structure, i.e., decays only to the excited states of the daughter nuclei already known in α decay [10] and ^{14}C decay [9, 7].

The interest in studying these new decay modes lies in the quantitative estimations of the lifetimes and branching ratios on one hand and in the construction of models regarding nuclear clustering, nucleus-nucleus potentials, the nature of spontaneous fission processes, etc., on the other hand. In the present paper we touch both the above aims of study concerning the fine structure of cluster decay. First, we construct a mechanism of clustering by using the enlarged superfluid model (ESM) [11, 12] for the nuclear states, which is an extension of the JINR-Dubna quasiparticle phonon nuclear model (QPNM) [13]. The spectroscopic factors are expanded [14] in terms of products of cluster overlaps and intrinsic overlap integrals. The cluster overlaps are equivalents of the generalized coefficients of fractional parentage [15, 16], while for the intrinsic overlap integrals we construct a model which is an extension of the usual models for light particle decay such as deuteron, triton, and α decay [17]. Secondly, we calculate several hindrance factors for the ^{4}He, ^{14}C, and ^{20}O radioactivity of some translead nuclei and compare them with the experimental data and some previous calculations. In the case of α decay a relatively good agreement with the experiment is obtained.

II. HINDRANCE FACTORS AND RELATIVE INTENSITIES

A. Hindrance factors

The hindrance factor (HF) is defined [18] as the ratio of the actual half life for a given cluster transition characterized by the energy release Q to the half life obtained from the Geiger-Nuttal [19] law at the same energy Q.

Such a definition of the hindrance factor determines a quantity which is almost independent of energy Q. This quantity measures the attenuation of the decay probability due to the mechanism of cluster formation and not to the barrier penetration. The microscopic description of preformation has a key role in the understanding of the decay process and requires a precise knowledge of the initial and final quantum states. The HF plays the same role as the $\ln ft$ plays in β decay.

When we are dealing with a cluster transition from the initial state $(I_iK_i\pi_i)$ of a heavy deformed nucleus with axial symmetry to the ground or excited states $(I_1K_1\pi_1)$ and $(I_2K_2\pi_2)$ of the residual axial deformed nuclei, the corresponding theoretical hindrance factor has the following expression [7, 20]:

$$\mathrm{HF}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)} = \left[\sum_l F_l \mid d_{l\ K=|K_1+K_2-K_i|}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)} \mid^2\right]^{-1} . \quad (1)$$

When any of the residual nuclei is a spherical nucleus, the K quantum number disappears from the sets $IK\pi$ of quantum numbers that identify the nuclear state (see, e.g., the case of the α decay),

$$d_{l\ K=|K_1+K_2-K_i|}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)} = \frac{\gamma_{l\ K=|K_1+K_2-K_i|}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}}{\gamma_{l=0\ K=0}^{[0^+(\mathrm{g.s.})\to 0^+(\mathrm{g.s.});0^+(\mathrm{g.s.})]}} . \quad (2)$$

The quantity γ_l^2 is the reduced width [10, 21, 22], while

$$F_l = \frac{P_l(Q)}{P_{l=0}(Q)} = \exp\left(\frac{2}{\hbar}\int_{r_i}^{r_o}[q_{l=0}(r) - q_l(r)]dr\right) , \quad (3)$$

where $P_l(Q)$ stands for the penetrability [23]. In the right-hand-side expression of the penetrability ratio F_l we have used the JWKB approximation [24] of the penetrability. "r_o" and "r_i" stand for the outer and inner turning points, respectively, and

$$q_l(r) = \sqrt{2m_0A_{\mathrm{red}}(V_l^{\mathrm{Coul+nucl}} - Q)}, \quad (4)$$

where $A_{\mathrm{red}} = \frac{A_1A_2}{A_1+A_2}$ and $V_l^{\mathrm{Coul+nucl}}$ $[V_l^{\mathrm{Coul+nucl}} = \tilde{V}_{000}(R) + \frac{\hbar^2}{2\mu}l(l+1)]$ is the sum of the Coulomb, nuclear $[\tilde{V}_{000}(R) = (4\pi)^{-\frac{1}{2}}V_{000}(R)]$, and centrifugal one body potentials acting between the emitted cluster and the daughter nucleus. $Q = Q_c = Q_0 - E^*_{I_1K_1\pi_1} - E^*_{I_2K_2\pi_2}$. In the R-matrix theory [22] when calculating the penetrability the inner turning point is replaced by the channel radius (R_c). There can be cases when the inner turning point and the channel radius do not coincide. Usually [4, 10], the Coulomb part of this potential is replaced by a pointlike Coulomb potential, and the nuclear part by a square well or a Saxon-Woods one. Within these simple prescriptions the penetrability ratio F_l becomes

$$F_l = \exp\left(2\int_{z_o}^{z_i} dz[\sqrt{x^{-1}\rho_i z - 1} - \sqrt{l(l+1)z^2 + x^{-1}\rho_i z - 1}]\right) \approx \exp[-\zeta(Z_1, A_1, Z_2, A_2)l(l+1)], \quad (5)$$

where $z_i = z_c = \rho_c^{-1}$, $z_o = \rho_o^{-1}$, $\rho_c = kR_c$, $\rho_o = kR_o$, and $x = QR_c/Z_1Z_2e^2$:

$$\zeta(Z_1, Z_2, A_1, A_2, l) \approx \beta\frac{\sqrt{x}}{\rho_c} = \beta\sqrt{\frac{\hbar^2}{2\mu R_cZ_1Z_2e^2}} = \beta\delta\sqrt{\frac{1}{A_c^{1/3}Z_1Z_2A_{\mathrm{red}}}}, \quad (6)$$

in which β may be considered a constant close to 5 [plus terms $\simeq O(l(l+1)x^2/\rho_c)$] for the case studied above. This constant becomes smaller for the case of the square well being replaced by any other realistic potential. $\delta = \sqrt{\frac{\hbar c}{2r_0mc^2\alpha}} \approx 2.45$, in which $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.036}$ is the fine structure constant. $A_c^{1/3} = A_1^{1/3} + A_2^{1/3}$. For α decay the ζ function has the following approximate expression [25]: $\zeta(Z_1 = 2, A_1 = 4, Z_2 = Z, A_2 = A) = \frac{2.027}{\sqrt{Z}A^{\frac{1}{6}}}$.

For deformed nuclei there is a matrix part of the penetrability [26, 27, 21, 28, 10] $(\mathbf{K}_{cc'})$, the Fröman-Nosov matrix,

$$\mathbf{K}_{cc'} = \left\langle \Phi_{c'} \middle| \exp\left(\frac{i}{\hbar}S_c^1(\mathbf{R})\right) \middle| \Phi_c \right\rangle \quad (7)$$

responsible for the channel coupling. Here $\mid \Phi_c\rangle$ is the channel spin function [22] including the angular momentum part in the relative motion of the decay products. The function $S_c^1(\mathbf{R})$ is the JWKB expression [24] of the noncentral part of the action,

$$\frac{i}{\hbar}S_c^1(\mathbf{R}) = -\frac{1}{\hbar}\int_{R_c}^{R} dR\frac{m_0A_{\mathrm{red}}V_1(\mathbf{R}, \theta_i^{(1)}, \theta_i^{(2)})}{q_{l_c}(R)} , \quad (8)$$

where $q_l(r)$ is given by Eq. (4). The noncentral part of the action is generated by the noncentral part $V_1(\mathbf{R}, \theta_i^{(1)}, \theta_i^{(2)})$ of the double-folded $M3Y$ interaction potential (see Refs. [29–32]):

$$V(\mathbf{R}, \theta_i^{(1)}, \theta_i^{(2)}) = \sum_{\lambda_1,\lambda_2,\lambda_3}\sum_{\mu_1,\mu_2,\mu_3} C_{\mu_1\mu_2\mu_3}^{\lambda_1\lambda_2\lambda_3}C_{0\ 0\ 0}^{\lambda_1\lambda_2\lambda_3}C_{\mu_3-\mu_3\ 0}^{\lambda_3\lambda_3\ 0}0, \quad (9)$$

$$\mathcal{D}_{\mu_1\ 0}^{\lambda_1}(\theta_i^{(1)})\mathcal{D}_{\mu_1\ 0}^{\lambda_2}(\theta_i^{(2)})Y_{\lambda_3-\mu_3}(\hat{R})V_{\lambda_1,\lambda_2,\lambda_3}(R) = \tilde{V}_{000}(R) + V_1(\mathbf{R}, \theta_i^{(1)}, \theta_i^{(2)}),$$

acting between the residual nuclei. Here $\tilde{V}_{000}(R) = (4\pi)^{-\frac{1}{2}}V_{000}(R)$, while

$$V_{\lambda_1,\lambda_2,\lambda_3}(R) = \frac{4}{\sqrt{\pi}}(-1)^{\frac{\lambda_1+\lambda_2+\lambda_3}{2}}\hat{\lambda}_1\hat{\lambda}_2\int_0^\infty dx_1\rho_{\lambda_1} \times \int_0^\infty dx_2\rho_{\lambda_2}F^v_{\lambda_1,\lambda_2,\lambda_3}(x_1,x_2,x_3)\ , \quad (10)$$

$$F^v_{\lambda_1,\lambda_2,\lambda_3}(x_1,x_2,x_3) = \int_0^\infty dqq^2v(q)\prod_{i=1}^3 j_{\lambda_i}(qx_i)\ , \quad (11)$$

where $v(q)$ is the Fourier transform of the effective NN potential $v(r)$, e.g., the $M3Y$ potential reads $v(r) = \sum_{s=0}^2 \kappa_s\frac{1}{r}\exp[-\mu_s r]$, in which $\kappa_0 = e^2$, $\mu_0 = 0$ stand for the Coulomb part, while $\kappa_1 = 1528.75$ MeV, $\mu_1 = 4$ fm^{-1}, $\kappa_2 = -784.4$ MeV, and $\mu_2 = 25$ fm^{-1} stand for the nuclear part, and in addition $v(q) = \sum_{s=0}^2 \kappa_s\frac{1}{q^2+\mu_s^2}$ (see Refs. [29–32]). ρ_{λ_i} are the multipoles of the nuclear densities. $j_{\lambda_i}(qx_i)$ are the spherical Bessel functions. $\mathcal{D}^\lambda_{\mu\mu'}(\theta_i)$ are the Wigner rotation matrices describing the orientation of the involved nuclei, while $Y_{\lambda\mu}(\hat{R})$ is the λth-order spherical harmonics.

The nuclear densities used in our folding procedure are described by Fermi functions [32] $\left(\rho(\mathbf{r}) = \rho_0\left[1+\exp\left\{\frac{r-r_0A^{1/3}(1+\beta_2Y_{20})}{a}\right\}\right]\right)$. The parameters r_0 and a are 1.2 fm and 0.5 fm, respectively, if not stated otherwise. They are used in order to reproduce the elastic scattering and nuclear reaction cross sections [33].

The shapes of different multipole terms $V_{\lambda_1,\lambda_2,\lambda_3}(R)$ of the potential (9) are analogous to those plotted in Fig. 2 of Ref. [32] (see also Ref. [29]).

The Fröman-Nosov matrix (7) for the α decay of a zero spin axially symmetric deformed nucleus into the α plus axially symmetric deformed nucleus becomes [21, 28, 10, 34]

$$\mathbf{K}_{ll'} = \int d\Omega Y^*_{l'0}(\Omega)e^{-BP_2(\cos\theta)}Y_{l0}(\Omega). \quad (12)$$

Neglecting the nuclear part and taking a pointlike Coulomb potential only the expression of the parameter B becomes [34]

$$B = \sqrt{\frac{2\mu}{\hbar^2}}\int_{R_c}^{r_\circ}\frac{1}{2}\frac{Q_0e^2}{R^3\sqrt{V_c-E}}\ . \quad (13)$$

The approximate expression of the parameter B calculated by Fröman [26] is

$$B = \eta\beta_2\sqrt{\frac{5kR_0}{4\pi\eta}\left(1-\frac{kR_0}{\eta}\right)}\left[\frac{4}{5}-\frac{2}{5}\frac{kR_0}{\eta}\right], \quad (14)$$

while Nosov [27] got a slightly different expression but close to Fröman's:

$$B = \eta\beta_2\sqrt{\frac{5}{4\pi}}\left\{\sqrt{\frac{kR_0}{\eta}\left(1-\frac{kR_0}{\eta}\right)}\left[\frac{4}{5}-i\frac{2}{5}\frac{kR_0}{\eta}\right]\right\}. \quad (15)$$

$\eta = 4Ze^2/(\hbar v)$ is the Sommerfeld parameter times factor 2, k is the wave number, β_2 is the quadrupole deformation, and R_0 stands for the nuclear radius.

Taking into account the channel coupling within the JWKB approximation the quantity $d^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}_{l\ K=|K_1+K_2-K_i|}$ from the expression of the hindrance factor (1) becomes

$$d^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}_{l\ K} = \frac{\tilde{\gamma}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}_{l\ K=|K_1+K_2-K_i|}}{\tilde{\gamma}^{[00+(\mathrm{g.s.})\to 00+(\mathrm{g.s.});00+(\mathrm{g.s.})]}_{l=0\ K=0}}, \quad (16)$$

where

$$\tilde{\gamma}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}_{l\ K=|K_1+K_2-K_i|} = \sum_{c'}\mathbf{K}_{cc'}\gamma^{(I_iK_i\pi_i\to I'_1K'_1\pi'_1;I'_2K'_2\pi'_2)}_{l'K'=|K'_1+K'_2-K_i|}\ , \quad (17)$$

in which $c = l, I_1, K_1, \pi_1, I_2, K_2, \pi_2$ and $c' = l', I'_1, K'_1, \pi'_1, I'_2, K'_2, \pi'_2$.

B. Relative intensities

The general expression for the relative intensities in cluster decay or cold fission leading to a definite channel quantum state $(I_1K_1\pi_1;I_2K_2\pi_2)$ is defined as a ratio of two quantities. The numerator represents the partial decay width that defines a given channel of the cluster radioactivity $(A,Z)\to(A_1,Z_1)+(A_2,Z_2)$. The denominator is the sum of all partial decay widths.

Thus

$$I^{(I_1K_1\pi_1;I_2K_2\pi_2)}_{\mathrm{rel}} = \frac{\Gamma^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}(Q_c)}{\sum_{I_1K_1\pi_1;I_2K_2\pi_2}\Gamma^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}(Q_c)}$$

$$= \frac{\sum_l F_l d^2_l(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)R(Q_c,Q_0)}{\sum_{I_1K_1\pi_1;I_2K_2\pi_2}\sum_l F_l d^2_l(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)R(Q_c,Q_0)}$$

$$= \frac{\left(\mathrm{HF}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}\right)^{-1}R(Q_c,Q_0)}{\sum_{I_1K_1\pi_1I_2K_2\pi_2}\left(\mathrm{HF}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}\right)^{-1}R(Q_c,Q_0)}\ , \quad (18)$$

where $Q_0 = M(A,Z) - M_1(A_1,Z_1) - M_2(A_2,Z_2)$ and $Q_c = Q_0 - E^*_{I_1K_1\pi_1} - E^*_{I_2K_2\pi_2}$. The quantities F_l and $d_{l\,K}^{(I_iK_i\pi_i\to I_1K_1\pi_1;I_2K_2\pi_2)}$ are defined in Sec. II A. The quantity $R(Q,Q_0)$ is defined as follows:

$$R(Q,Q_0) = \frac{P_{l=0}(Q)}{P_{l=0}(Q_0)} \ . \tag{19}$$

For the α-decay relative intensities one set of quantum numbers $IK\pi$ is replaced by I=0, π=+:

$$I_{\rm rel}^{(I_fK_f\pi_f)} = \frac{\sum_l F_l d_l^2(I_iK_i\pi_i \to I_fK_f\pi_f)R(Q_c,Q_0)}{\sum_{I_fK_f\pi_f}\sum_l F_l d_l^2(I_iK_i\pi_i \to I_fK_f\pi_f)R(Q_c,Q_0)} = \frac{\left(\mathrm{HF}^{(I_iK_i\pi_i\to I_fK_f\pi_f)}\right)^{-1}R(Q_c,Q_0)}{\sum_{I_fK_f\pi_f}\left(\mathrm{HF}^{(I_iK_i\pi_i\to I_fK_f\pi_f)}\right)^{-1}R(Q_c,Q_0)} \ , \tag{20}$$

where $Q_0 = M(A,Z) - M_f(A_f,Z_f) - M_\alpha$ and $Q_c = Q_0 - E^*_{I_fK_f\pi_f}$.

III. ENLARGED SUPERFLUID MODEL

The enlarged superfluid model (ESM) Hamiltonian for nonrotational states of deformed nuclei includes an average field of neutron and proton systems in the form of the axially symmetric Saxon-Woods (or Hartree-Fock), monopole pairing, isoscalar and isovector particle-hole and particle-particle multipole, and spin-multipole interactions between quasiparticles as well as the so-called α-like four nucleon interaction [11]. For the particle-hole and particle-particle multipole and spin-multipole interaction parts we use a separable interaction [11] of rank $N > 1$:

$$H = H_0 + H' \ , \tag{21}$$

where

$$H_0 = \sum_\tau [H^{\rm av}_{\rm s.p.}(\tau) - G_\tau P^\dagger_\tau P_\tau] + H_4 \ , \tag{22}$$

in which

$$H^{\rm av}_{\rm s.p.}(\tau) = \sum_{s\sigma} E_s a^\dagger_{s\sigma}a_{s\sigma} \ , \tag{23}$$

$$P_\tau = \sum_s a_{s-}a_{s+} \ , \tag{24}$$

$$H_4 = -G_4 P^\dagger_p P^\dagger_n P_n P_p \ , \tag{25}$$

and

$$H' = \sum_\tau \left[-\frac{1}{2}\sum_{\lambda\mu\sigma}\sum_{n=1}^{N}\left(\sum_{\eta=\pm1}(\kappa^{\lambda\mu}_{0\tau} + \eta\kappa^{\lambda\mu}_{1\tau})Q^\dagger_{n\lambda\mu\sigma}(\tau)Q_{n\lambda\mu\sigma}(\eta\tau) + G^{\lambda\mu}_\tau P^\dagger_{n\lambda\mu\sigma}(\tau)P_{n\lambda\mu\sigma}(\tau)\right) - \frac{1}{2}\sum_{L\lambda\mu\sigma}\sum_{n=1}^{N}\left(\sum_{\eta=\pm1}(\kappa^{L\lambda\mu}_{0\tau} + \eta\kappa^{L\lambda\mu}_{1\tau})T^\dagger_{nL\lambda\mu\sigma}(\tau)T_{nL\lambda\mu\sigma}(\eta\tau) + G^{L\lambda\mu}_\tau P^\dagger_{nL\lambda\mu\sigma}(\tau)P_{nL\lambda\mu\sigma}(\tau)\right)\right] \ , \tag{26}$$

where

$$\begin{pmatrix} P_{n\lambda\mu\sigma} \\ P_{nL\lambda\mu\sigma}\end{pmatrix} = \sum_{ss'}\sum_{\sigma\sigma'}\left\langle s\sigma\left|\begin{pmatrix}\Re^{\lambda\mu}_{n(pp)}(r)Y_{\lambda,\sigma\mu}(\hat r)\\ \Re^{L\lambda\mu}_{n(pp)}(r)\{\hat\sigma Y_L\}_{\lambda,\sigma\mu}\end{pmatrix}\right|s'\sigma'\right\rangle a_{s\sigma}a_{s'\sigma'} \tag{27}$$

and

$$\begin{pmatrix} Q_{n\lambda\mu\sigma} \\ T_{nL\lambda\mu\sigma}\end{pmatrix} = \sum_{ss'}\sum_{\sigma\sigma'}\left\langle s\sigma\left|\begin{pmatrix}\Re^{\lambda\mu}_{n(ph)}(r)Y_{\lambda,\sigma\mu}(\hat r)\\ \Re^{L\lambda\mu}_{n(ph)}(r)\{\hat\sigma Y_L\}_{\lambda,\sigma\mu}\end{pmatrix}\right|s'\sigma'\right\rangle a^\dagger_{s\sigma}a_{s'\sigma'} . \tag{28}$$

The properties of the above particle-particle and particle-hole matrix elements are discussed in Refs. [11, 13]. The symbol $\tau = -\frac{1}{2}$ stands for the proton system and $\tau = +\frac{1}{2}$ stands for the neutron system, $a^\dagger_{s_\tau\sigma_\tau}$ ($a_{s_\tau\sigma_\tau}$) are the fermion operators which create (destroy) a nucleon in (from) the single particle state $|\ s_\tau\sigma_\tau\rangle$, where σ_τ is the sign of the projection of the angular momentum of the state onto the nuclear symmetry axis, s_τ being the rest $(N_\tau, n_{z_\tau}, \Omega_\tau, \pi_\tau, \ldots)$ of the quantum numbers that label the single particle energy levels. The term H_4 from Eq. (5) is an effective, coherent two pairs (four nucleon) interaction term, which induces the dynamical α-like four nucleon correlations in the superfluid phases of atomic nuclei [12]. G_τ are the pairing coupling strengths, $G^{\lambda\mu}_\tau$ and $G^{L\lambda\mu}_\tau$ are the coupling constants of the particle-particle interaction [11], and $\kappa^{\lambda\mu}_{0\tau}$, $\kappa^{\lambda\mu}_{1\tau}$ and $\kappa^{L\lambda\mu}_{0\tau}$, $\kappa^{L\lambda\mu}_{1\tau}$ are the isoscalar and isovector coupling constants of the particle-hole multipole-multipole and spin-multipole interactions [13]. G_4 is the four nucleon inter-

action constant and $\sigma = \pm 1$. The functions $\Re^{\lambda\mu}_{n(pp)}(r)$ and $\Re^{L\lambda\mu}_{n(pp)}(r)$ stand for the radial parts of the particle-particle multipole-multipole and spin-multipole separable interactions, while $\Re^{\lambda\mu}_{n(ph)}(r)$ and $\Re^{L\lambda\mu}_{n(ph)}(r)$ stand for the radial parts of the particle-hole multipole-multipole and spin-multipole separable interactions.

To find the superfluid solutions we first should deal with the mean field. As a trial wave function for the ground state of the atomic nucleus we use a BCS-type wave function, and the mean field is described by the H_0 part of the total Hamiltonian:

$$| \text{BCS}\rangle = \prod_{\tau s_\tau}(u_{s_\tau} + v_{s_\tau} a^\dagger_{s_\tau +} a^\dagger_{s_\tau -}) \mid 0\rangle \ , \tag{29}$$

where $u^2_s + v^2_s = 1$ and $\mid 0\rangle$ denotes the absolute vacuum.

Thus the constrained energy functional is

$$W = \langle \text{BCS} \mid H_0 - \sum_\tau \lambda_\tau \hat{N}_\tau \mid \text{BCS}\rangle$$

$$= \sum_\tau \left[\sum_{s_\tau} 2(\tilde{E}_{s_\tau} - \lambda_\tau) v^2_{s_\tau} - G_\tau \chi^2_\tau \right] - G_4 \chi^2_p \chi^2_n \ . \tag{30}$$

Here λ_τ denotes the nucleon Fermi level, $\hat{N}_\tau$ is the nucleon number operator,

$$\chi_\tau = \langle \text{BCS} \mid \sum_{s_\tau} a^\dagger_{s_\tau +} a^\dagger_{s_\tau -} \mid \text{BCS}\rangle = \sum_{s_\tau} u_{s_\tau} v_{s_\tau} \tag{31}$$

is the so-called pairing correlation function or order parameter, and

$$\tilde{E}_{s_{p(n)}} = E_{s_{p(n)}} - \frac{1}{2}(G_{p(n)} + G_4 \chi^2_{n(p)}) v^2_{s_{p(n)}} - \frac{1}{4} G_4 v^2_{s_{p(n)}} \sum_{s_{n(p)}} v^4_{s_{n(p)}} \tag{32}$$

are the modified (from the values $E_{s_{p(n)}}$) single particle energies. Usually these self-consistent field corrections are omitted [13].

The minimization of the function W given by Eq. (30) with respect to the variational parameters leads to the following gap and constraint equations:

$$\frac{1}{2}(G_{p(n)} + G_4 \chi^2_{n(p)}) \sum_{s_{p(n)}} \epsilon^{-1}_{s_{p(n)}} = 1$$

$$\sum_{s_\tau} [1 - (\tilde{E}_{s_\tau} - \lambda_\tau)\epsilon^{-1}_{s_\tau}] = N_\tau \ ,$$

$$\epsilon_{s_\tau} = \sqrt{(\tilde{E}_{s_\tau} - \lambda_\tau)^2 + \Delta^2_\tau} \ , \tag{33}$$

for the doubly even mass deformed superfluid nuclei. For odd and odd-odd mass deformed superfluid nuclei the above equations are modified according to the blocking effect [13].

The Bogoliubov-Valatin u_s and v_s parameters are parametrized according to the formulas

$$\begin{pmatrix} u^2_{s_\tau} \\ v^2_{s_\tau} \end{pmatrix} = \frac{1}{2}\left(1 \pm \frac{\tilde{E}_{s_\tau} - \lambda_\tau}{\epsilon_{s_\tau}}\right) \tag{34}$$

and the correlation function becomes

$$\chi_\tau = \frac{1}{2}\Delta_\tau \sum_{s_\tau} \epsilon^{-1}_{s_\tau} \ . \tag{35}$$

A simple inspection of the gap equations shows that the proton and neutron equations are coupled, i.e., it is possible that the superfluidities of the proton and neutron systems may be generated by one another, even in the case when for one system, in the absence of four nucleon interactions, Belyaev's condition [12] is not satisfied. Due to the additional term $G_4\chi^2$ the strengths $G_{p(n)} + G_4\chi^2_{n(p)}$ may increase and fulfill Beliaev's condition for the superfluid solutions. This mechanism explained [11] in several cases the origin of the odd-even staggering of the charge radii of isotopes of one element (see Ref. [35] also). Moreover, the gap equations can have [11] for some nuclei more than one set of solutions, a fact which opened a new area of research—the superfluid isomers.

To find the excitation spectrum and the corresponding wave functions we add to the H_0 the H' part and use the recipe from Refs. [11, 13, 36, 37]. Within the ESM the wave function for any ground or excited state of the atomic nucleus is given by

$$\chi_{K\pi} = \Omega^\dagger_{Z,N}(K\pi) \mid \tilde{0}\rangle \quad Q_i(\lambda\mu) \mid \tilde{0}\rangle = 0 \ , \tag{36}$$

where

$$\Omega^\dagger_{ee}(K\pi) = 1 \tag{37}$$

for the ground state of an even-even nucleus,

$$\Omega^\dagger_{ee}(K\pi) = Q^\dagger_i(\lambda\mu) \tag{38}$$

for an excited collective state of an even-even nucleus,

$$\Omega^\dagger_{\text{odd}}(K\pi) = \sum_{q\delta} C_q(K\pi)\alpha^\dagger_{q\delta} + \sum_{\lambda\mu i}\sum_{\beta\gamma} D^{\lambda\mu i}_{\beta\gamma}(K\pi)\alpha^\dagger_{\beta\gamma} Q^\dagger_i(\lambda\mu) \tag{39}$$

for the ground state or excited state of an odd-mass nucleus, and

$$\Omega^\dagger_{\text{odd-odd}}(K\pi) = \sum_{q_p\delta_p q_n\delta_n} C_{q_p q_n}(K\pi)\alpha^\dagger_{q_p\delta_p}\alpha^\dagger_{q_n\delta_n} + \sum_{\lambda\mu i}\sum_{\beta_p\gamma_p\beta_n\gamma_n} D^{\lambda\mu i}_{\beta_p\gamma_p\beta_n\gamma_n}(K\pi) \times \alpha^\dagger_{\beta_p\gamma_p}\alpha^\dagger_{\beta_n\gamma_n} Q^\dagger_i(\lambda\mu) \tag{40}$$

for the ground state or excited state of an odd-odd nucleus.

In terms of quasiparticle operators the phonon $Q_i(\lambda\mu)$ operator is given by the expression

$$Q_i(\lambda\mu) = Q^{(p)}_i(\lambda\mu) + Q^{(n)}_i(\lambda\mu) \ , \tag{41}$$

where

$$Q_i^{(p \text{ or } n)}(\lambda\mu) = \frac{1}{2}\sum_{ss'}[\psi_{ss'}^{\lambda\mu i}A(ss') - \phi_{ss'}^{\lambda\mu i}A^{\dagger}(ss') + \bar{\psi}_{ss'}^{\lambda\mu i}\bar{A}(ss') - \bar{\phi}_{ss'}^{\lambda\mu i}\bar{A}^{\dagger}(ss')] \quad (42)$$

with

$$A(ss') = 2^{-\frac{1}{2}}\sum_{\mu}\mu\alpha_{s'\mu}\alpha_{s-\mu}, \quad \bar{A}(ss') = 2^{-\frac{1}{2}}\sum_{\mu}\alpha_{s'\mu}\alpha_{s\mu}, \quad (43)$$

$$\alpha_{s\mu} = u_s a_{s-\mu} + \mu v_s a_{s\mu}^{\dagger}. \quad (44)$$

When the collective energy $\omega_i^{\lambda\mu} = \langle Q_i(\lambda\mu)HQ_i^{\dagger}(\lambda\mu)\rangle$ is very close to a two quasiparticle energy $\epsilon_{s_\tau} + \epsilon_{s'_\tau}$ then the structure of the operator $Q_i(\lambda\mu)$ is dominated by the corresponding two quasiparticle component, and the state becomes mainly a two quasiparticle state. The expressions for $\psi_{ss'}^{\lambda\mu i}$, $\phi_{ss'}^{\lambda\mu i}$, $\bar{\psi}_{ss'}^{\lambda\mu i}$, $\bar{\phi}_{ss'}^{\lambda\mu i}$, $C_q(K\pi)$, $D_{\beta\gamma}^{\lambda\mu i}(K\pi)$, $C_{q_pq_n}(K\pi)$, $D_{\beta_p\gamma_p\beta_n\gamma_n}^{\lambda\mu i}(K\pi)$ are given in Ref. [13]; however, the energies ω, η and the pairing gap parameters entering these expressions are modified [11] due to the new H_4 additional term introduced in the ESM as compared to the quasiparticle phonon nuclear model (QPNM) [13, 37, 36].

IV. THE CLUSTER SPECTROSCOPIC FACTOR

In the previous paper [7] we derived the expressions for the hindrance factors of cluster radioactive decays including the cluster spectroscopic factors and their internal structure: the cluster overlaps and the intrinsic overlap integrals. The overall approach is analogous to that developed for the α decay [10, 21, 28, 14]. Within this approach we view the decay process as composed of two main steps. First the mother nucleus makes a kind of phase transition from the initial state, which could be of any structure (Fermi liquid [14], superfluid [13,38], spherical or deformed, one or many α cluster states [39–41], one or many combined heavy cluster states, etc.), to the final state composed of at least one cluster, which is going to be emitted, and the residual nucleus, which may also have any structure as above. One mechanism of such a transition could be cluster condensation, e.g., α condensation [38], cluster formation [42, 43], or what is usually assumed the formation of the cluster (e.g., the α cluster) from the already formed condensates of smaller clusters (e.g., Cooper pairs, bosons in the interacting-boson model, etc.) [7,11–13,17,10,21,28,14,38,42,39,43]. Another (less studied) mechanism could be the slow shape deformation [44] from any initial shape configuration of the studied many particle system through shapes that are energetically very unfavored (a large amplitude collective motion) to a shape corresponding to the two daughter nuclei in contact.

Second, the two daughter nuclei tunnel through the potential barrier in their relative motion, without further change in shape.

The favored cluster transitions follow the Geiger-Nuttal [19] law, which can be qualitatively explained by a mechanism according to which the emitted cluster is constructed mainly from strongly correlated pairs of nucleons situated around the proton and neutron Fermi levels. This mechanism leads to almost the same magnitude of the reduced widths for the isotopes of the element studied and consequently to small hindrance factors [10, 21, 7].

The unfavored transitions do not follow the Geiger-Nuttal law, because of the large variations of the reduced widths [10, 21, 28, 14], which have a key role in the understanding of the decay process and require a precise knowledge of the structures of the initial and final quantum states. From such transitions we can learn much about the structure of atomic nuclei and the mechanism of the decay phenomenon. The unfavored transitions have large hindrance factors and are characterized by the fact that the nucleons used to build the cluster are collected from different groups of strongly correlated and/or uncorrelated nucleons entering the structure of the initial state. In this last case it is necessary, first, to break up the correlated groups of nucleons and then to build the cluster which is going to be emitted.

There are clusters containing $4n$ particles, with n an integer. As examples we have the already observed ^{20}O, ^{24}Ne, etc. nuclei emitted from some translead nuclei. There are other clusters, such as for instance ^{14}C, which may be composed of groups of $4n$ particles (α-type groups) and one group containing less than $4n$ particles (deuteron- or triton-type groups). In order to use Moshinsky-type transformation brackets [45] we have to follow Ref. [17].

In the next subsection we study the ^{20}O case and afterwards we give the general procedure for calculating the intrinsic overlap integrals.

A. Intrinsic overlap integrals for ^{20}O decay

The ^{20}O cluster contains five α-type groups of nucleons. After a sequence of Moshinsky [45] transformations and orthogonal transformations for rearranging angular momenta, the spatial part $\tilde{\Psi}_\beta$ of the wave function Ψ_β from Eq. (18) of Ref. [7] (defined in terms of products of 20 $[\psi_{nlm}(\mathbf{r}\omega)]$ spatial harmonic oscillator wave functions) can be expressed in terms of 20 harmonic oscillator wave functions of the same frequency ω. Some (19) of these harmonic oscillator wave functions are functions of Jacobi coordinates ($\boldsymbol{\rho}_k$):

$$\rho_1^{(\alpha_1)} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}, \quad \rho_2^{(\alpha_1)} = \frac{\mathbf{r}_3 - \mathbf{r}_4}{\sqrt{2}}, \quad \rho_3^{(\alpha_1)} = \frac{\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4}{2},$$

$$\rho_1^{(\alpha_2)} = \frac{\mathbf{r}_5 - \mathbf{r}_6}{\sqrt{2}}, \quad \rho_2^{(\alpha_2)} = \frac{\mathbf{r}_7 - \mathbf{r}_8}{\sqrt{2}}, \quad \rho_3^{(\alpha_2)} = \frac{\mathbf{r}_5 + \mathbf{r}_6 - \mathbf{r}_7 - \mathbf{r}_8}{2},$$

$$\rho_1^{(\alpha_3)} = \frac{\mathbf{r}_9 - \mathbf{r}_{10}}{\sqrt{2}}, \quad \rho_2^{(\alpha_3)} = \frac{\mathbf{r}_{11} - \mathbf{r}_{12}}{\sqrt{2}}, \quad \rho_3^{(\alpha_3)} = \frac{\mathbf{r}_9 + \mathbf{r}_{10} - \mathbf{r}_{11} - \mathbf{r}_{12}}{2},$$
$$\rho_1^{(\alpha_4)} = \frac{\mathbf{r}_{13} - \mathbf{r}_{14}}{\sqrt{2}}, \quad \rho_2^{(\alpha_4)} = \frac{\mathbf{r}_{15} - \mathbf{r}_{16}}{\sqrt{2}}, \quad \rho_3^{(\alpha_4)} = \frac{\mathbf{r}_{13} + \mathbf{r}_{14} - \mathbf{r}_{15} - \mathbf{r}_{16}}{2},$$
$$\rho_1^{(\alpha_5)} = \frac{\mathbf{r}_{17} - \mathbf{r}_{18}}{\sqrt{2}}, \quad \rho_2^{(\alpha_5)} = \frac{\mathbf{r}_{19} - \mathbf{r}_{20}}{\sqrt{2}}, \quad \rho_3^{(\alpha_5)} = \frac{\mathbf{r}_{17} + \mathbf{r}_{18} - \mathbf{r}_{19} - \mathbf{r}_{20}}{2},$$
$$\rho_{T_1} = \frac{\sum_{n=1}^{4} \mathbf{r}_n - \sum_{m=5}^{8} \mathbf{r}_m}{2\sqrt{2}}, \qquad \rho_{T_2} = \frac{\sum_{n=1}^{8} \mathbf{r}_n - 2\sum_{m=9}^{12} \mathbf{r}_i}{2\sqrt{6}},$$
$$\rho_D = \frac{\sum_{i=13}^{16} \mathbf{r}_i - \sum_{j=17}^{20} \mathbf{r}_i}{2\sqrt{2}}, \quad \rho = \frac{2\sum_{i=1}^{12} \mathbf{r}_i - 3\sum_{i=13}^{20} \mathbf{r}_i}{\sqrt{120}}, \tag{45}$$

and one of them is a function of $\mathbf{R} = \sqrt{20}\mathbf{R}_{^{20}\mathrm{O}}$, where

$$\mathbf{R}_{^{20}\mathrm{O}} = \frac{\sum_{i=1}^{20} \mathbf{r}_i}{20} \tag{46}$$

is the center-of-mass coordinate.

Further, we may express the wave function $\tilde{\Psi}_\beta$ as products of four particle spatial wave functions $\mid \nu_{\alpha_i} l_{\alpha_i} m_{\alpha_i} \rangle$ coupled to a given angular momentum l_{α_i} (where $\nu_{\alpha_i} = n_1^{\alpha_i}, l_1^{\alpha_i}, n_2^{\alpha_i}, l_2^{\alpha_i}, n_3^{\alpha_i}, l_3^{\alpha_i}, n_4^{\alpha_i}, l_4^{\alpha_i}, l_{12}^{\alpha_i}, l_{34}^{\alpha_i}$),

$$\mid \tilde{\Psi}_\beta \rangle = \mid \nu_{\alpha_1} l_{\alpha_1} \nu_{\alpha_2} l_{\alpha_2} (l_{\alpha_{12}}) \nu_{\alpha_3} l_{\alpha_3} [l_{\alpha_T}] \times \nu_{\alpha_4} l_{\alpha_4} \nu_{\alpha_5} l_{\alpha_5} [l_{\alpha_D}]; lm \rangle. \tag{47}$$

The spatial part of the ^{20}O nucleus intrinsic wave function we express, as in Ref. [7], as follows:

$$\psi_{\mathrm{int}}^{(^{20}\mathrm{O})} = \sum_k C_k \left(\frac{\beta_k}{\sqrt{\pi}}\right)^{57} \exp\left(-\frac{\beta_k}{2} \sum_i^{19} \rho_i^2\right). \tag{48}$$

The coefficients C_k are the overlap integrals $\langle \psi_{\mathrm{int}}^a \mid \Psi_{\beta_{\omega_a}^a} \rangle$ from Eq. (21) of Ref. [7].

Let us write down the overlap integral

$$I_{^{20}\mathrm{O}}(\alpha\mathbf{R}) = \langle \psi_{\mathrm{int}}^{^{20}\mathrm{O}} \mid \tilde{\Psi}_\beta \rangle. \tag{49}$$

This overlap integral can be expressed in terms of α-type overlap integrals

$$I_{\alpha_i}(\alpha\mathbf{R}) = \langle \varphi_{\mathrm{int}}^{\alpha_i} \mid \nu_{\alpha_i} l_{\alpha_i} m_{\alpha_i} \rangle = \left(\frac{2}{\sqrt{\pi}}\right)^3 \left(\frac{2\alpha\beta}{\beta^2 + \alpha^2}\right)^{\frac{9}{2}}, \tag{50}$$

$$\sum_{N_{\alpha_i}} \left(\frac{\beta^2 - \alpha^2}{\beta^2 + \alpha^2}\right)^{\frac{1}{2}\left[\sum_{s=1}^{4}\left(2n_s^{(\alpha_i)} + l_s^{(\alpha_i)}\right) - \left(2N_{\alpha_i} + l_{\alpha_i}\right)\right]} \times X_{N_{\alpha_i} l_{\alpha_i}}(\nu_{\alpha_i}) \Psi_{N_{\alpha_i} l_{\alpha_i} m_{\alpha_i}}(\alpha\mathbf{R}_{\alpha_i}),$$

where $\varphi_{\mathrm{int}}^{\alpha_i} = \prod_{k=1}^{3} \Psi_{000}(\beta\boldsymbol{\rho}_k^{\alpha_i})$ is the intrinsic α-particle oscillator-type wave function. Here, however, the oscillator parameter β is equal to β_k from Eq. (48). The oscillator parameter α ($\alpha = \sqrt{\frac{m\omega}{\hbar}}$) is defined by the mother nucleus oscillator frequency. The quantity $X_{N_{\alpha_i} l_{\alpha_i}}(\nu_{\alpha_i})$ contains three Moshinsky [45] brackets:

$$\begin{aligned} X_{N_{\alpha_i} l_{\alpha_i}}(\nu_{\alpha_i}) = & \sum_{n_{\rho_1^{(\alpha_i)}} n_{\rho_2^{(\alpha_i)}} n_{\rho_3^{(\alpha_i)}}} \prod_{k=1}^{3} \sqrt{\frac{(n_{\rho_k^{(\alpha_i)}} + \frac{1}{2})!}{n_{\rho_k^{(\alpha_i)}}!}} \langle n_{\rho_1^{(\alpha_i)}} 0 N_1^{(\alpha_i)} l_{12}^{\alpha_i} \mid n_1^{(\alpha_i)} l_1^{(\alpha_i)} n_2^{(\alpha_i)} l_2^{(\alpha_i)} l_{12}^{(\alpha_i)} \rangle \\ & \times \langle n_{\rho_2^{(\alpha_i)}} 0 N_2^{(\alpha_i)} l_{34}^{(\alpha_i)} \mid n_3^{(\alpha_i)} l_3^{(\alpha_i)} n_4^{(\alpha_i)} l_4^{(\alpha_i)} l_{34}^{(\alpha_i)} \rangle \langle n_{\rho_3^{(\alpha_i)}} 0 N_{\alpha_i} l_{\alpha_i} \mid N_1^{(\alpha_i)} l_{12}^{(\alpha_i)} N_2^{(\alpha_i)} l_{34}^{(\alpha_i)} l_{34}^{(\alpha_i)} \rangle \\ = & (-1)^{N_{\alpha_i}} \left(\frac{\sqrt{\pi}}{2}\right)^3 \sqrt{\frac{(N_{\alpha_i} + l_{\alpha_i} + \frac{1}{2})!}{N_{\alpha_i}!(2l_{\alpha_i} + 1)}} \sum_{k_{\alpha_i}} \mathcal{B}_{k_{\alpha_i}}^{\alpha_i}(\nu_{\alpha_i}) \frac{k_{\alpha_i}!}{2^{l_{\alpha_i} + 2k_{\alpha_i}}} \binom{k_{\alpha_i} + l_{\alpha_i} + \frac{1}{2}}{k_{\alpha_i} - N_{\alpha_i}}, \end{aligned} \tag{51}$$

where

$$\mathcal{B}_{k_{\alpha_i}}^{\alpha_i}(\nu_{\alpha_i}) = C_{0\ 0\ 0}^{l_1^{(\alpha_i)} l_2^{(\alpha_i)} l_{12}^{(\alpha_i)}} C_{0\ 0\ 0}^{l_3^{(\alpha_i)} l_4^{(\alpha_i)} l_{34}^{(\alpha_i)}} C_{0\ 0\ 0}^{l_{12}^{(\alpha_i)} l_{34}^{(\alpha_i)} l_{\alpha_i}} \times \prod_{s=1}^{4} \left[\sqrt{\frac{n_s^{(\alpha_i)}!(2l_s^{(\alpha_i)} + 1)}{\left(n_s^{(\alpha_i)} + l_s^{(\alpha_i)} + \frac{1}{2}\right)!}} \sum\nolimits_{k_s^{(\alpha_i)}=0}^{\tilde{n}_s^{(\alpha_i)}} \binom{n_s^{(\alpha_i)} + l^{(\alpha_i)} + \frac{1}{2}}{n_s^{(\alpha_i)} - k_s^{(\alpha_i)}} \frac{(-1)^{k_s^{(\alpha_i)}}}{k_s^{(\alpha_i)}!} \right]. \tag{52}$$

The tilde above the sum in the above expression stands for the restriction $\sum_{s=1}^{4} (2k_s^{(\alpha_i)} + l_s^{(\alpha_i)}) = 2k_{\alpha_i} + l_{\alpha_i}$ in performing the summation.

The intrinsic overlap integral for ^{20}O decay can now be expressed in terms of the $X_{N_{\alpha_i} l_{\alpha_i}}(\nu_{\alpha_i})$ quantities:

$$I_{^{20}\mathrm{O}}(\alpha\mathbf{R}) = \sum_k C_k \left[\frac{2}{\sqrt{\pi}}\left(\frac{2\alpha\beta_k}{\alpha^2+\beta_k^2}\right)^{\frac{3}{2}}\right]^{19} \sum_N \left[\frac{\alpha^2-\beta_k^2}{\alpha^2+\beta_k^2}\right]^{\frac{1}{2}\left[\sum_{i=1}^{5}\sum_{s=1}^{4}\left(2n_s^{(\alpha_i)}+l_s^{(\alpha_i)}\right)-(2N+l)\right]}$$
$$\times X_{Nl}^{^{20}\mathrm{O}}(\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_T l_D)\Psi_{Nlm}(\alpha\mathbf{R}), \tag{53}$$

where

$$X_{Nl}^{^{20}\mathrm{O}}(\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_T l_D)$$
$$= \sum_{N_{\alpha_1}\cdots N_{\alpha_5}N_T N_D} X_{N_T l_T}(N_{\alpha_1}l_{\alpha_1}N_{\alpha_2}l_{\alpha_2}l_{\alpha_{12}}N_{\alpha_3}l_{\alpha_3}) X_{N_D l_D}(N_{\alpha_4}l_{\alpha_4}N_{\alpha_5}l_{\alpha_5}) \prod_{i=1}^{5}\left[X_{N_{\alpha_i}l_{\alpha_i}}(\nu_{\alpha_i})\right], \tag{54}$$

where $X_{N_D L_D}$ has the expression [17]

$$X_{N_D L_D}(\nu_D) = \sum_{n_{\rho_D}} \sqrt{\frac{(n_{\rho_D}+\frac{1}{2})!}{n_{\rho_D}!}}\langle n_{\rho_D}0N_{D_1}l_{12} \mid n_1^{(D)}l_1^{(D)}n_2^{(D)}l_2^{(D)}l_{12}^{(D)}\rangle$$
$$= (-1)^{N_D}\left(\frac{\sqrt{\pi}}{2}\right)\sqrt{\frac{(N_D+l_D+\frac{1}{2})!}{N_D!(2l_D+1)}}\sum_{k_D}\mathcal{B}_{k_D}^{D}(\nu_D)\frac{k_D!}{(\sqrt{2})^{l_D+2k_D}}\binom{k_D+l_D+\frac{1}{2}}{k_D-N_D}, \tag{55}$$

with

$$\mathcal{B}_{k_D}^{D}(\nu_D) = C_{0\ 0\ 0}^{l_1^{(D)}l_2^{(D)}l_{12}^{(D)}}\prod_{s=1}^{2}\left[\sqrt{\frac{n_s^{(D)}!(2l_s^{(D)}+1)}{\left(n_s^{(D)}+l_s^{(D)}+\frac{1}{2}\right)!}}\tilde{\sum}_{k_s^{(D)}=0}^{n_s^{(D)}}\binom{n_s^{(D)}+l^{(D)}+\frac{1}{2}}{n_s^{(D)}-k_s^{(D)}}\frac{(-1)^{k_s^{(D)}}}{k_s^{(D)}!}\right]. \tag{56}$$

Here $\nu_D = n_1^{(D)}, l_1^{(D)}, n_2^{(D)}, l_2^{(D)}$. The tilde above the sum in the above expression stands for the restriction $\sum_{s=1}^{2}(2k_s^{(D)}+l_s^{(D)}) = 2k_D+l_D$ in performing the summation.

$X_{N_T L_T}$ has the expression [17]

$$X_{N_T L_T}(\nu_T) = \sum_{n_{\rho_{T_1}}n_{\rho_{T_2}}}\prod_{k=1}^{2}\sqrt{\frac{(n_{\rho_{T_k}}+\frac{1}{2})!}{n_{\rho_{T_k}}!}}\langle n_{\rho_{T_1}}0N_{T_1}l_{12} \mid n_1l_1n_2l_2l_{12}\rangle\langle n_{\rho_{T_2}}0N_T l_T \mid N_{T_1}l_{12}n_3l_3l_T\rangle$$
$$= (-1)^{N_T}\left(\frac{\sqrt{\pi}}{2}\right)^2\sqrt{\frac{(N_T+l_T+\frac{1}{2})!}{N_T!(2l_T+1)}}\sum_{k_T}\mathcal{B}_{k_T}^{T}(\nu_T)\frac{k_T!}{(\sqrt{3})^{l_T+2k_T}}\binom{k_T+l_T+\frac{1}{2}}{k_T-N_T}, \tag{57}$$

where

$$\mathcal{B}_{k_T}^{T}(\nu_T) = C_{0\ 0\ 0}^{l_1^{(T)}l_2^{(T)}l_{12}^{(T)}}C_{0\ 0\ 0}^{l_{12}^{(T)}l_3^{(T)}l^{(T)}}\prod_{s=1}^{3}\left[\sqrt{\frac{n_s^{(T)}!(2l_s^{(T)}+1)}{\left(n_s^{(T)}+l_s^{(T)}+\frac{1}{2}\right)!}}\tilde{\sum}_{k_s^{(T)}=0}^{n_s^{(T)}}\binom{n_s^{(T)}+l^{(T)}+\frac{1}{2}}{n_s^{(T)}-k_s^{(T)}}\frac{(-1)^{k_s^{(T)}}}{k_s^{(T)}!}\right]. \tag{58}$$

Here $\nu_T = n_1^{(T)}, l_1^{(T)}, n_2^{(T)}, l_2^{(T)}, l_{12}^{(T)}, n_3^{(T)}, l_3^{(T)}$. The tilde above the sum in the above expression stands for the restriction $\sum_{s=1}^{3}(2k_s^{(T)}+l_s^{(T)}) = 2k_T+l_T$ in performing the summation. The expression for $X_{N_{\alpha_i}l_{\alpha_i}}(\nu_{\alpha_i})$ is given in Eqs. (50) and (51).

The expression for $X_{Nl}^{^{20}\mathrm{O}}$ ($\nu_{\alpha_1}, \nu_{\alpha_2}, \nu_{\alpha_3}, \nu_{\alpha_4}, \nu_{\alpha_5}, l_{\alpha_{12}}, l_T l_D$) can be directly obtained from the following equation:

$$\mid \nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_T l_D; lm\rangle = \sum_{n_{\rho_1^{(\alpha_1)}}, l_{\rho_1^{(\alpha_1)}}, \ldots, n_\rho, l_\rho, N, L} \langle n_{\rho_1^{(\alpha_1)}}, l_{\rho_1^{(\alpha_1)}}, \ldots, n_\rho, l_\rho, N, L; l \mid \nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_T l_D; l\rangle$$
$$\times \mid n_{\rho_1^{(\alpha_1)}}, l_{\rho_1^{(\alpha_1)}}, \ldots, n_\rho, l_\rho, N, L; lm\rangle . \tag{59}$$

In the above equation in the left-hand side is the wave vector containing 20 oscillator wave functions $\Psi_{n_i l_i m_i}(\alpha\mathbf{r}_i)$, whose angular momenta l_i are coupled as follows:

$$\boldsymbol{l}_1^{(\alpha_i)} + \boldsymbol{l}_2^{(\alpha_i)} = \boldsymbol{l}_{12}^{(\alpha_i)}, \qquad \boldsymbol{l}_3^{(\alpha_i)} + \boldsymbol{l}_4^{(\alpha_i)} = \boldsymbol{l}_{34}^{(\alpha_i)}, \qquad \boldsymbol{l}_{12}^{(\alpha_i)} + \boldsymbol{l}_{34}^{(\alpha_i)} = \boldsymbol{l}_{\alpha_i},$$
$$\boldsymbol{l}_{\alpha_1} + \boldsymbol{l}_{\alpha_2} = \boldsymbol{l}_{\alpha_{12}}, \qquad \boldsymbol{l}_{\alpha_{12}} + \boldsymbol{l}_{\alpha_3} = \boldsymbol{l}_T, \qquad \boldsymbol{l}_{\alpha_4} + \boldsymbol{l}_{\alpha_5} = \boldsymbol{l}_D, \qquad \boldsymbol{l}_T + \boldsymbol{l}_D = \boldsymbol{l}. \tag{60}$$

The right-hand side contains 19 oscillator wave functions $\Psi_{n_{\rho_k}l_{\rho_k}m_{\rho_k}}(\alpha\boldsymbol{\rho}_k)$, that depend on Jacobi coodinates ρ_k and one $\Psi_{NLM}(\alpha\mathbf{R})$. From Eq. (59) by assuming $\mathbf{r}_1 = \mathbf{r}_2 = \cdots = \mathbf{r}_{20} = \mathbf{x}$ we obtain

$$X^{^{20}\mathrm{O}}_{Nl}(\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_Tl_D) = \sum_{n_{\rho_1^{(\alpha_1)}},l_{\rho_1^{(\alpha_1)}},\ldots,n_\rho,l_\rho} \langle n_{\rho_1^{(\alpha_1)}}, l_{\rho_1^{(\alpha_1)}}, \ldots, n_\rho, l_\rho, N, L; l \mid \nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_Tl_D; l\rangle$$
$$\times \prod_{i=1}^{5}\prod_{k=1}^{3}\sqrt{\frac{(n_{\rho_k^{(\alpha_i)}}+\frac{1}{2})!}{n_{\rho_k^{(\alpha_i)}}!}} \prod_{m=1}^{2}\sqrt{\frac{(n_{\rho_{T_m}}+\frac{1}{2})!}{n_{\rho_{T_m}}!}}\sqrt{\frac{(n_{\rho_D}+\frac{1}{2})!}{n_{\rho_D}!}}\sqrt{\frac{(n_\rho+\frac{1}{2})!}{n_\rho!}}$$
$$= (-1)^N\left(\frac{\sqrt{\pi}}{2}\right)^{19}\sqrt{\frac{(N+l+\frac{1}{2})!}{N!(2l+1)}}\sum_k \mathcal{B}_k(\nu)\frac{k!}{\sqrt{20}^{l+2k}}\binom{k+l+\frac{1}{2}}{k-N}, \tag{61}$$

where

$$\mathcal{B}_k(\nu) = \prod_{i=1}^{5}\prod_{s=1}^{4}\left[\sqrt{\frac{n_s^{(\alpha_i)}!(2l_s^{(\alpha_i)}+1)}{\left(n_s^{(\alpha_i)}+l_s^{(\alpha_i)}+\frac{1}{2}\right)!}}\tilde{\sum}_{k_s^{(\alpha_i)}=0}^{n_s^{(\alpha_i)}}\binom{n_s^{(\alpha_i)}+l^{(\alpha_i)}+\frac{1}{2}}{n_s^{(\alpha_i)}-k_s^{(\alpha_i)}}\frac{(-1)^{k_s^{(\alpha_i)}}}{k_s^{(\alpha_i)}!}\right]$$
$$\times \prod_{i=1}^{5} C^{l_1^{(\alpha_i)}l_2^{(\alpha_i)}l_{12}^{(\alpha_i)}}_{0\ 0\ 0}C^{l_3^{(\alpha_i)}l_4^{(\alpha_i)}l_{34}^{(\alpha_i)}}_{0\ 0\ 0}C^{l_{12}^{(\alpha_i)}l_{34}^{(\alpha_i)}l_{(\alpha_i)}}_{0\ 0\ ;\ 0}C^{l_{(\alpha_1)}l_{(\alpha_2)}l_{(\alpha_{12})}}_{0\ 0\ ;0}C^{l_{(\alpha_{12})}l_{(\alpha_3)}l_T}_{0\ 0\ 0}C^{l_{(\alpha_4)}l_{(\alpha_5)}l_D}_{0\ 0\ 0}C^{l_Tl_Dl}_{0\ 0\ 0}. \tag{62}$$

Here $\nu = \nu_{\alpha_1}, \nu_{\alpha_2}, \nu_{\alpha_3}, \nu_{\alpha_4}, \nu_{\alpha_5}, l_{\alpha_{12}}, l_T, l_D$. The tilde above the sum from the above expression stands for the restriction $\sum_{i=1}^{5}\sum_{s=1}^{4}(2k_s^{(\alpha_i)}+l_s^{(\alpha_i)}) = 2k+l$ in performing the summation.

B. Cluster overlaps for ^{20}O decay

According to Eq. (19) from Ref. [7] the cluster overlaps are defined as an expansion in terms of the generalized coefficients of fractional parentage (GCFP) [15, 16],

$$\mathrm{GCFP} = \langle \Psi_{\beta^A_{\omega_{A+a}}}\Psi_{\beta^a_{\omega_{A+a}}} \mid\} \Phi^{I_iK_i\pi_i}_{A+a}\rangle, \tag{63}$$

which is an overlap integral of three totally antisymmetric wave functions with given total spin and parity (eventually isospin). The above functions are defined in terms of the products of single particle oscillator wave functions with the same frequency (here ω_{A+a}). The final nucleus wave function $\mid \Psi^{I_fK_f\pi_f}_A\rangle$ may have, especially for heavy cluster decay, a different oscillator frequency ($\omega_A \neq \omega_{A+a}$). If so, we may expand the $\mid \Psi^{I_fK_f\pi_f}_A\rangle$ in terms of an analogous wave function, but with the oscillator frequency ω_{A+a}. The expansion coefficients $C_\beta = \langle \Psi^{I_fK_f\pi_f}_A \mid \Psi_{\beta^A_{\omega_A}}\rangle \langle \Psi_{\beta^A_{\omega_A}} \mid \Psi_{\beta^A_{\omega_{A+a}}}\rangle$ are given in Eq. (20) of Ref. [7]. Here the $\langle \Psi^{I_fK_f\pi_f}_A \mid \Psi_{\beta^A_{\omega_A}}\rangle$ coefficients depend on the chosen basis $\mid \Psi_{\beta^A_{\omega_A}}\rangle$, while the $\langle \Psi_{\beta^A_{\omega_A}} \mid \Psi_{\beta^A_{\omega_{A+a}}}\rangle$ coefficients are the oscillator frequency dilatation [46] coefficients and they can be expressed in terms of polynomials in $x = (\omega_{A+4}+\omega_A)/(\omega_{A+4}-\omega_A)$ [$I^{(l)}_{nn'} = \langle nlm(\omega_{A+4}) \mid n'lm(\omega_A)\rangle = N^{(l)}_{nn'}(x)\ x^{n'-n}F(-n,-n';-n-n'-l-\frac{1}{2};x^2)$; $N^{(l)}_{nn'}(x) = \sqrt{\frac{\Gamma(n+1)\Gamma(n'+1)}{\Gamma(n+l+\frac{3}{2})\Gamma(n'+l+\frac{3}{2})}}\frac{\Gamma(n+n'+l+\frac{3}{2})}{\Gamma(n+1)\Gamma(n'+1)}\left(\sqrt{1-\frac{1}{x^2}}\right)^{l+\frac{3}{2}}$ [47]]. Thus we may include C_β in the nuclear structure coefficients from Eqs. (36)–(40).

The calculation of the amplitude of the reduced widths can be reduced [7] to the calculation of the matrix elements

$$M_{LM}(K_i\pi_i \to K_f\pi_f) = \langle \tilde{0} \mid \Omega(K_i\pi_i)\Phi^\dagger_{K_f\pi_f} \mid \tilde{0}\rangle, \tag{64}$$

where

$$\Phi^\dagger_{K_f\pi_f} \mid \tilde{0}\rangle = \sum_{\nu_1\cdots\nu_8}\sum_{\tau_1\cdots\tau_8}\sum_{\omega_1\cdots\omega_{12}}\sum_{\sigma_1\cdots\sigma_{12}} A^{LM}_{\tau_1\cdots\tau_8;\sigma_1\cdots\sigma_{12}}(\nu_1\cdots\nu_8 \mid \omega_1\cdots\omega_{12})$$
$$\times a^\dagger_{\nu_1\tau_1}\cdots a^\dagger_{\nu_8\tau_8}a^\dagger_{\omega_1\sigma_1}\cdots a^\dagger_{\omega_{12}\sigma_{12}}\Omega^\dagger(K_f\pi_f) \mid \tilde{0}\rangle. \tag{65}$$

The dominant term in the matrix element (64) contains the favored ^{20}O cluster transitions. The expressions for this term in the case of the transitions between doubly even, odd, and odd-odd mass nuclei are

$$M^f_{LM}(0^+ \to 0^+)_{ee} = \sum_{\nu_1\cdots\nu_4}\sum_{\omega_1\cdots\omega_6}$$
$$\times_{ee}A^{LM}_{+-+-+-+-;+-+-+-+-+-+-}(\nu_1\nu_1\nu_2\nu_2\nu_3\nu_3\nu_4\nu_4 \mid \omega_1\omega_1\omega_2\omega_2\omega_3\omega_3\omega_4\omega_4\omega_5\omega_5\omega_6\omega_6)$$
$$\times \prod_{s=1}^{4}u^f_{\nu_s}v^i_{\nu_s}\prod_{j=1}^{6}u^f_{\omega_j}v^i_{\omega_j}\ , \tag{66}$$

$$M^f_{LM}(K_i\pi_i \to K_f = K_i\pi_f = \pi_i)_{\rm odd}$$
$$= \left[\sum_q C_q(K_i\pi_i)C_q(K_f\pi_f) + \sum_{\lambda\mu i}\sum_{\beta\gamma} D^{\lambda\mu i}_{\beta\gamma}(K_i\pi_i)D^{\lambda\mu i}_{\beta\gamma}(K_f\pi_f)\right]\sum_{\nu_1\cdots\nu_4}\sum_{\omega_1\cdots\omega_6}$$
$$\times_{\rm odd} A^{LM}_{+-+-+-+-;+-+-+-+-+-}(\nu_1\nu_1\nu_2\nu_2\nu_3\nu_3\nu_4\nu_4 \mid \omega_1\omega_1\omega_2\omega_2\omega_3\omega_3\omega_4\omega_4\omega_5\omega_5\omega_6\omega_6)$$
$$\times \prod_{s=1}^{4} u^f_{\nu_s}v^i_{\nu_s}\prod_{j=1}^{6} u^f_{\omega_j}v^i_{\omega_j}\ , \tag{67}$$

$$M^f_{LM}(K_i\pi_i \to K_f = K_i\pi_f = \pi_i)_{\rm odd\text{-}odd}$$
$$= \left[\sum_{q_p} C_{q_pq_n}(K_i\pi_i)C_{q_pq_n}(K_f\pi_f) + \sum_{\lambda\mu i}\sum_{\beta_p\gamma_p\beta_n\gamma_n} D^{\lambda\mu i}_{\beta_p\gamma_p\beta_n\gamma_n}(K_i\pi_i)D^{\lambda\mu i}_{\beta_p\gamma_p\beta_n\gamma_n}(K_f\pi_f)\right]\sum_{\nu_1\cdots\nu_4}\sum_{\omega_1\cdots\omega_6}$$
$$\times_{\rm odd\text{-}odd} A^{LM}_{+-+-+-+-;+-+-+-+-+-}(\nu_1\nu_1\nu_2\nu_2\nu_3\nu_3\nu_4\nu_4 \mid \omega_1\omega_1\omega_2\omega_2\omega_3\omega_3\omega_4\omega_4\omega_5\omega_5\omega_6\omega_6)$$
$$\times \prod_{s=1}^{4} u^f_{\nu_s}v^i_{\nu_s}\prod_{j=1}^{6} u^f_{\omega_j}v^i_{\omega_j}\ . \tag{68}$$

Here the summations are performed within the condition that any two indices should not be equal and, moreover, none of them should be equal either to q or to β in the case of odd mass nuclei and to q_p, q_n, β_p, or to β_n in the case of odd-odd mass nuclei, respectively.

These formulas represent a straightforward generalization of the formulas presented in Ref. [20] that correspond to the favored α transitions. The coefficients $A^{LM}_{\tau_1\cdots\tau_8;\sigma_1\cdots\sigma_{12}}(\nu_1\cdots\nu_8 \mid \omega_1\cdots\omega_{12})$ are the cluster transfer amplitudes analogous to the α transfer amplitudes [see $\Gamma^{\Omega_1\Omega_2\Omega_3\Omega_4}_{LM}$ in Ref. [28] or $A^{LM}_{\tau_1\tau_2,\sigma_1\sigma_2}(\nu_1\nu_2 \mid \omega_1\omega_2)$ in Ref. [20]]. These cluster transfer amplitudes are the intrinsic overlap integrals determined by the ESM single particle basis. Taking into account the expansion of the Nilsson-like wave function in terms of oscillator shell model wave functions ($\chi_{s\sigma} = \sum_{nlj} a^{\Omega_s}_{nlj} \mid nlj\sigma\Omega_s\rangle$), the coefficients $A^{LM}_{\tau_1\cdots\tau_8;\sigma_1\cdots\sigma_{12}}(\nu_1\cdots\nu_8 \mid \omega_1\cdots\omega_{12})$ can be expressed in terms of intrinsic overlap integrals (49) determined by a spherical basis (59). The cluster overlaps for the favored transitions are defined, e.g., in the case of the favored transitions between doubly even nuclei, by $\prod_{s=1}^{4} u^f_{\nu_s}v^i_{\nu_s}\prod_{j=1}^{6} u^f_{\omega_j}v^i_{\omega_j}$.

The explicit expressions of the matrix elements $M^{\rm uf}_{LM}(K_i\pi_i \to K_f\pi_f)$, etc. from Ref. [20] can also be generalized in the case of unfavored ^{20}O cluster transitions; however, we do not reproduce them here.

C. General remarks concerning the calculation of the intrinsic overlap integrals for any cluster decay

The approach presented above for the ^{20}O cluster decay could be generalized for other cluster decays in which the emitted cluster is a spherical doubly even nucleus. For example, in the case of ^{14}C decay, the X_{Nl} matrix entering the intrinsic overlap integral

$$I_{^{14}\rm C}(\alpha\mathbf{R}) = \sum_k C_k\left[\frac{2}{\sqrt{\pi}}\left(\frac{2\alpha\beta_k}{\alpha^2+\beta_k^2}\right)^{\frac{3}{2}}\right]^{13}\sum_N\left[\frac{\alpha^2-\beta_k^2}{\alpha^2+\beta_k^2}\right]^{\frac{1}{2}\left[\sum_{i=1}^{3}\sum_{s=1}^{4}\left(2n_s^{(\alpha_i)}+l_s^{(\alpha_i)}\right)+\sum_{s'=1}^{2}\left(2n_{s'}^{(D)}+l_{s'}^{(D)}\right)-(2N+l)\right]}$$
$$\times X^{^{14}\rm C}_{Nl}(\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}\nu_{\alpha_4}\nu_{\alpha_5}l_{\alpha_{12}}l_Tl_D)\Psi_{Nlm}(\alpha\mathbf{R}) \tag{69}$$

can be built from 3 α-type $X^{\alpha_i}_{Nl}$ matrices, one triton-type X^{T}_{Nl} matrix, and two deuteron-type X^{D}_{Nl} matrices, finally arriving at the following expression:

$$X^{^{14}\rm C}_{Nl}(\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}l_{\alpha_{12}}l_Tn_1^Dl_1^Dn_2^Dl_2^Dl_D) = \sum_{n_{\rho_1^{(\alpha_1)}},l_{\rho_1^{(\alpha_1)}},\ldots,n_\rho,l_\rho}\langle n_{\rho_1^{(\alpha_1)}},l_{\rho_1^{(\alpha_1)}},\ldots,n_\rho,l_\rho,N,L;l \mid \nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3}l_{\alpha_{12}}l_Tn_1^Dl_1^Dn_2^Dl_D;l\rangle$$
$$\times\prod_{i=1}^{3}\prod_{k=1}^{3}\sqrt{\frac{(n_{\rho_k^{(\alpha_i)}}+\frac{1}{2})!}{n_{\rho_k^{(\alpha_i)}}!}}\prod_{m=1}^{2}\sqrt{\frac{(n_{\rho_{T_m}}+\frac{1}{2})!}{n_{\rho_{T_m}}!}}\sqrt{\frac{(n_{\rho_D}+\frac{1}{2})!}{n_{\rho_D}!}}\sqrt{\frac{(n_\rho+\frac{1}{2})!}{n_\rho!}}$$
$$= (-1)^N\left(\frac{\sqrt{\pi}}{2}\right)^{13}\sqrt{\frac{(N+l+\frac{1}{2})!}{N!(2l+1)}}\sum_k\mathcal{B}_k(\nu)\frac{k!}{\sqrt{14}^{l+2k}}\binom{k+l+\frac{1}{2}}{k-N}\ , \tag{70}$$

where

$$\mathcal{B}_k(\nu)=\prod_{i=1}^{3}\prod_{s=1}^{4}\left[\sqrt{\frac{n_s^{(\alpha_i)}!(2l_s^{(\alpha_i)}+1)}{\left(n_s^{(\alpha_i)}+l_s^{(\alpha_i)}+\frac{1}{2}\right)!}}\tilde{\sum}_{k_s^{(\alpha_i)}=0}^{n_s^{(\alpha_i)}}\begin{pmatrix}n_s^{(\alpha_i)}+l^{(\alpha_i)}+\frac{1}{2}\\ n_s^{(\alpha_i)}-k_s^{(\alpha_i)}\end{pmatrix}\frac{(-1)^{k_s^{(\alpha_i)}}}{k_s^{(\alpha_i)}!}\right]$$

$$\times\prod_{s=1}^{2}\left[\sqrt{\frac{n_s^{(D)}!(2l_s^{(D)}+1)}{\left(n_s^{(D)}+l_s^{(D)}+\frac{1}{2}\right)!}}\tilde{\sum}_{k_s^{(D)}=0}^{n_s^{(D)}}\begin{pmatrix}n_s^{(D)}+l^{(D)}+\frac{1}{2}\\ n_s^{(D)}-k_s^{(D)}\end{pmatrix}\frac{(-1)^{k_s^{(D)}}}{k_s^{(D)}!}\right]$$

$$\times\prod_{i=1}^{3}C_{0\ 0\ 0}^{l_1^{(\alpha_i)}l_2^{(\alpha_i)}l_{12}^{(\alpha_i)}}C_{0\ 0\ 0}^{l_3^{(\alpha_i)}l_4^{(\alpha_i)}l_{34}^{(\alpha_i)}}C_{0\ 0\ 0}^{l_{12}^{(\alpha_i)}l_{34}^{(\alpha_i)}l_{(\alpha_i)}}C_{0\ 0\ 0}^{l_{(\alpha_1)}l_{(\alpha_2)}l_{(\alpha_{12})}}C_{0\ 0\ 0}^{l_{(\alpha_{12})}l_{(\alpha_3)}l_T}C_{0\ 0\ 0}^{l_1^{(D)}l_2^{(D)}l_D}C_{0\ 0\ 0}^{l_Tl_Dl}\,. \tag{71}$$

Here $\nu=\nu_{\alpha_1},\nu_{\alpha_2},\nu_{\alpha_3},l_{\alpha_{12}},l_T,l_D$. The tilde above the sum in the above expression stands for the restriction $\sum_{i=1}^{3}\sum_{s=1}^{4}(2k_s^{(\alpha_i)}+l_s^{(\alpha_i)})+\sum_{j=1}^{2}(2k_j^{(D)}+l_j^{(D)})=2k+l$ in performing the summation.

In performing the above integral we have used the transformations from the nucleon coordinates $\mathbf{r}_i$ to the Jacobi (ρ_s) and center-of-mass ($\mathbf{R}$) coordinates

$$\rho_1^{(\alpha_1)}=\frac{\mathbf{r}_1-\mathbf{r}_2}{\sqrt{2}},\quad \rho_2^{(\alpha_1)}=\frac{\mathbf{r}_3-\mathbf{r}_4}{\sqrt{2}},\quad \rho_3^{(\alpha_1)}=\frac{\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4}{2},$$

$$\rho_1^{(\alpha_2)}=\frac{\mathbf{r}_5-\mathbf{r}_6}{\sqrt{2}},\quad \rho_2^{(\alpha_2)}=\frac{\mathbf{r}_7-\mathbf{r}_8}{\sqrt{2}},\quad \rho_3^{(\alpha_2)}=\frac{\mathbf{r}_5+\mathbf{r}_6-\mathbf{r}_7-\mathbf{r}_8}{2},$$

$$\rho_1^{(\alpha_3)}=\frac{\mathbf{r}_9-\mathbf{r}_{10}}{\sqrt{2}},\quad \rho_2^{(\alpha_3)}=\frac{\mathbf{r}_{11}-\mathbf{r}_{12}}{\sqrt{2}},\quad \rho_3^{(\alpha_3)}=\frac{\mathbf{r}_9+\mathbf{r}_{10}-\mathbf{r}_{11}-\mathbf{r}_{12}}{2},$$

$$\rho_{T_1}=\frac{\sum_{n=1}^{4}\mathbf{r}_n-\sum_{m=5}^{8}\mathbf{r}_m}{2\sqrt{2}},\qquad \rho_{T_2}=\frac{\sum_{n=1}^{8}\mathbf{r}_n-2\sum_{m=9}^{12}\mathbf{r}_m}{2\sqrt{6}},$$

$$\rho_D=\frac{\mathbf{r}_{13}-\mathbf{r}_{14}}{\sqrt{2}},\qquad \rho=\frac{2\sum_{i=1}^{12}\mathbf{r}_i-6\sum_{i=13}^{14}\mathbf{r}_i}{\sqrt{84}},\qquad \mathbf{R}=\frac{\sum_{i=1}^{14}\mathbf{r}_i}{\sqrt{14}}, \tag{72}$$

in an analogous way as in the ^{20}O case.

V. α DECAY

By using the enlarged superfluid model (ESM) [11], we calculated the quasiparticle phonon structure of the ground state of the ^{255}Fm nucleus, which emits α and ^{20}O clusters. We calculated also the ground and several excited states of the daughter nuclei ^{251}Cf and ^{235}U, respectively. The results are reproduced in Tables I and II. The structures of these states are very close to the structures given within the quasiparticle phonon nuclear model (see Refs. [36] and [37], respectively).

In calculating the ^{255}Fm and ^{251}Cf excited state structure the ESM parameters used are $G_p=0.14$ MeV, $G_n=0.12$ MeV, and $G_4=0.25$ keV. The parameters of the average field are taken from Ref. [36]. The deformation parameters used are $\beta_{20}=0.26$ and $\beta_{40}=0.035$. They are taken from the calculations of ^{251}Cf static deformations (see Ref. [36]). The particle-hole quadrupole and octupole parameters used [see Eq. (26)] are $\kappa_{n\tau}^{\lambda\mu}=\kappa_{0\tau}^{2\mu}=0.664$ keV fm^{-4}, $\kappa_{n\tau}^{\lambda\mu}=\kappa_{1\tau}^{2\mu}=62.4$ eV fm^{-4}, $\kappa_{n\tau}^{\lambda\mu}=\kappa_{0\tau}^{3\mu}=8.6$ eV fm^{-6}, and $\kappa_{n\tau}^{\lambda\mu}=\kappa_{1\tau}^{3\mu}=1.2$ eV fm^{-6}. The particle-particle quadrupole parameters used [see Eq. (26)] are $G_{n\tau}^{L\lambda\mu}=G_{0\tau}^{L2\mu}=12$ eV fm^{-4}. All the other coupling constants entering Eq. (26) and not mentioned here have been taken equal to zero. For ^{235}U the above parameters are the same except the deformation parameters ($\beta_{20}=0.23$ and $\beta_{40}=0.08$), which are taken from experiment [48,49]. These values are close to the values obtained in the calculations of ^{235}U static deformations (see Ref. [37]).

In the calculated structures we restricted the valence single particle space to 52 proton levels and 52 neutron levels centered around the Fermi levels and the number of the quadrupole and octupole phonons with $\lambda\mu=20,22,30,31,32$ and i=1,2 (see Ref. [11]), following the recipe used within the quasiparticle phonon model developed in Ref. [36].

Within the R-matrix approximation [7] we calculated the HF's for the favored and some unfavored α decays of ^{255}Fm to the ground and some excited states in ^{251}Cf nucleus. The expressions of the reduced widths within the superfluid model are given in Ref. [20]. The results have been compared with the calculations of Ref. [28] and the experimental data [50] [see Table III(a)]. They are not far from our previous calculations [10]. A relatively good agreement with the experimental data is obtained. The data denoted by HF$_{\rm MPR}$ have been obtained by using the reduced widths from Ref. [28]; however, the penetrability ratios and the Fröman-Nosov matrices have been calculated with the $M3Y$ double-folding potential (see Sec. II A). From Tables I and III(a) we conclude that the α decay of the ^{255}Fm ground state to the $\frac{7}{2}^+$ 106.33 keV state of ^{251}Cf can be considered as the favored α transition. The explanation of small (close to unity) HF's in this case is based on the picture according to which

TABLE I. The calculated, within ESM, structure of some ground and excited states entering the α decays of ^{255}Fm(g.s.).

Nucleus	I^π K	$E_{\rm expt}$ (MeV) [63, 54]	$E_{\rm theor}$ (MeV)	Structure
^{255}Fm	$\frac{7}{2}^+$ $\frac{7}{2}$	0.0	0.0	**97.91**% [613] $\frac{7}{2}^+$ + 2.1% [624] $\frac{7}{2}^+$ +2.1% [613] $\frac{7}{2}^+$ Q_{20} + 2.5% [611] $\frac{3}{2}^+$ Q_{22}
^{251}Cf	$\frac{1}{2}^+$ $\frac{1}{2}$	0.0	0.0	**84.21**% [620] $\frac{1}{2}^+$ + 0.04% [631] $\frac{1}{2}^+$ +4.1% [622] $\frac{3}{2}^+$ Q_{22} + 2.5% [620] $\frac{1}{2}^+$ Q_{20}
^{251}Cf	$\frac{7}{2}^+$ $\frac{7}{2}$	0.10633	0.116	**87.23**% [613] $\frac{7}{2}^+$ + 6.04% [624] $\frac{7}{2}^+$ +2.1% [611] $\frac{3}{2}^+$ Q_{22} + 2.5% [725] $\frac{11}{2}^-$ Q_{32}
^{251}Cf	$\frac{3}{2}^+$ $\frac{3}{2}$	0.1777	0.176	**82.92**% [622] $\frac{3}{2}^+$ + 1.09% [611] $\frac{3}{2}^+$ +8.04% [620] $\frac{1}{2}^+$ Q_{22} + 2.5% [752] $\frac{3}{2}^-$ Q_{30}
^{251}Cf	$\frac{11}{2}^-$ $\frac{11}{2}$	0.3704	0.380	**87.88**% [725] $\frac{11}{2}^-$ +6% [613] $\frac{7}{2}^+$ Q_{32} + 4.04% [615] $\frac{9}{2}^+$ Q_{31}
^{235}U	$\frac{7}{2}^+$ $\frac{7}{2}$	0.445716	0.458	**71.03**% [624] $\frac{7}{2}^+$ + 7.09% [613] $\frac{7}{2}^+$ +9.04% [743] $\frac{7}{2}^-$ Q_{30} + 1.05% [725] $\frac{11}{2}^-$ Q_{32}
^{235}U	$\frac{7}{2}^+$ $\frac{7}{2}$	1.236	1.458	**61.02**% [613] $\frac{7}{2}^+$ + 9.09% [624] $\frac{7}{2}^+$ +19.04% [624] $\frac{7}{2}^+$ Q_{20} + 1.05% [725] $\frac{11}{2}^-$ Q_{32}

the cluster (in this case an α particle) is built from the fermions just situated at the Fermi surface, where strong pairing correlations occur and, in addition, one may neglect the differences in structure of the parent and daughter states. On the other hand, for the other α transitions to the intrinsic states [see Table III(a)], the hindrance factors are large, and this is explained by the fact that during the formation process of the α cluster at least one Cooper pair is destroyed and one nucleon from this Cooper pair is coupled with the uncoupled nucleon of the mother nucleus in order to participate in the formation of the α cluster. The theoretical HF's for unfavored α transitions are, as a rule, larger than the experimental ones, except those corresponding to the α transitions to the members of the ^{251}Cf ground state rotational band. Within the ESM picture small spins of the final states are more favored than large ones.

The channel radial regular and irregular wave functions have been calculated by using the Coulomb potential plus the realistic $M3Y$ double-folding potential [51], in which one uses an effective interaction derived from the G-matrix elements based on the Reid soft-core NN potential [52] in the form assuming only one pion exchange potential (OPEP) force between the states with odd relative angular momentum [33]. This potential is obtained numerically, and then is interpolated by cubic spline functions to improve the accuracy of the numerical integration. It is well known that the barrier penetra-

TABLE II. The calculated, within ESM, excitation energies (in keV) of ^{251}Cf and ^{235}U ground and several excited states.

$I_f^{\pi f}$	$E_{\rm expt}^{^{235}{\rm U}}$ [64, 63]	$E_{\rm QPNM}^{^{235}{\rm U}}$ [37]	$E_{\rm ESM}^{^{235}{\rm U}}$	$I_f^{\pi f}$	$E_{\rm expt}^{^{251}{\rm Cf}}$ [64, 63]	$E_{\rm QPNM}^{^{251}{\rm Cf}}$ [36]	$E_{\rm ESM}^{^{251}{\rm Cf}}$
$\frac{7}{2}^-$	0.0	0.0	0.0	$\frac{1}{2}^+$	0.0	0.0	0.0
$\frac{1}{2}^+$	0.0768	5.0	0.2	$\frac{7}{2}^+$	105.73	50.0	116.0
$\frac{5}{2}^+$	129.297	100	140.0	$\frac{3}{2}^+$	177.69	170.0	176.0
$\frac{5}{2}^+$	332.841	310	360.0	$\frac{9}{2}^+$	426.0	370.0	414.0
$\frac{3}{2}^+$	393.211	450	410.0	$\frac{9}{2}^-$	434.0	320.0	474.0
$\frac{7}{2}^+$	445.745	490	458.0	$\frac{11}{2}^-$	370.4	380.0	380.0
$\frac{5}{2}^-$	633.088	300	660.0	$\frac{7}{2}^+$		390.0	421.0
$\frac{3}{2}^-$	637.786	680	663.0	$\frac{1}{2}^-$		630.0	525.0

TABLE III. (a) The calculated, within ESM, hindrance factors for favored, weak unfavored, and unfavored α transitions from ^{255}Fm(g.s.) to the members of the rotational bands of several intrinsic states of ^{251}Cf. (b) The calculated l=0 penetrabilities, penetrability ratios (F_l) for the favored ^{255}Fm(g.s.) $\rightarrow \alpha +$ ^{251}Cf transition, and the inner (r_i) and outer (r_o) turning points (in fm). For further explanations, see the text.

(a)

E_f (keV) [63, 65]	$I_f^{\pi f}$	HF_{expt} [25, 48]	HF_{MPR} [28]	HF_{ESM}	E_f (keV) [63, 65]	$I_f^{\pi f}$	HF_{expt} [25, 48]	HF_{MPR} [28]	HF_{ESM}
106.33	$\frac{7}{2}^+$	1.24	0.62	0.95	0.0	$\frac{1}{2}^+$	4500	1621	2100
166.31	$\frac{9}{2}^+$	12.9	6.34	8.75	24.82	$\frac{3}{2}^+$	2800	1003	1300
239.33	$\frac{11}{2}^+$	52	21.07	28	47.83	$\frac{5}{2}^+$	500	246	265
325.3	$\frac{13}{2}^+$	125	178.3	210	105.7	$\frac{7}{2}^+$	120	416	475
424.1	$\frac{15}{2}^+$	390	544	570	146.5	$\frac{9}{2}^+$	610	399	455
	$\frac{17}{2}^+$				237.7	$\frac{11}{2}^+$	3300	1416	1710
177.7	$\frac{3}{2}^+$	2700	15604	18150	370.4	$\frac{11}{2}^-$	540	3179	3355
211.6	$\frac{5}{2}^+$	2500	14719	17305	442.0	$\frac{13}{2}^-$	840	3704	3950
258.4	$\frac{7}{2}^+$	3300	16084	18955		$\frac{15}{2}^-$		804687	847315
319.4	$\frac{9}{2}^+$	7300	24440	28540		$\frac{17}{2}^-$			

(b)

l	P_l, F_l, r_i, r_0	Rasm.	MPR	$r_0 = 1.2$, $a = 0.5$	$r_0 = 1.2$, $a = 0.7$	$r_0 = 1.3$, $a = 0.7$
0	P_0 (r_i)			0.829×10^{-32}	0.383×10^{-30}	0.389×10^{-30}
0	P_0 (8.25 fm)		0.461×10^{-24}	0.258×10^{-27}	0.304×10^{-27}	0.307×10^{-27}
2	F_2	0.613	0.605	0.438	0.534	0.534
4	F_4	0.195	0.189	0.082	0.102	0.102
6	F_6	0.032	0.031	0.007	0.009	0.009
	r_i		9.4	6.5	6.1	6.1
	r_o		44.8	38.9	38.9	38.9

bilities are very sensitive to assumptions about nuclear potential (radius, slope, etc.) and channel radius. The nuclear densities used in our folding procedure are described by Fermi functions (see Sec. II A and Ref. [32]) with $r_0^{(\alpha)} = 1.2$ fm, and $a^{(\alpha)} = 0.5$ fm. For ^{251}Cf we used different sets of density parameters and concluded [see Table III(b)] that the penetrability ratios (F_l) are relatively stable, i.e., they do not depend very much on the parameters entering the partner densities and they are close to the values given by Rasmussen's formula [see Eq. (5) and Refs. [25, 18]). Since in the expressions of the hindrance factors we need the penetrability ratios (F_l) only, we believe the predictions we give are more credible. In Table III(b) the l= 0 $M3Y$ penetrabilities calculated at the inner turning point and at the channel radius $R_c = 8.25$ fm as suggested in Ref. [28] are given. The penetrability ratios (F_l) entering the hindrance factor for the favored transition are calculated with Rasmussen's formula [see Table III(b), column 4] and as ratios of the penetrabilities for different density parameters of the ^{251}Cf nucleus [see Table III(b), columns 5, 6, and 7). The deformation parameter of the ^{251}Cf density was taken as above, i.e., $\beta_2 = 0.26$. We also report in Table III(b) the values for the inner and outer turning points for the $l = 0$ penetrabilities. Our Fröman-Nosov matrix is very close to the K_{lL}^m obtained in Ref. [28] (see Table IV for the K_{lL}^0 entering the expression of the hindrance factor for the favored transition).

TABLE IV. The calculated Fröman-Nosov matrix for the favored ^{255}Fm(g.s.) $\rightarrow \alpha +$ ^{251}Cf transition.

$l \setminus L$	0	2	4	6	8
0	1.0437	0.3215	0.0576	0.0022	0.0002
2	0.3215	1.3869	0.2351	0.0575	0.0012
4	0.0576	0.2351	1.3122	0.2154	0.0471
6	0.0022	0.0575	0.2154	1.2998	0.1933
8	0.0002	0.0012	0.0471	0.1933	1.2855

The radial scattering wave functions are calculated at the experimental resonance energies using the Numerov algorithm. At a distance of 15 fm the nuclear folding potential V_n makes practically no contribution, and the regular solution is normalized to have the asymptotic behavior of the Coulomb functions [53]. The value of the irregular solution at this distance is obtained from the Wronskian relation, and then the whole irregular solution is obtained integrating backwards to the origin. However, at small distances the fragments interact strongly, and this asymptotic solution is gradually replaced [14, 7] by the "internal" wave function supposed to describe the compound system before decay.

VI. CLUSTER DECAY

We also calculated [see Table V(a)] the favored and weak unfavored ^{20}O cluster transitions from the ^{255}Fm

TABLE V. (a) The calculated, within ESM, hindrance factors for several ^{255}Fm(g.s.) $\rightarrow$ ^{20}O + ^{235}U transitions. (b) The same as Table III(b), but for the favored ^{255}Fm(g.s.) $\rightarrow$ ^{20}O + ^{235}U cluster transition. For further information, see the text.

(a)

Favored band			Weak unfavored band		
E_f (keV) [54]	$I_f^{\pi_f}$	HF_{ESM}	E_f (keV) [54]	$I_f^{\pi_f}$	HF_{ESM}
1236.0	$\frac{7}{2}^+$	≈ 5	445.716	$\frac{7}{2}^+$	≈ 185
	$\frac{9}{2}^+$	≈ 11	509.92	$\frac{9}{2}^+$	≈ 428
	$\frac{11}{2}^+$	≈ 18	587.82	$\frac{11}{2}^+$	≈ 729
	$\frac{13}{2}^+$	≈ 31	682.57	$\frac{13}{2}^+$	≈ 1224

(b)

l	P_l, F_l, r_i, r_o	Rasm.	$r_0 = 1.2$, $a = 0.5$	$r_0 = 1.3$, $a = 0.7$
0	P_0 (r_i)		0.117×10^{-90}	0.149×10^{-90}
0	P_0 (10.41 fm)		0.125×10^{-57}	0.124×10^{-57}
2	F_2	0.809	0.859	0.861
4	F_4	0.494	0.605	0.610
6	F_6	0.227	0.350	0.357
-	r_i		5.7	6.3
-	r_o		27.1	27.1

nucleus to some excited states in the ^{235}U nucleus by using the approach presented in Sec. IV. Due to the fact that a relatively large number of nucleons (eight protons and 12 neutrons) occupies the levels around the Fermi proton and neutron levels, the single particle level [613] $\frac{7}{2}^+$, which has the main contribution in the structure of the ground state of the ^{255}Fm nucleus, can be found at the relatively high excitation energy of the ^{235}U nucleus.

From Table I we learn that within the ESM [11] the structure of the ^{255}Fm ground state contains contributions from two single quasiparticle states, namely, 97.9% [613] $\frac{7}{2}^+$ and 2.1% [624] $\frac{7}{2}^+$ emerging from $1i_{\frac{11}{2}}$ and $2g_{\frac{9}{2}}$, respectively. These states occur also in the structure of ^{235}U excited states lying at 446 keV and 1236 keV excitation energy, respectively (see Table I and Ref. [37]). By using the ESM structure for the above initial and final odd mass nuclei, the expression (1) for the hindrance factor becomes

$$\mathrm{HF}\left[^{255}\mathrm{Fm}(I_i^{\pi_i}K_i) \rightarrow ^{20}\mathrm{O} + ^{235}\mathrm{U}(I_f^{\pi_f} = I_i^{\pi_i}K_f = K_i)\right] \approx \left\{\sum_l F_l \left| C^{I_i\ l\ I_i}_{K_i K K_i} C_{\rho_i}(K_i\pi_i) C_{\rho_f=\rho_i}(K_i\pi_i)(\mathrm{RSA})_{lK}^{(i\rightarrow f)}\right|^2\right\}^{-1}, \quad (73)$$

where

$$(\mathrm{RSA})_{LK}(K_i\pi_i \rightarrow K_f\pi_f) = \frac{\sum_{\nu_1\cdots\nu_4}^{\mathrm{odd}} \sum_{\omega_1\cdots\omega_6}^{\mathrm{odd}} {}_{\mathrm{odd}}A^{LM}_{\mathrm{fav}}(\nu_1\cdot\nu_4 \mid \omega_1\cdots\omega_6) \prod_{s=1}^4 u^f_{\nu_s} v^i_{\nu_s} \prod_{j=1}^6 u^f_{\omega_j} v^i_{\omega_j}}{\sum_{\nu_1\cdots\nu_4}^{ee} \sum_{\omega_1\cdots\omega_6}^{ee} {}_{ee}A^{LM}_{\mathrm{fav}}(\nu_1\cdots\nu_4 \mid \omega_1\cdots\omega_6) \prod_{s=1}^4 u^f_{\nu_s} v^i_{\nu_s} \prod_{j=1}^6 u^f_{\omega_j} v^i_{\omega_j}}, \quad (74)$$

in which ${}_{ee}A^{LM}_{\mathrm{fav}}$ and ${}_{\mathrm{odd}}A^{LM}_{\mathrm{fav}}$ are defined in Eqs. (66) and (67), respectively. $C_{\rho_{i(f)}}$ are the weights of the single quasiparticle state in the structure of the $i(f)$ state. The only difference (see also Ref. [20]) between the cases corresponding to the odd mass and doubly even nuclei is that in the first case the sum in the above equation excludes the common quasiparticle state of both the mother and daughter nuclear states (e.g., $[613]\frac{7}{2}^+$ for ^{255}Fm$\rightarrow^{20}$O+^{235}U).

In the calculations presented in Table V(a) we neglected the collective state contributions in the structure of the initial and final states, which may affect the presented predictions a little. The $l = 0$ penetrabilities calculated at the inner turning point and at the channel radius $R_c = r_0(A_1^{1/3} + A_2^{1/3}) = 10.5$ fm are given [see Table V(b)]. The penetrability ratios (F_l) entering the hindrance factor for the favored transition are calculated with Rasmussen's formula [Table V(b), column 3] and as ratios of the penetrabilities for different parameters for the density of the ^{235}U nucleus [Table V(b), columns 4 and 5]. We also report in Table V(b) the values for the inner and outer turning points for the $l = 0$ penetrability.

The suggestion given in Ref. [54], pages 406 and 407, that the levels lying at 445.71 keV and 509.82 keV should

belong to the rotational band built on the intrinsic state with the structure ground state ⊗ octupole phonon does not fit our results. The suggested structures contribute only 9%, the dominant contribution coming from the single quasiparticle state $[624]\frac{7}{2}^+$.

Unfortunately, the above discussed ^{20}O radioactivity cases have half-lives greater than the maximum half-life ($10^{25.75}$ sec) among the experimentally measured [4] cluster decay half-lives and, hence, would be hard to measure.

The ^{223}Ra and ^{229}Th nuclei belong [55, 56] to the well-known region of soft nuclei with $Z \approx 88$ and $N \approx 134$, with strong octupole correlations in the ground and low lying excited states, where the $1j_{\frac{15}{2}}$ intruder orbital interacts strongly with the $2g_{\frac{9}{2}}$ natural parity orbital. The HF's for both the α and ^{14}C decays of the ground state of ^{223}Ra are very difficult to calculate at present, due to the lack of accurate structure of the mother and daughter nuclei. Studying the experimental HF for α decays to ^{219}Rn ground and low lying excited states [57] we learn that ≈ 15 [57] transitions have small (≤ 100) HF's and of these transitions five have HF's ≤ 10. The corresponding excited states [158.64 keV, $(\frac{3}{2}, \frac{5}{2})^+$, HF = 7.9; 269.48 keV, $(\frac{3}{2}^+, \frac{5}{2}^+)$, HF = 4.5; 338.27 keV, $(\frac{3}{2})^+$, HF = 5.6; 446.83 keV; $(\frac{3}{2})^-$, HF = 7.9; and 515.1 keV; HF = 4.5] have very different structure, and this fact tells us that the structure of the ground state of ^{223}Ra is not as simple as, e.g., the ^{255}Fm case. It may contain many more or less equal components of single quasiparticle or quasiparticle phonon states. Unfortunately, not all the spins and parities of the ^{219}Rn excited states populated by α decay are known. Thus it is a difficult problem to describe the quantum states involved in the α and ^{14}C decay of ^{223}Ra. In our opinion, it is not sufficient to have a description of these states within an independent particle model only [58, 59]. Residual interactions could play an important role [10]. The restrictions concerning the valence single particle space (52 proton levels and 52 neutron levels centered around the Fermi levels) and the number of quasiparticles and phonons (as, e.g., in the case of ^{255}Fm→α+^{251}Cf decay, where only $\lambda\mu$ = 20,22,30,31,32 and i=1,2 phonons have been used), may lead to an inaccurate structure of the ^{223}Ra nucleus. First, the valence single particle space should be extended and, secondly, at the next step, when incorporating the quasiparticle phonon interaction, the number of quasiparticles and phonons should be increased. Such a task is as hard to perform as the calculations within the OXBASH shell model code with realistic residual interactions [7].

Within the ESM the expression for the hindrance factor becomes

$$\mathrm{HF}\left[^{223}\mathrm{Ra}(I_i^{\pi_i}K_i) \rightarrow ^{14}\mathrm{C} + ^{209}\mathrm{Pb}(I_f^{\pi_f}K_f)\right] \approx \left\{\sum_l F_l \left| C^{I_i\ l\ I_f}_{K_i K K_f} C_{\rho_i} C_{\rho_f} a^{\Omega_i=K_i}_{N_i l_i j_i} a^{\Omega_f=K_f}_{N_f l_f j_f} u^f_{K_f\pi_f} v^i_{K_i\pi_i} (\mathrm{RSA})_l^{(i\rightarrow f)} \right|^2 \right\}^{-1} . \tag{75}$$

We may obtain an analogous expression for ^{229}Th(g.s.) → ^{20}O + ^{209}Pb cluster transitions also. Here, the ratio RSA

$$(\mathrm{RSA})_l^{(i\rightarrow f)} = \frac{\sum^{\mathrm{odd}}_{\nu_1\nu_2\nu_3} \sum^{\mathrm{odd}}_{\omega_2\omega_3\omega_4} {}_{\mathrm{odd}}A^{LM}_{\mathrm{uf}}(K_f\pi_f, K_i\pi_i) \prod_{s=1}^3 u^f_{\nu_s} v^i_{\nu_s} \prod_{j=2}^4 u^f_{\omega_j} v^i_{\omega_j}}{\sum^{ee}_{\nu_1\cdots\nu_4} \sum^{ee}_{\omega_1\cdots\omega_6} {}_{ee}A^{LM}_{\mathrm{fav}} \prod_{s=1}^3 u^f_{\nu_s} v^i_{\nu_s} \prod_{j=1}^4 u^f_{\omega_j} v^i_{\omega_j}} \tag{76}$$

of the spectroscopic amplitudes has some analogy with that given in Eq. (74); however it does not contain the Nilsson-like coefficients ($a^{\Omega_i=K_i}_{N_i l_i j_i}$, $a^{\Omega_f=K_f}_{N_f l_f j_f}$) that characterize the single particle contribution in the structure of the initial and final states and the corresponding Bogoliubov-Valatin ($u^f_{K_f\pi_f} v^i_{K_i\pi_i}$) transformation amplitudes. The coefficients ${}_{\mathrm{odd}}A^{LM}_{\mathrm{uf}}(K_f\pi_f, K_i\pi_i)$ and ${}_{ee}A^{LM}_{\mathrm{fav}}$ are generalizations of the α transfer amplitudes [$A^{LM}_{\tau\tau',\sigma\sigma'}(\nu\nu' \mid \omega\omega')$, see Ref. [20]] and contain the overlap integrals given in Eq. (69).

Within such an approximation we calculated the hindrance factors for ^{223}Ra(g.s.) → ^{14}C + ^{209}Pb and ^{229}Th(g.s) → ^{20}O + ^{209}Pb cluster transitions.

In Table VI we reproduce the calculated structure of some states entering the above mentioned cluster transitions. Here $\lambda\mu$ = 20,22,30,31,32 and i=1, 2, and 3 phonons have been used. In calculating the ^{223}Ra and ^{229}Th ground state structure the ESM parameters used are $G_p = 0.14$ MeV, $G_n = 0.10$ MeV, and $G_4 = 0.26$ keV. The parameters of the average field (see Ref. [37]) are V_{0p} = 55.536 98 MeV, $r_{0,p}$ = 1.309 75 fm, a_p = 0.700 71, and $\kappa_{s\text{-}o,p}$= 5.564 79 MeV, V_{0n} = 37.786 83 MeV, $r_{0,n}$ = 1.396 28 fm, a_n = 0.700 71, $\kappa_{s\text{-}o,n}$= 7.319 07 MeV. The deformation parameters used are β_{20} = 0.15 and β_{40} = 0.10. The particle-hole quadrupole and octupole parameters used [see Eq. (26)] are $\kappa^{\lambda\mu}_{n\tau} = \kappa^{2\mu}_{0\tau} = 0.67$ keV fm^{-4}, $\kappa^{\lambda\mu}_{n\tau} = \kappa^{2\mu}_{1\tau} = 0.06$ keV fm^{-4}, $\kappa^{\lambda\mu}_{n\tau} = \kappa^{3\mu}_{0\tau} = 0.01$ keV fm^{-6}, and $\kappa^{\lambda\mu}_{n\tau} = \kappa^{3\mu}_{1\tau} = 1.0$ eV fm^{-6}. The particle-particle quadrupole parameters used [see Eq. (26)] are $G^{\lambda\mu}_{n\tau} = G^{2\mu}_{\tau} = 15$ eV fm^{-4}. All the other coupling constants entering Eq. (26) and not mentioned here have been taken equal to zero. The theoretical HF's together with the experimental ones when available and several intermediate quantities necessary in the calculations are given in Tables VII(a), VII(b), VIII(a), and VIII(b). The explanations for Tables VII(b) and VIII(b) are the same as for Table V(b) given in the text. In the calculations from the last column we have used the density parameters for the ^{209}Pb nucleus indicated in the top. The abbreviation DSPC in Tables VII(a) and VIII(b) means the dominant

TABLE VI. The calculated, within ESM, structure of the ground and some excited states entering the cluster transitions ^{229}Th(g.s.) $\rightarrow$ ^{20}O + ^{209}Pb and ^{223}Ra(g.s.) $\rightarrow$ ^{14}C + ^{209}Pb.

Nucleus	I^π K	E_{expt} [66] (MeV)	E_{theor} (MeV)	Structure
^{229}Th	$\frac{5}{2}^+$ $\frac{5}{2}$	0.0	0.0	**87.91**% [633] $\frac{5}{2}^+$ + 1.1% [622] $\frac{5}{2}^+$ +2.1% [743] $\frac{7}{2}^-$ Q_{31} + 2.5% [631] $\frac{1}{2}^+$ Q_{22}
^{223}Ra	$\frac{3}{2}^+$ $\frac{3}{2}$	0.0	0.0	**78.21**% [631] $\frac{3}{2}^+$ + 2.04% [642] $\frac{3}{2}^+$ +13.1% [752] $\frac{5}{2}^-$ Q_{31} + 2.5% [761] $\frac{3}{2}^-$ Q_{30} +3.1% [631] $\frac{1}{2}^+$ Q_{22} + 2.5% [501] $\frac{1}{2}^-$ Q_{31}
^{209}Pb	$\frac{11}{2}^+$ $\frac{11}{2}$	0.7788	1.116	**97.23**% [606] $\frac{11}{2}^+$ + 1.04% [615] $\frac{11}{2}^+$ +2.1% [743] $\frac{7}{2}^-$ Q_{32} + 2.5% [725] $\frac{11}{2}^-$ Q_{30}
^{209}Pb	$\frac{9}{2}^+$ $\frac{9}{2}$	0.0	0.0	**92.92**% [615] $\frac{9}{2}^+$ + 1.09% [624] $\frac{9}{2}^+$ +1.04% [615] $\frac{9}{2}^+$ Q_{20} + 2.5% [624] $\frac{9}{2}^+$ Q_{20}

single particle configuration.

A few more comments may be in order here. First of all, our ESM is not to be taken too seriously for very complex structures, which would be the case of ^{223}Ra. This would be not true even for structures close to single quasiparticle states, because the assumption of ^{223}Ra as an axially symmetric deformed nucleus seems to be not realistic [56,55]. On the other hand, in order to have realistic structures for both the initial and final nuclear states, calculations within shell model codes like OXBASH or the ESM are practically impossible for present computers. Therefore simple schematic models like that presented above would be useful.

Within the ESM the calculation of the hindrance factor HF=3, experimentally observed [9] in the case of the transition from the ground state of ^{223}Ra ($\frac{3}{2}^+$) to the $\frac{15}{2}^-$, 1423 keV excited state in ^{209}Pb, can be performed by using the parity admixture only [60, 61]. There is an excited level $\frac{3}{2}^-$ in ^{223}Ra, lying at 50 keV excitation energy, which can be admixed [61] in the ground state. We roughly calculated (i.e., assuming a parity-mixed doublet [61]) the admixture coefficient of this first excited state into the ground state of ^{223}Ra by using the technique developed in Ref. [60] and the parity nonconserving potential used in Ref. [62]. This coefficient is found to be of the order of 10^{-5}, but higher lying states could change this value. With this value, the hindrance factor for the above mentioned transition is of the order of 10^6, far away from the experimental value. A more realistic result, however, could be obtained by increasing the number of single particle valence levels and the number of phonons and single quasiparticle states used to describe the structure of the above nuclear states.

TABLE VII. (a) The calculated, within ESM, hindrance factors for two ^{223}Ra(g.s.) $\rightarrow$ ^{14}C + ^{209}Pb transitions. (b) The same as Table III(b), but for the ^{223}Ra(g.s.) $\rightarrow$ ^{14}C + ^{209}Pb cluster transition.

(a)

E_f (keV)	$I_f^{\pi_f}$(DSPC)	$[Nn_z\Lambda]_i$	$[Nn_z\Lambda]_f$	$a_{nlj}^{\Omega_i}$	$a_{nlj}^{\Omega_f}$	C_{Ω_i}	C_{Ω_f}	HF_{expt} [9]	HF_{ESM}
0.0	$\frac{9}{2}^+$ $(2g_{9/2})$	[642]	[615]	0.8	1.0	2%	98%	600.0	$\approx$ 668.0
779.0	$\frac{11}{2}^+$ $(1i_{11/2})$	[631]	[606]	0.8	1.0	78%	97%	3.0	$\approx$ 28.0

(b)

l	P_l, F_l, r_i, r_o	Rasm.	r_0 = 1.2, a = 0.5
0	P_0 (r_i)		0.268×10^{-49}
0	P_0 (10.41 fm)		0.102×10^{-35}
2	F_2	0.728	0.799
4	F_4	0.348	0.485
6	F_6	0.109	0.225
	r_i		5.7
	r_o		22.2

TABLE VIII. (a) The calculated, within ESM, hindrance factors for two ^{229}Th(g.s) $\rightarrow$ ^{20}O + ^{209}Pb transitions. (b) The same as Table III(b), but for the ^{229}Th (g.s.) $\rightarrow$ ^{20}O + ^{209}Pb cluster transition.

(a)

E_f (keV)	$I_f^{\pi_f}$ (DSPC)	$[Nn_z\Lambda]_i$	$[Nn_z\Lambda]_f$	$a_{nlj}^{\Omega_i}$	$a_{nlj}^{\Omega_f}$	C_{Ω_i}	C_{Ω_f}	HF_{ESM}
0.0	$\frac{9}{2}^+$ $(2g_{9/2})$	[633]	[615]	0.72	1.0	1%	98%	≈ 1070
779.0	$\frac{11}{2}^+$ $(1i_{11/2})$	[622]	[606]	0.70	1.0	87%	97%	≈ 20

(b)

l	P_l, F_l, r_i, r_o	Rasm.	$r_o = 1.2$, $a = 0.5$
0	P_0 (r_i)		0.553×10^{-65}
0	P_0 (10.41 fm)		0.602×10^{-47}
2	F_2	0.796	0.858
4	F_4	0.467	0.601
6	6	0.202	0.552
	r_i		5.7
	r_o		21.7

VII. CONCLUSIONS

In this work we reported the results of several calculations performed within the enlarged superfluid model [11], for some selected (favored and weak hindered) α transitions in the ^{255}Fm(g.s.) $\rightarrow$ α + ^{251}Cf process. The α decay mechanism used is derived [21, 10, 20] directly from the R-matrix approach to nuclear reactions, i.e., it is assumed that the α cluster can be found with some probability in the structure of the decaying nucleus. The external wave functions are calculated from a cluster nucleus double-folding model potential obtained with the $M3Y$ interaction. A relatively good agreement with the experimental data has been obtained. An analogous mechanism has been applied for the ^{223}Ra(g.s.) $\rightarrow$ ^{14}C + ^{209}Pb, ^{255}Fm(g.s.) $\rightarrow$ ^{20}O + ^{235}U and ^{229}Th(g.s.) $\rightarrow$ ^{20}O + ^{209}Pb processes. For these three processes the spectroscopic factors are expanded in terms of products of cluster overlaps and intrinsic overlap integrals. Explicit expressions for the cluster overlaps (equivalents of the generalized coefficients of fractional parentage) have been derived. For the intrinsic overlap integrals we construct a model, which is an extension of the usual models for simple particle decay such as deuteron, triton, and α decay. In these cases difficulties arise with increasing complexity of the structure of the nuclear states involved and due to lack of realistic structure for the ^{223}Ra and ^{229}Th ground states.

Within the ESM we overestimate the experimental HF corresponding to the ^{223}Ra(g.s.) $\rightarrow$ ^{14}C + ^{209}Pb process leading to the ground state of ^{209}Pb and to the first excited state of ^{209}Pb. We cannot explain the experimental HF corresponding to the $\frac{15}{2}^-$ (1423 keV) state, but our approach does not use a very large basis of either single particle or phonon states. The HF corresponding to the $\frac{15}{2}^-$, 1423 keV state in ^{209}Pb, within the ESM, has been calculated by using the parity admixture of the $\frac{3}{2}^-$ first excited state in the $\frac{3}{2}^+$ ground state of ^{223}Ra.

Predictions have been made for the hindrance factors corresponding to the following cluster transitions (see Tables III and VII): ^{255}Fm (ground state) $\rightarrow$ ^{20}O + ^{235}U (445.716 keV, 1236 keV, and their rotational bands) and ^{229}Th (ground state) $\rightarrow$ ^{20}O + ^{209}Pb (ground state and $\frac{11}{2}^+$, 779 keV excited state).

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