Analysis of compound and quasicompound resonances in a multichannel, finite-rank model

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Within a multichannel, multirank model of wide applicability, resonances in complex systems are studied. In particular, compound resonances, which are well observed in nuclear systems can be clearly examined as arising from bound states in subsystems of the coupled-channel system. A less well known, and until now little understood, form of resonance is the quasicompound resonance (or structure). These are seen as arising from resonances in subsystems of the multichannel system. The nature of the two types of resonances is contrasted, in their behavior as the strength of coupling increases. Compound resonances arise infinitely narrow, but of fixed height and increase in width, whereas quasicompound structures arise as small "bumps" of finite width, and increase in height. After presentation of the general theory, these properties are illustrated in a simple numerical model calculation.

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I. INTRODUCTION

Resonances play an important role in nuclear physics (as in other branches of physics). Usually the word "resonance" is interpreted in the sense of "single-particle resonance," a quantum effect resulting from a trapping of the wave function inside a potential (which may be purely nuclear or nuclear plus centrifugal and, perhaps, Coulomb). By contrast, the concept of a compound resonance in nuclear physics implies a resonance in a complex nucleus resulting from underlying structure in its subsystems. Compound resonances (or bound states embedded in the continuum) arise out of bound states in subsystems, are typical of a multichannel problem, and are usually very narrow (proportional to the square of the coupling between the elastic channel and the channel carrying the bound state). They have been extensively studied [1], and are also well confirmed as existing in nature, in many nuclear systems. In particular compound resonances have been certainly identified in nucleon-^{12}C low-energy scattering, through a mechanism of excitation of the 2^+, 4.43 MeV state of ^{12}C (see, for example, Ref. [2]). Another kind of resonance in a complex nucleus, a quasicompound resonance, can also result from subsystem structure. Quasicompound resonances, which are due to resonances in subsystems, were originally defined by Beregi and Lovas [3] and have been investigated by others [4,5]. While they have been the subject of theoretical study, they are more difficult to identify in nature.

In this paper, we study compound and quasicompound resonances in a unified and quite general framework. In this way, we are able to compare the behavior of the two types of resonances and see the differences. It is also a framework in which exact numerical calculations are readily performed. While compound resonances have been more extensively studied, both experimentally and theoretically, quasicompound resonances (or structures) have a firm theoretical basis and should also be observable. We give clues how to identify such structures. The model is a multichannel, multirank system that has been developed in detail elsewhere [6]. The model, though simple, has very general features. In particular, the restriction to multirank interactions is no limitation, since it has been demonstrated that any interaction can be reduced to a multirank one by the Sturmian expansion [7] or as in the work of the Gratz group [8].

Section II develops the theoretical framework in detail. Section III deals with the cases of the compund and quasicompound resonances within this framework. Section IV contains a calculation in a simple model system, to illustrate the points made in the previous section about the behavior of the compound and quasicompound resonances. It is possible that a physical example for a quasicompound resonances can be found in the same ^{12}C system as in Ref. [2]. This is a subject for future work. Finally, there are two appendixes with some mathematical details.

II. THEORY

Consider a two-body, multichannel scattering problem, defined by the following multirank potential (represented in momentum space k):

$$V_{cc'}(kk') = \sum_{nn'=1}^{N} \sqrt{k} w_c^n(k) b_{cc'}^{nn'} w_{c'}^{n'}(k')\sqrt{k'}, \qquad (2.1a)$$

where $c = 1, 2, \ldots, C$ is the channel label, and stands for all quantum numbers defining the channel; angular momentum expansion is intended, label l being inside c; $c = 1$ means the elastic channel; $n = 1, 2, \ldots, N$ is the rank label; and w indicates the form factors in momentum space, as defined in Ref. [6].

Since the game is played mainly in space $c \otimes n$ and partially in subspace n, we adopt from now on the shorthand notations outlined in Appendix A. Although a little cumbersome, the notations lead to great formal simplifications in the following. Following these notations, Eq. (2.1a) may be written in compact form as follows:

$$V_{cc'}(kk') = \sqrt{k}\langle w_c(k)|\mathbf{b}_{cc'}|w_{c'}(k')\rangle\sqrt{k'}. \qquad (2.1b)$$

As shown in Ref. [6], the elastic scattering matrix may be written as follows:

$$S_{11} = 1 - 2ia\langle w_1|(\mathbf{M})^{-1}_{11}|w_1\rangle, \qquad (2.2)$$

where i is the imaginary unit and

$$\mathbf{M}_{cc'} = (\mathbf{b}^{-1})_{cc'} - \delta_{cc'}\mu_c \int \frac{|w_c(x)\rangle\langle w_c(x)|}{k_c^2 + i\epsilon - x^2} x\, dx \qquad (2.3a)$$

is the Fredholm matrix, while

$$a_c = \frac{\pi}{2}\mu_c, \quad a_1 = a, \qquad (2.3b)$$

$$\mu_c = \frac{2m_c}{\hbar^2}, \quad \mu_1 = \mu \qquad (2.3c)$$

are real constants, m_c being the reduced mass of the channel. In Eq. (2.3a) and in the following, the outgoing solution is intended if not otherwise stated, namely, $\mathbf{M}_{cc'} = \mathbf{M}^{(+)}_{cc'}$. The channel momentum k_c is defined as follows:

$$k_c = \sqrt{\mu_c \mathcal{E}_c}, \quad k_1 = k, \qquad (2.4a)$$

where

$$\mathcal{E}_c = E - \eta_c, \quad \mathcal{E}_1 = E \qquad (2.4b)$$

is the channel energy, E the total energy, and $\eta_1 = 0, \eta_2, \ldots, \eta_C$ are the threshold energies.

If channel c is open, k_c is real and the complex Fredholm matrix assumes the following expression:

$$\mathbf{M}_{cc'} = \mathbf{P}_{cc'} + \delta_{cc'} i a_c |w_c\rangle\langle w_c|, \qquad (2.5a)$$

$$\mathbf{P}_{cc'} = (\mathbf{b}^{-1})_{cc'} + \delta_{cc'}\mu_c \mathrm{PP} \int \frac{|w_c(x)\rangle\langle w_c(x)|}{x^2 - k_c^2} x\, dx, \qquad (2.5b)$$

where PP means principal part. Here and in the following, when the momentum variable is not explicitly shown, the physical channel momentum is meant, namely,

$$|w_c\rangle = |w_c(k_c)\rangle \; .$$

If channel c is closed, k_c is pure imaginary and the real Fredholm matrix may be written as follows:

$$\mathbf{M}_{cc'} = \mathbf{P}_{cc'} = (\mathbf{b}^{-1})_{cc'} + \delta_{cc'}\mu_c \int \frac{|w_c(x)\rangle\langle w_c(x)|}{x^2 + h_c^2} x\, dx, \qquad (2.6a)$$

$$ih_c = k_c = \sqrt{\mu_c \mathcal{E}_c}. \qquad (2.6b)$$

We are interested in a factorization of S_{11} of the type $S_{11} = N/D$. This can be obtained by extracting in $\mathbf{M}$ the imaginary part of the elastic channel, namely,

$$\mathbf{M}_{cc'} = \mathbf{R}_{cc'} + ia\delta_{c1}|w_c\rangle\langle w_{c'}|\delta_{c'1}, \qquad (2.7)$$

where

$$\mathbf{R}_{11} = \mathbf{P}_{11}, \qquad (2.8a)$$

if $c = c' = 1$, and

$$\mathbf{R}_{cc'} = \mathbf{M}_{cc'}, \qquad (2.8b)$$

otherwise. By writing

$$\delta_{1c}|w_c\rangle = |\Phi_c\rangle$$

or, more explicitly,

$$\delta_{1c} w_c^n = \Phi_c^n,$$

Eq. (2.7) may be put into the form

$$\mathbf{M} = \mathbf{R} + ia|\Phi\rangle\langle\Phi|, \qquad (2.7')$$

which is of the type (A10). We may therefore invert $\mathbf{M}$ by means of Eq. (A11b), and then substitute into Eq. (2.2). After some algebra one gets

$$S_{11} = \frac{1 - ia\langle w_1|(\mathbf{R}^{-1})_{11}|w_1\rangle}{1 + ia\langle w_1|(\mathbf{R}^{-1})_{11}|w_1\rangle}, \qquad (2.9)$$

which is an expression of the desired form. Note that, if only the elastic channel ($c = 1$) is open, $\mathbf{R}$ is real and S_{11} unitary ($|S_{11}|^2 = 1$), as expected.

In order to analyze the resonant behavior of the S matrix (2.9), we make now the fundamental assumption of the weak coupling limit: *All couplings between channels are small of order ϵ, and only terms up to ϵ^2 are retained throughout.* This amounts to writing the strength as follows:

$$\mathbf{b}_{cc'} = \delta_{cc'}\mathbf{B}_c + \epsilon(1 - \delta_{cc'})\mathbf{B}'_{cc'}. \qquad (2.10)$$

By inverting $\mathbf{b}$ up to ϵ^2, and taking into account that the first term of the right-hand side of Eq. (2.10) is diagonal in c [see Eq. (A5a)], one gets

$$(\mathbf{b}^{-1})_{cc'} = \delta_{cc'}\mathbf{B}_c^{-1} - \epsilon(1 - \delta_{cc'})\mathbf{B}_c^{-1}\mathbf{B}'_{cc'}\mathbf{B}_{c'}^{-1} + \epsilon^2 \sum_{c'' \neq cc'} \mathbf{B}_c^{-1}\mathbf{B}'_{cc''}\mathbf{B}_{c''}^{-1}\mathbf{B}'_{c''c'}\mathbf{B}_{c'}^{-1}. \qquad (2.11)$$

Then the matrix $\mathbf{R}$, defined by Eqs. (2.7), (2.8), may be written as follows:

$$\mathbf{R}_{cc'} = \delta_{cc'}\mathbf{R}'_c - \epsilon(1-\delta_{cc'})\mathbf{B}_c^{-1}\mathbf{B}'_{cc'}\mathbf{B}_{c'}^{-1} + \epsilon^2 \sum_{c''\neq cc'} \mathbf{B}_c^{-1}\mathbf{B}'_{cc''}\mathbf{B}_{c''}^{-1}\mathbf{B}'_{c''c'}\mathbf{B}_{c'}^{-1} , \qquad (2.12)$$

where

$$\mathbf{R}'_1 = \mathbf{B}_1^{-1} + \mu \mathrm{PP}\int \frac{|w_1(x)\rangle\langle w_1(x)|}{x^2-k_c^2} x\, dx \equiv \mathbf{P}'_1, \qquad (2.13a)$$

for $c=1$, and

$$\mathbf{R}'_c = \mathbf{B}_c^{-1} + \mu_c \int \frac{|w_c(x)\rangle\langle w_c(x)|}{x^2-k_c^2-i\epsilon} x\, dx \equiv \mathbf{M}'_c, \qquad (2.13b)$$

for any $c \neq 1$. In the latter case note that

$$\mathbf{M}'_c = \mathbf{P}'_c, \qquad (2.13c)$$

is real if c is closed and

$$\mathbf{M}'_c = \mathbf{P}'_c + i a_c |w_c\rangle\langle w_c| \qquad (2.13d)$$

is complex if c is open. It is also worthwhile to note that (i) $\mathbf{R}'_1 = \mathbf{P}'_1$ is the real part of the Fredholm matrix of the elastic channel, in the zero-coupling limit ($\epsilon = 0$) and (ii) $\mathbf{R}'_c = \mathbf{M}'_c$, for any $c \neq 1$ is the Fredholm matrix of the uncoupled c channel.

Inversion of $\mathbf{R}$ up to ϵ^2 gives (remember that, following Appendix A, although we display the term cc', the inversion of $\mathbf{R}$ is meant in the whole space $c \otimes n$, while only for the parts diagonal in c, namely, $\mathbf{R}'$ and $\mathbf{B}$, is the inversion reduced to subspace n)

$$(\mathbf{R}^{-1})_{cc'} = \delta_{cc'}\mathbf{R}'^{-1}_c + \epsilon(1-\delta_{cc'})\mathbf{R}'^{-1}_c\mathbf{B}_c^{-1}\mathbf{B}'_{cc'}\mathbf{B}_{c'}^{-1}\mathbf{R}'^{-1}_{c'} + \epsilon^2 \sum_{c''\neq cc'} \mathbf{R}'^{-1}_c\mathbf{B}_c^{-1}\mathbf{B}'_{cc''}(\mathbf{B}_{c''}^{-1}\mathbf{R}'^{-1}_{c''}\mathbf{B}_{c''}^{-1} - \mathbf{B}_{c''}^{-1})\mathbf{B}'_{c''c'}\mathbf{B}_{c'}^{-1}\mathbf{R}'^{-1}_{c'} . \qquad (2.14)$$

In order to derive the scattering matrix (2.9), we need

$$\langle w_1|(\mathbf{R}^{-1})_{11}|w_1\rangle = \langle w_1|(\mathbf{P}'^{-1})_{11}|w_1\rangle + \epsilon^2 \sum_{c\neq 1} \langle w_1|\mathbf{P}'^{-1}_1\mathbf{B}_1^{-1}\mathbf{B}'_{1c}(\mathbf{B}_c^{-1}\mathbf{M}'^{-1}_c\mathbf{B}_c^{-1} - \mathbf{B}_c^{-1})\mathbf{B}'_{c1}\mathbf{B}_1^{-1}\mathbf{P}'^{-1}_1|w_1\rangle . \qquad (2.15)$$

Note that the term linear in ϵ disappears, because of the factor $1-\delta$ in Eq. (2.12).

It is convenient now to write down the expression of the scattering matrix for any open channel c, in the zero-coupling approximation ($\epsilon = 0$). This can be done by employment of Eqs. (A11a), (A11b), (A12a), (A12b):

$$\bar{S}_{cc} = e^{2i\delta_c} = \frac{\det \mathbf{M}'^*_c}{\det \mathbf{M}'_c} = \frac{1 - i a_c \langle w_c|(\mathbf{P}'^{-1})_c|w_c\rangle}{1 + i a_c \langle w_c|(\mathbf{P}'^{-1})_c|w_c\rangle}. \qquad (2.16a)$$

Here δ_c is the corresponding (uncoupled) phase shift, and $\mathbf{M}'$ the (uncoupled) Fredholm matrix, namely,

$$\det \mathbf{M}'_c = \rho_c e^{-i\delta_c} = \det \mathbf{P}'_c[1 + i a_c\langle w_c|(\mathbf{P}'^{-1})_c|w_c\rangle] = \det \mathbf{P}'_c + i a_c \langle w_c|\hat{\mathbf{P}}'_c|w_c\rangle , \qquad (2.16b)$$

$$\rho_c = \sqrt{(\det \mathbf{P}'_c)^2 + a_c^2 \langle w_c|\hat{\mathbf{P}}'_c|w_c\rangle^2}, \qquad (2.16c)$$

$$\sin\delta_c = -a_c \frac{\langle w_c|\hat{\mathbf{P}}'_c|w_c\rangle}{\rho_c}, \qquad (2.16d)$$

$$\cos\delta_c = \frac{\det \mathbf{P}'_c}{\rho_c}, \qquad (2.16e)$$

where we have introduced the matrix of minors, namely, for any matrix $\mathbf{A}$:

$$\mathbf{A}^{-1} \det \mathbf{A} = \hat{\mathbf{A}}. \qquad (2.16f)$$

Of course, the above expressions hold in particular for the elastic channel $c=1$, which is always open. By substitution of Eq. (2.15) into Eq. (2.9), and introducing the uncoupled elastic phase shift δ_1, the elastic scattering matrix in the weak-coupling approximation may be written as follows:

$$S_{11} = \frac{e^{i\delta_1} - i a\epsilon^2 \cos\delta_1 \sum_{c\neq 1}\langle W_c|(\mathbf{M}'^{-1})_c - \mathbf{B}_c|W_c\rangle}{e^{-i\delta_1} + i a\epsilon^2 \cos\delta_1 \sum_{c\neq 1}\langle W_c|(\mathbf{M}'^{-1})_c - \mathbf{B}_c|W_c\rangle}, \qquad (2.17)$$

where we have introduced the vectors

$$\langle W_c| = \langle w_1|\mathbf{P}'^{-1}_1\mathbf{B}_1^{-1}\mathbf{B}'_{1c}\mathbf{B}_c^{-1}, \qquad (2.18a)$$

$$|W_c\rangle = \mathbf{B}_c^{-1}\mathbf{B}'_{c1}\mathbf{B}_1^{-1}\mathbf{P}'^{-1}_1|w_1\rangle . \qquad (2.18b)$$

Note that in the uncoupled limit ($\epsilon = 0$), $S_{11} = \bar{S}_{11} = e^{2i\delta_1}$ as expected. If all ($c \neq 1$) coupled channels are closed, all $\mathbf{M}'_c$ are real, so that (W_c and B_c being real anyway) $|S_{11}|^2 = 1$ as expected.

III. RESONANCES

A. Compound resonance (the bound state embedded in the continuum)

Let us consider the channel $c = \alpha$. Assume that at the channel energy $\mathcal{E}_{\alpha 0}$ (or equivalently at the total energy

$E_0 = \eta_\alpha + \mathcal{E}_{\alpha 0}$) the channel α is closed ($E_0 < \epsilon_\alpha$, $\mathcal{E}_{\alpha 0} < 0$), and exhibits a proper bound state in the zero-coupling limit. The label 0 is meant to characterize completely the bound state (among others possibly present in the same channel at different energies). This means that

$$\det \mathbf{M}'_\alpha(\mathcal{E}_{\alpha 0}) = \det \mathbf{P}'_\alpha(\mathcal{E}_{\alpha 0}) = 0, \tag{3.1a}$$

where

$$\mathbf{M}'_\alpha(\mathcal{E}_{\alpha 0}) = \mathbf{P}'_\alpha(\mathcal{E}_{\alpha 0}) = (\mathbf{B}^{-1})_\alpha + \mu_\alpha \int \frac{|w_\alpha(x)\rangle\langle w_\alpha(x)|}{x^2 + h_{0\alpha}^2} x\, dx \tag{3.1b}$$

is obviously real and where

$$ih_{\alpha 0} = k_{\alpha 0} = \sqrt{\mu_\alpha \mathcal{E}_{\alpha 0}}. \tag{3.1c}$$

We assume that the level is well insulated, and expand the function $\det \mathbf{M}'_\alpha$ in the neighborhood of $\mathcal{E}_0$, in the spirit of a one-level Breit-Wigner approximation, namely,

$$\det \mathbf{M}'_\alpha(E) = \det \mathbf{P}'_\alpha(E) = \gamma_\alpha(E - E_0), \tag{3.2a}$$

$$\gamma_\alpha = \left[\frac{d}{dE} \det \mathbf{M}'_\alpha\right]_{E=E_0}. \tag{3.2b}$$

Now consider Eq. (2.17), separate the resonant channel α from other channels, introduce the parametrizations (3.2a), (3.2b), and write

$$\sum_{c\neq 1} \langle W_c|(\mathbf{M}'^{-1})_c - \mathbf{B}_c|W_c\rangle = \frac{X_\alpha}{E - E_0} + Z_\alpha, \tag{3.3a}$$

$$X_\alpha = \frac{1}{\gamma_\alpha}\langle W_\alpha|\hat{\mathbf{P}}'_\alpha|W_\alpha\rangle, \tag{3.3b}$$

$$Z_\alpha = \sum_{c\neq 1,\alpha} \langle W_c|(\mathbf{M}'^{-1})_c - \mathbf{B}_c|W_c\rangle - \langle W_\alpha|\mathbf{B}_\alpha|W_\alpha\rangle, \tag{3.3c}$$

where, according to the Breit-Wigner philosophy, all quantities are intended to be calculated at resonance ($E = E_0$).

Substitute Eq. (3.3a) into (2.17), manipulate, and compare with the standard Breit-Wigner-like scattering matrix, namely,

$$S_{11} = S_{11}^{\text{bg}} \frac{E - E_0 - \Delta - i\frac{\Gamma}{2}}{E - E_0 - \Delta + i\frac{\Gamma}{2}}. \tag{3.4}$$

From comparison, the following expressions of the background scattering matrix S_{11}^{bg}, width Γ, and shift factor Δ are obtained:

$$S_{11}^{\text{bg}} \equiv e^{2i\Phi_1} = \frac{e^{i\delta_1} - ia\epsilon^2 \cos\delta_1 Z_\alpha}{e^{-i\delta_1} + ia\epsilon^2 \cos\delta_1 Z_\alpha} = \frac{1 - ia(\langle w_1|\mathbf{P}_1'^{-1}|w_1\rangle + \epsilon^2 Z_\alpha)}{1 + ia(\langle w_1|\mathbf{P}_1'^{-1}|w_1\rangle + \epsilon^2 Z_\alpha)} = \frac{\det \mathbf{M}'^* - ia\epsilon^2 \det \mathbf{P}'_1 Z_\alpha}{\det \mathbf{M}' + ia\epsilon^2 \det \mathbf{P}'_1 Z_\alpha}, \tag{3.5a}$$

$$\Gamma = 2a\epsilon^2 \cos^2\delta_1 X_\alpha = \frac{2a\epsilon^2 X_\alpha}{1 + a^2\langle w_1|(\mathbf{P}'_1)^{-1}|w_1\rangle^2}, \tag{3.5b}$$

$$\Delta = a\epsilon^2 \sin\delta_1 \cos\delta_1 X_\alpha = -\frac{a\epsilon^2 X_\alpha \langle w_1|(\mathbf{P}'_1)^{-1}|w_1\rangle}{1 + a^2\langle w_1|(\mathbf{P}'_1)^{-1}|w_1\rangle^2}. \tag{3.5c}$$

We are interested in the cross section (normalized to 1), namely,

$$\sigma = \frac{1}{4}|1 - S_{11}|^2 . \tag{3.6a}$$

The following should be noted.

(i) In absence of background ($\Phi_1 = 0$), the cross section becomes

$$\sigma = \frac{(\frac{\Gamma}{2})^2}{(E - e_0)^2 + (\frac{\Gamma}{2})^2}, \tag{3.6b}$$

with $e_0 = E_0 + \Delta$, and shows a maximum at $E = e_0$, of magnitude 1, and width Γ.

(ii) If and only if all channels $c \neq 1, \alpha$ are closed, the background is unitary (Φ_1 is real and, in the particular case of 2 channels only, $\Phi_1 = \delta_1$).

(iii) Up to ϵ^2, the "spectator" channels ($c \neq 1, \alpha$) do not influence the width and shift, but only the background (3.5a), through Z_α.

(iv) In the case of real Φ_1, the cross section becomes

$$\sigma = \frac{[(E - e_0)\sin\Phi_1 - \frac{\Gamma}{2}\cos\Phi_1]^2}{(E - e_0)^2 + (\frac{\Gamma}{2})^2} \tag{3.6c}$$

and exhibits a maximum (where $\sigma = 1$) at $E = e_0 - \frac{\Gamma}{2}\tan\Phi_1$ and a minimum (where $\sigma = 0$) at $E = e_0 + \frac{\Gamma}{2}\cot\Phi_1$.

(v) The width and shift are proportional to ϵ^2. The compound resonance starts infinitely narrow at the resonant energy, as the coupling is switched on. Then it shifts and broadens as the coupling increases.

It is interesting to consider the particular case of rank-1 potentials. In this case all matrices in space n become numbers, and we have, in particular,

$$M'_c = P'_c + ia_c w_c^2 = \rho_c e^{-i\delta_c}, \tag{3.7a}$$

$$\rho_c = \sqrt{P_c'^2 + a_c^2 w_c^4}, \tag{3.7b}$$

$$\tan\delta_c = -\frac{a_c w_c^2}{P'_c}, \tag{3.8}$$

$$W_c = \frac{B'_{1c}}{B_1 B_c}\frac{w_1}{P'_1}, \tag{3.9}$$

$$X_\alpha = \frac{W_\alpha^2}{\gamma_\alpha}. \tag{3.10}$$

We obtain finally the width and shift:

$$\Gamma = \left(\frac{B'_{1\alpha}}{B_1 B_\alpha}\right)^2 \frac{2a\epsilon^2 w_1^2}{\gamma_\alpha |M'_{11}|^2}, \tag{3.11a}$$

$$\Delta = -\left(\frac{B'_{1\alpha}}{B_1 B_\alpha}\right)^2 \frac{a^2\epsilon^2 w_1^4}{\gamma_\alpha P'_1 |M'_{11}|^2}, \tag{3.11b}$$

where all functions are calculated at E_0.

The Feshbach theory (see, for example, Ref. [9]) gives in the weak-coupling limit the following general expression:

$$\Gamma = \frac{\pi\mu}{k_0}|\langle\phi_\alpha Q V P \Psi_1^{(+)}\rangle|^2. \tag{3.12}$$

It is interesting to apply Eq. (3.12) to our case, and rederive Eq. (3.11a). In what follows remember that angular momentum expansion is implied, and that the channel index contains all quantum numbers, including l.

The normalized bound-state wave function in the unperturbed channel α, with respect to the momentum coordinate p, is

$$\phi_\alpha(\mathcal{E}_{\alpha 0}, p) = -C_{\alpha 0}\frac{w_\alpha(p)\sqrt{p}}{k_0^2 + p^2}, \tag{3.13a}$$

$$C_{\alpha 0}^2 = \int \frac{w_\alpha^2(q) q\, dq}{(k_0^2 + p^2)^2}. \tag{3.13b}$$

The scattering wave function of the unperturbed elastic channel is

$$\Psi_1(\mathcal{E}_{\alpha 0}, p) = \delta(p-k) + \frac{\mu}{M'_{11}}\frac{w_1(p)w_1(k)\sqrt{pk}}{k^2 + i\epsilon - p^2}. \tag{3.14}$$

The detailed expression of Eq. (3.12) is the following:

$$\Gamma = \frac{\pi\mu}{k_0}\left|\int \phi_\alpha(\mathcal{E}_{\alpha 0}, p) w_\alpha(p)\sqrt{p}\, dp B'_{\alpha 1} w_1(q)\sqrt{q}\, dq \Psi_1^{(+)}(\mathcal{E}_{\alpha 0}, q)\right|^2. \tag{3.15}$$

From Eqs. (3.13a), (3.13b), (3.14), (3.15), Eq. (3.11a) can be found, provided one takes into account that

$$M'_\alpha(E) = \frac{1}{B_\alpha} + \int \frac{w_\alpha^2(p) p\, dp}{\frac{p^2}{\mu_\alpha} + \eta_\alpha - E},$$

and therefore

$$\gamma_\alpha\left[\frac{d}{dE}M'_\alpha\right]_{E_0} = \mu_\alpha^2 \int \frac{w_\alpha^2(p)p\, dp}{(p^2 + h_{\alpha 0}^2)^2} = \left(\frac{\mu_\alpha}{C_{\alpha 0}}\right)^2.$$

Take also into account that, at resonance,

$$M'_\alpha(E_{\alpha 0}) = 0,$$

and therefore

$$\int \frac{w_\alpha^2(p)p\, dp}{p^2 + h_{\alpha 0}^2} = -\frac{1}{\mu_\alpha B_\alpha}.$$

So the identity between our expression (3.11a) and the general Feshbach equation (3.12) is demonstrated.

B. Quasicompound structure

Let us consider the channel $c = \alpha$. Consider the case where, at the channel energy $\mathcal{E}_{\alpha 0}$ (or equivalently at the total energy $E_0 = \eta_\alpha + \mathcal{E}_{\alpha 0}$), the channel α is open ($E_0 > \eta_\alpha$, $\mathcal{E}_{\alpha 0} > 0$) and exhibits a resonance. We assume that the uncoupled S matrix of the resonant channel can be separated in a background times a resonant part, written in the usual Breit-Wigner form, namely,

$$\bar{S}_{\alpha\alpha} = e^{2i\phi_\alpha}\frac{E - E_0 - i\frac{\Gamma_\alpha}{2}}{E - E_0 + i\frac{\Gamma_\alpha}{2}}, \tag{3.16a}$$

$$\det \mathbf{M}'_\alpha(E_0) \propto \left(E - E_0 + i\frac{\Gamma_\alpha}{2}\right)e^{-i\phi_\alpha}. \tag{3.16b}$$

This means that, at resonance,

$$\det \mathbf{M}'_\alpha(E_0) = [\det \mathbf{P}'_\alpha + i a_\alpha \langle w_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle]_{E_0} \propto i\frac{\Gamma_\alpha}{2}e^{-i\phi_\alpha}. \tag{3.17}$$

In the framework of the Breit-Wigner philosophy, to calculate all slowly varying functions at resonance, we may write

$$\cos\phi_\alpha = \frac{a_\alpha\langle w_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle}{\rho_\alpha}, \tag{3.18a}$$

$$\sin\phi_\alpha = \frac{\det \mathbf{P}'_\alpha}{\rho_\alpha}, \tag{3.18b}$$

$$\rho_\alpha = \sqrt{(\det \mathbf{P}')_\alpha^2 + a_\alpha^2\langle w_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle^2}. \tag{3.18c}$$

Now it is convenient to extract in Eq. (2.17) the resonant channel $c = \alpha$, and use the properties (A11a), (A11b), (A12a), (A12b) to write down the fundamental quantity

$$\langle W_\alpha|(\mathbf{M}'^{-1})_\alpha|W_\alpha\rangle = e^{i\phi_\alpha}\frac{X'_\alpha + iY'_\alpha}{E - E_0 + i\frac{\Gamma_\alpha}{2}}, \tag{3.19}$$

$$X'_\alpha = \langle W_\alpha|\hat{\mathbf{P}}'_\alpha|W_\alpha\rangle, \tag{3.20a}$$

$$Y'_\alpha = a_\alpha \frac{\langle W_\alpha|\hat{\mathbf{P}}'_\alpha|W_\alpha\rangle\langle w_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle - \langle W_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle^2}{\det \mathbf{P}'_\alpha}. \tag{3.20b}$$

It is convenient to write

$$e^{i\phi_\alpha}(X'_\alpha + iY'_\alpha) = (X_\alpha + iY_\alpha), \tag{3.21}$$

and after a certain amount of algebra one gets

$$X_\alpha = \frac{a_\alpha\langle W_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle^2}{\rho_\alpha}, \tag{3.22a}$$

$$Y_\alpha = \cot\phi_\alpha\left(\frac{\rho_\alpha\langle W_\alpha|\hat{\mathbf{P}}'_\alpha|W_\alpha\rangle}{a_\alpha\langle w_\alpha|\hat{\mathbf{P}}'_\alpha|w_\alpha\rangle} - X_\alpha\right). \tag{3.22b}$$

Let us consider the case $\phi_\alpha \to 0$, which means no background in the "mother" resonance. In this case, $X_\alpha \to X'_\alpha$, $Y_\alpha \to Y'_\alpha$. Particular care must be taken with Y'_α [Eq. (3.20b)], because without background, $\det \mathbf{P}'_\alpha = 0$ at resonance, and therefore a divergence seems to appear in Y'_α. Nevertheless, it will be shown in the Appendix B that at resonance also the numerator in Eq. (3.20b) vanishes under broad conditions, and Y'_α is therefore finite.

After substitution Eq. (2.17) may be put in the canonical form

$$S_{11} = S_{11}^{\rm bg}\frac{E - e_0 - \Delta + i\frac{\Gamma_r - \Gamma_e}{2}}{E - e_0 + i\frac{\Gamma_r+\Gamma_e}{2}}. \tag{3.23}$$

The meaning of the parameters in Eq. (3.23) is better understood by neglecting the background ($S_{11}^{\rm bg} = 1$):

$$S_{11} = \frac{E - e_0 - \Delta + i\frac{\Gamma_r - \Gamma_e}{2}}{E - e_0 + i\frac{\Gamma_r+\Gamma_e}{2}}, \tag{3.24}$$

and considering the normalized cross section [see Eq. (3.6a)], namely,

$$\sigma = \frac{1}{4}|1 - S_{11}|^2 = \frac{1}{4}\frac{\Delta^2 + \Gamma_e^2}{(E-e_0)^2 + \frac{(\Gamma_e+\Gamma_r)^2}{4}}. \tag{3.25}$$

It is immediately seen that this cross section has a maximum in $E = e_0$, of height

$$\sigma(E = e_0) = \frac{\Delta^2 + \Gamma_e^2}{(\Gamma_e + \Gamma_r)^2} \tag{3.26}$$

and width $\Gamma = \Gamma_e + \Gamma_r$. In this sense e_0 may be defined as the true resonance energy, and following this definition, $e_0 - E_0$ is the shift between the true and the unperturbed resonance. The parameter Δ is a further shift factor, while Γ_e, Γ_r, and Γ have the meaning of elastic, reaction, and total width.

In Eq. (3.23) the background has still the form (3.5a), with the position (3.3c), while by expansions up to ϵ^2, the following expressions are found for all widths and shifts:

$$\Gamma_r = \Gamma_\alpha - 2a\epsilon^2 \sin\delta_1 \cos\delta_1 Y_\alpha, \tag{3.27a}$$

$$\Gamma_e = 2a\epsilon^2\cos^2\delta_1 X_\alpha, \tag{3.27b}$$

$$e_0 - E_0 = a\epsilon^2\cos\delta_1(\sin\delta_1 X_\alpha + \cos\delta_1 Y_\alpha), \tag{3.28a}$$

$$\Delta = -2a\epsilon^2\cos^2\delta_1 Y_\alpha. \tag{3.28b}$$

The following should be noted.

(i) In the zero-coupling limit, the resonant width Γ_r approaches the natural width Γ_α, $\Gamma_r - \Gamma_\alpha$, and Γ_e approaching zero as ϵ^2 [note that Γ_r is positive in the weak-coupling limit, because of Γ_α, while Γ_e is positive by virtue of Eq. (3.22a), (3.27b)]. Both shifts approach zero as ϵ^2.

(ii) The resonant (Breit-Wigner) part of S_{11} is obviously not unitary because at least channel α is open. Let us analyze its behavior in detail. From Eq. (3.26) it is seen that in the weak-coupling limit, the height of the cross section at resonance becomes

$$\sigma(E = e_0) = \left(\frac{\Gamma_e}{\Gamma_\alpha}\right)^2 = \epsilon^4\left(\frac{2aX_\alpha}{\Gamma_\alpha}\right)^2\cos^4\delta_1, \tag{3.29}$$

and is therefore infinitesimal like ϵ^4. This means that in the very-weak-coupling limit the quasicompound structure looks like a small bump in the cross section and not like a resonance (at least in the absence of background, but when the background is added the shape is changed and the maximum is substituted by a maximum-minimum sequence, but the perturbation is still very low as long as the coupling is small).

We may render a little more quantitative these considerations, by saying that Eq. (3.23) describes a "true" resonance if and only if there is an energy near e_0, where ${\rm Im}S_{11} = 0$ and ${\rm Re}S_{11} < 0$, because in this case the phase shift δ_1 crosses the value $\frac{\pi}{2}$. In absence of background it may be easly shown that this may happen if and only if

$$\Gamma_r < \Gamma_e, \tag{3.30}$$

a condition never satisfied in the weak-coupling limit.

(iii) In conclusion, the quasicompound structure starts as a small bump in the cross section, as the coupling is switched on, and increases in height as the coupling increases (quite differently from the compound resonance, which starts as a sharp resonance in the cross section, as the coupling is switched on, and increases in width as the coupling increases).

We give finally the formulas for the particular case of rank-1 potential, namely,

$$X_\alpha = \left(\frac{B_{1\alpha}w_1}{B_1 B_\alpha P'_1}\right)^2\frac{2a_\alpha}{\Gamma_\alpha}, \tag{3.31a}$$

$$Y_\alpha = \left(\frac{B_{1\alpha}w_1}{B_1 B_\alpha P'_1}\right)^2\cot\phi_\alpha\left(\frac{\Gamma_\alpha}{2a_\alpha w_\alpha^2} - \frac{2a_\alpha w_\alpha^2}{\Gamma_\alpha}\right). \tag{3.31b}$$

IV. EXAMPLE: A MODEL CALCULATION

We give here a purely numerical calculation, with the aim of showing the features of the cross section in the

TABLE I. Parameters of the interactions.

Channel	Parameter[a]	Compound resonance	Quasicompound
1	β_1	0.70	0.70
(elastic)	B_1	-20.88406	-20.88406
2	β_2^1	1.0	1.0
	β_2^2	0.5	0.5
	B_2^{11}	1.0	-0.39525
	B_2^{22}	-0.62462	0.39187
threshold	η_2	3.0	1.0
	$\mathcal{E}_{20}$	-1.0	0.9975
	Γ_2	–	0.20
	E_0	2.0	1.9975

[a]The off-diagonal B's [Eq. (2.10)] are all set to 1.0: $B_{12}^{\prime 12} = B_{21}^{\prime 12} = B_{12}^{\prime 21} = B_{21}^{\prime 21} = 1$.

case of compound resonances and quasicompound structures. We assume a two-channel model, S-wave separable potential of rank 1 in channel 1 (the elastic channel that gives the background) and of rank two in channel 2 (this channel carries the "mother" bound state or resonance). The form factors are chosen (for the sake of simplicity) as suggested in Ref. [10], namely,

$$w_1(x) = \sqrt{x}\frac{1}{\beta_1^2 + x^2}, \quad (4.1a)$$

$$w_2^1(x) = \sqrt{x}\frac{1}{(\beta_2^1)^2 + x^2}, \quad (4.1b)$$

$$w_2^2(x) = \sqrt{x}\frac{2\beta_2^2}{[(\beta_2^2)^2 + x^2]^2}. \quad (4.1c)$$

The philosophy is to have the simplest model able to carry the structures whose behavior we plan to analyze. We choose $\mu_c = 1, a_c = \frac{\pi}{2}$ [see Eqs. (2.3b), (2.3c)], and $B_2^{12} = B_2^{21} = 0$; this assumes diagonality in the rank index. As far as the notations are concerned, remember that lower labels refer to channel, while upper labels refer to rank (or dropped if the rank is 1). For the definition of strength parameters in particular, see Eq. (2.10) and Appendix A. The above assumptions lead to the following Fredholm matrices:

$$M_1 = \frac{1}{B_1} + \frac{1}{2\beta_1(\beta_1 - ik)^2}, \quad (4.2a)$$

$$M_2^{11} = \frac{1}{B_2^1} + \frac{1}{2\beta_2^1(\beta_2^1 - ik_2)^2}, \quad (4.2b)$$

$$M_2^{22} = \frac{1}{B_2^2} + \frac{(2\beta_2^2 - ik_2)^2 + (\beta_2^2)^2}{4(\beta_2^2)^2(\beta_2^2 - ik_2)^4}, \quad (4.2c)$$

$$M_2^{12} = M_2^{21} = \frac{\beta_2^1 + 2\beta_2^2 - ik_2}{(\beta_2^1 - ik_2)(\beta_2^2 - ik_2)^2(\beta_2^1 + \beta_2^2)^2}. \quad (4.2d)$$

The range parameters β_c^n [Eqs. (4.1)] and strength parameters B_c^{nn} and $B_{cc'}^{'nn'}$ [Eq. (2.10)] are given in Table I.

The parameters for the compound resonance give rise to a bound state at the energy $\mathcal{E}_{20} = -1.0$ in the (insulated) channel 2. By coupling the two channels together and putting the threshold at the energy $\eta_2 = 3.0$, cross sections [as defined in Eq. (3.6c)] are obtained, as shown in Fig. 1, for increasing coupling values. As expected we get a compound resonance whose position tends to $E_0 = \eta_2 + \mathcal{E}_{20} = 2$, and whose width tends to zero, as the coupling tends to zero. To check the conclusions drawn in the weak-coupling approximation, we have fitted the calculated cross sections (treated as if they were experimental data), by the Breit-Wigner form (3.6c). The

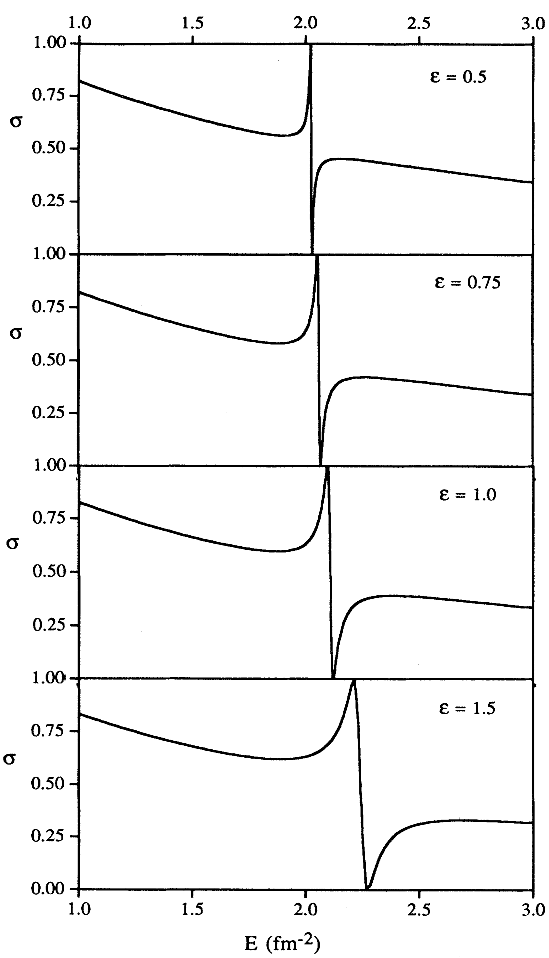

FIG. 1. Compound resonances: cross sections for increasing couplings. Only the lowest four values of ϵ from Table II are shown.

TABLE II. Compound resonances, Breit-Wigner parameters.

ϵ	Δ	$\frac{\Delta}{\epsilon^2}$	Γ	$\frac{\Gamma}{\epsilon^2}$	Φ_1 (degree)
0.5	0.0278	0.1112	0.0067	0.0268	45.8
0.75	0.0585	0.1040	0.0103	0.0183	45.8
1	0.1096	0.1096	0.0268	0.0268	44.6
1.5	0.2409	0.1071	0.0591	0.0263	42.9
2	0.4139	0.1035	0.1008	0.0252	41.2

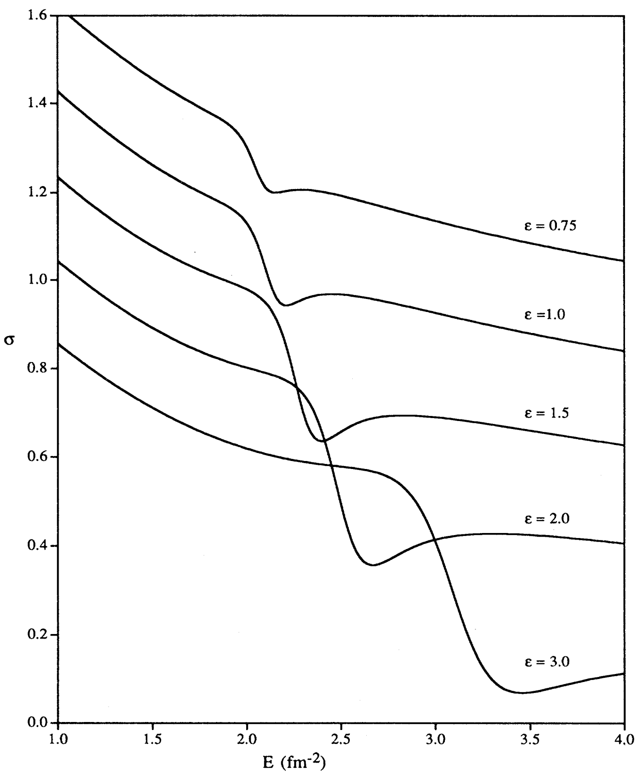

FIG. 2. Quasicompound structure: cross sections for increasing couplings. The curves for different ϵ are arbitrarily shifted relative to each other along the ordinate by 0.2 units, for clarity, the $\epsilon = 3$ case being unshifted. Only the lowest five values of ϵ from Table III are shown.

results are shown in Table II. In the fitting, the background Φ_1 was free to vary, but the fitted values show a slow and smooth variation. The proportionality of Γ and Δ to ϵ^2 is checked approximately up to $\epsilon = 2$.

For the quasicompound structure, the choice of parameters (Table I) gives rise to a (mother) single-particle resonance at the energy $\mathcal{E}_{20} = 0.9975$ and width $\Gamma_2 = 0.20$, in the (insulated) channel 2. We assume now $\eta_2 = 1$, in order to have the unperturbed quasicompound structure still at energy $E = 2$ (more precisely, at $E_0 = \eta_2 + \mathcal{E}_{20} = 1.9975$).

The quasicompound structures for increasing coupling values are shown in Fig. 2. From a comparison of Figs. 1 and 2 the different genesis of compound and quasicompound structures is clearly seen: the first ones becoming very narrow at constant height and the second very small at constant width, as the couplings tend to zero. These cross sections may be analyzed by a Breit-Wigner form of the type (3.23) (with $|S_{11}^{\rm bg}| = 1$, since we have only two channels in our model), and the results are shown in Table III. From observation of the table, one can draw the following conclusions.

(i) The derivation of the Breit-Wigner parameters by fitting the cross section seems to be much more difficult than in the previous case.

(ii) The fit begins at $\epsilon = 0.75$ because at smaller couplings, the determination of the Breit-Wigner parameters seems to be not reliable.

(iii) The shift Δ is small and difficult to determine from the fit; its behavior is not significant.

(iv) $E_0 - e_0$, Γ_e, $\Gamma_r - \Gamma_e$ seem to maintain approximate proportionality to ϵ^2 up to high coupling values.

(v) It is finally important to observe that, in the considered example, from $\epsilon = 3.0$ to $\epsilon = 4.0$, we have the transition from a bump to a "true" resonance, in the sense defined above. Figure 3 shows the Dalitz plot for $\epsilon = 3.0$ [Fig. 3(a)] and $\epsilon = 4.0$ [Fig. 3(b)], respectively. Only in the second case does the phase shift cross the value $\frac{\pi}{2}$. For lower coupling values the plot is similar to that of Fig. 3(a). It is immediately seen, from Table III, that from $\epsilon = 3.0$ to $\epsilon = 4.0$, we have the inversion between Γ_e and Γ_r (Γ_e becomes greater than Γ_r). It seems therefore that the rule given in Eq. (3.30) holds in general, in spite of the fact that it was demonstrated only in the case of no background.

TABLE III. Quasicompound states, Breit-Wigner parameters.

ϵ	$e_0 - E_0$	$\frac{e_0-E_0}{\epsilon^2}$	Δ	$\frac{\Delta}{\epsilon^2}$	Γ_e	$\frac{\Gamma_e}{\epsilon^2}$	Γ_r	$\frac{(\Gamma_r-\Gamma_2)}{\epsilon^2}$	Φ_1 (degree)
0.75	0.0635	0.1128	0.01003	0.0178	0.03631	0.0645	0.2810	0.1439	45.9
1.0	0.1243	0.1243	0.01028	0.0103	0.05487	0.0549	0.2734	0.0734	45.5
1.5	0.2724	0.1211	0.02350	0.0104	0.09867	0.0438	0.2797	0.0354	42.2
2.0	0.5035	0.1259	0.01935	0.0048	0.16180	0.0404	0.2971	0.0243	40.7
3.0	1.1124	0.1236	0.02252	0.0025	0.36013	0.0400	0.3908	0.0212	36.5
4.0	1.9206	0.1200	0.01020	0.0006	0.65650	0.0410	0.5165	0.0198	32.8

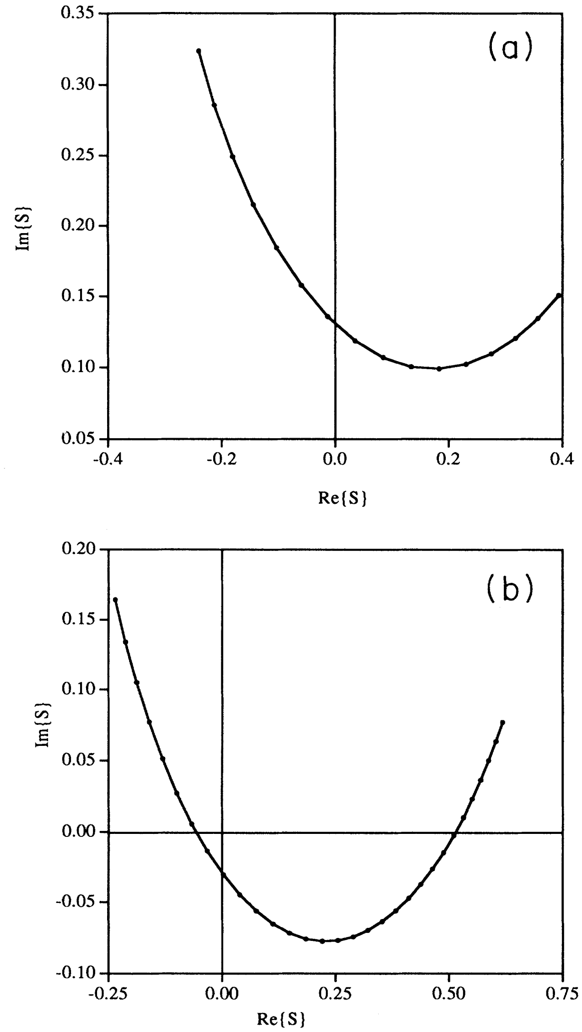

FIG. 3. Dalitz plot of the scattering matrices. In (a) the (nonresonant) S matrix for $\epsilon = 3.0$ is plotted for equally spaced energies, from E=3.0 up to E=3.3. In (b) the (resonant) S matrix for $\epsilon = 4.0$ is plotted for equally spaced energies, from E=3.8 up to E=4.0.

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APPENDIX A

Let $\mathbf{A}$ be a matrix in space $c\otimes n$, whose generic matrix element is indicated by $\mathbf{A}_{cc'}^{nn'}$. We can write matrix $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & .. \\ \mathbf{A}_{21} & \mathbf{A}_{22} & .. \\ .. & .. & .. \end{pmatrix}, \tag{A1a}$$

where

$$\mathbf{A}_{cc'} = \begin{pmatrix} A_{cc'}^{11} & A_{cc'}^{12} & .. \\ A_{cc'}^{21} & A_{cc'}^{22} & .. \\ .. & .. & .. \end{pmatrix} \tag{A1b}$$

is a matrix in subspace n. The inverse of $\mathbf{A}$ (in the complete space $c \otimes n$) may be written as

$$\mathbf{A}^{-1} = \begin{pmatrix} (\mathbf{A}^{-1})_{11} & (\mathbf{A}^{-1})_{12} & .. \\ (\mathbf{A}^{-1})_{21} & (\mathbf{A}^{-1})_{22} & .. \\ .. & .. & .. \end{pmatrix}, \tag{A2a}$$

where

$$(\mathbf{A}^{-1})_{cc'} = \begin{pmatrix} (\mathbf{A}^{-1})_{cc'}^{11} & (\mathbf{A}^{-1})_{cc'}^{12} & .. \\ (\mathbf{A}^{-1})_{cc'}^{21} & (\mathbf{A}^{-1})_{cc'}^{22} & .. \\ .. & .. & .. \end{pmatrix} \tag{A2b}$$

and where inversion is always intended in the complete space. This is obviously different from

$$(\mathbf{A}_{cc'})^{-1} = \begin{pmatrix} [(\mathbf{A}_{cc'})^{-1}]^{11} & [(\mathbf{A}_{cc'})^{-1}]^{12} & .. \\ [(\mathbf{A}_{cc'})^{-1}]^{21} & [(\mathbf{A}_{cc'})^{-1}]^{22} & .. \\ .. & .. & .. \end{pmatrix}, \tag{A3}$$

which is the inverse of $\mathbf{A}_{cc'}$ in subspace n.

If $\mathbf{A}$ is diagonal in space c, namely,

$$\mathbf{A}_{cc'} = \delta_{cc'}\mathbf{A}_{cc} \equiv \delta_{cc'}\mathbf{A}_c \tag{A4a}$$

or, equivalently,

$$A_{cc'}^{nn'} = \delta_{cc'}A_{cc}^{nn'} \equiv \delta_{cc'}A_c^{nn'}, \tag{A4b}$$

the inversion operation reduces from space $c\otimes n$ to space n, namely,

$$(\mathbf{A}^{-1})_{cc'} = \delta_{cc'}(\mathbf{A}^{-1})_{cc} \equiv \delta_{cc'}(\mathbf{A}^{-1})_c \tag{A5a}$$

or, equivalently,

$$(\mathbf{A}^{-1})_{cc'}^{nn'} = \delta_{cc'}[(\mathbf{A}^{-1})_{cc}]^{nn'} \equiv \delta_{cc'}[(\mathbf{A}^{-1})_c]^{nn'}. \tag{A5b}$$

Then we define the row vector in space $c \otimes n$,

$$\langle u| = (\langle u_1|\ \langle u_2|\ \cdots), \tag{A6a}$$

whose component

$$\langle u_c| = (u_c^1\ u_c^2\ \cdots) \tag{A6b}$$

is a row vector in space n. Similar definitions hold for column vectors, namely,

$$|u\rangle = \begin{pmatrix} |u_1\rangle \\ |u_2\rangle \\ \cdot \end{pmatrix}, \tag{A7a}$$

$$|u_c\rangle = \begin{pmatrix} u_c^1 \\ u_c^2 \\ \cdot \end{pmatrix}. \tag{A7b}$$

By these definitions the bracket

$$\langle u|v\rangle = \sum_c \langle u_c|v_c\rangle = \sum_{cn} u_c^n v_c^n \tag{A8}$$

is a scalar, while the ket-bra

$$|u\rangle\langle v| = \begin{pmatrix} |u_1\rangle\langle v_1| & |u_1\rangle\langle v_2| & .. \\ |u_2\rangle\langle v_1| & |u_2\rangle\langle v_2| & .. \\ .. & .. & .. \end{pmatrix} \tag{A9a}$$

is a matrix in space $c \otimes n$, and the ket-bra

$$|u_c\rangle\langle v_{c'}| = \begin{pmatrix} u_c^1 v_{c'}^1 & u_c^1 v_{c'}^2 & .. \\ u_c^2 v_{c'}^1 & u_c^2 v_{c'}^2 & .. \\ .. & .. & .. \end{pmatrix} \tag{A9b}$$

is a matrix in subspace n.

It is easy to recognize that both $|u\rangle\langle v|$ and $|u_c\rangle\langle v_{c'}|$, are rank-1 matrices in the space of definition.

We remember a useful theorem derived in Ref. [6], which may be formulated as follows.

Let

$$\mathbf{A} = \mathbf{B} + |u\rangle\langle v|, \tag{A10}$$

where $\mathbf{B}$ is assumed nonsingular and $|u\rangle\langle v|$ has rank 1, according to the above definition. Then the determinant and the inverse of $\mathbf{A}$ may be written as follows:

$$\det \mathbf{A} = \det \mathbf{B}(1 + \langle v|\mathbf{B}^{-1}|u\rangle), \tag{A11a}$$

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}|u\rangle\langle v|\mathbf{B}^{-1}}{1 + \langle v|\mathbf{B}^{-1}|u\rangle} = \mathbf{B}^{-1}\frac{(1 + \langle v|\mathbf{B}^{-1}|u\rangle)\mathbf{1} - |u\rangle\langle v|\mathbf{B}^{-1}}{1 + \langle v|\mathbf{B}^{-1}|u\rangle}. \tag{A11b}$$

By introducing the cofactors as in Eq. (2.16b), we may write, alternatively,

$$\det \mathbf{A} = \det \mathbf{B} + \langle v|\hat{\mathbf{B}}|u\rangle, \tag{A12a}$$

$$\hat{\mathbf{A}} = \hat{\mathbf{B}} + \frac{\langle v|\hat{\mathbf{B}}|u\rangle\hat{\mathbf{B}} - \hat{\mathbf{B}}|u\rangle\langle v|\hat{\mathbf{B}}}{\det \mathbf{B}}. \tag{A12b}$$

APPENDIX B

Let $\mathbf{A}$ be a real and symmetric matrix in space n, and $|V\rangle, |v\rangle$ ($\langle V|, \langle v|$) row (column) real vectors in the same space, as defined in the Appendix A. Consider the orthogonal transformation $\mathbf{T}$ which diagonalizes $\mathbf{A}$, namely,

$$\mathbf{A} = \tilde{\mathbf{T}}\mathbf{a}\mathbf{T}, \tag{B1a}$$

$$a_{nn'} = \delta_{nn'} a_n, \tag{B1b}$$

$$\det \mathbf{A} = \det \mathbf{a} = \Pi_n a_n, \tag{B1c}$$

where $\tilde{\mathbf{T}}$ means transpose and Π means product.

Let us write down the inverse matrix

$$\mathbf{A}^{-1} = \tilde{\mathbf{T}}\mathbf{a}^{-1}\mathbf{T}, \tag{B2a}$$

$$(\mathbf{a}^{-1})_{nn'} = \delta_{nn'} a_n^{-1}, \tag{B2b}$$

$$\det \mathbf{A}^{-1} = \det \mathbf{a}^{-1} = (\Pi_n a_n)^{-1}, \tag{B2c}$$

and the matrix of cofactors,

$$\hat{\mathbf{A}} = \tilde{\mathbf{T}}\hat{\mathbf{a}}\mathbf{T}, \tag{B3a}$$

$$\hat{a}_{nn'} = \delta_{nn'} \hat{a}_n, \tag{B3b}$$

$$\hat{a}_n = \frac{\det \mathbf{a}}{a_n} = \Pi_{n' \neq n} a_{n'}. \tag{B3c}$$

Then apply the same transformation to vectors, namely,

$$\mathbf{T}|V\rangle = |F\rangle, \tag{B4a}$$

$$\mathbf{T}|v\rangle = |f\rangle. \tag{B4b}$$

Now we are able to calculate all matrix elements, namely,

$$\langle V|\hat{\mathbf{A}}|V\rangle = \sum_n F_n^2 \Pi_{n' \neq n} a_{n'}, \tag{B5a}$$

$$\langle V|\hat{\mathbf{A}}|v\rangle = \sum_n F_n f_n \Pi_{n' \neq n} a_{n'}, \tag{B5b}$$

$$\langle V|\hat{\mathbf{A}}|v\rangle = \sum_n f_n^2 \Pi_{n' \neq n} a_{n'} \ . \tag{B5c}$$

Then it is easy to demonstrate that

$$Y = \frac{\langle V|\hat{\mathbf{A}}|V\rangle\langle v|\hat{\mathbf{A}}|v\rangle - \langle V|\hat{\mathbf{A}}|v\rangle^2}{\det \mathbf{A}} = \sum_{n,n'} F_n f_{n'}(F_n f_{n'} - F_{n'} f_n)\Pi_{n'' \neq n,n'} a_{n''}. \tag{B6}$$

Let $\det \mathbf{A} = 0$ [see Eq. (B1c)], and let this be a simple pole (this means that the eigenvelues are not degenerate, and this is true since $\mathbf{A}$ is assumed real and symmetric). Then assume $a_m = 0$ and $a_n \neq 0$ for all $n \neq m$. Then

$$Y = \sum_n F_m f_n (F_m f_n - F_n f_m)\Pi_{n' \neq m,n} a_{n'} \tag{B7}$$

is clearly nonsingular.

[1] L. Fonda and R. G. Newton, Ann. Phys. (N.Y.) **10**, 490 (1960); R. G. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, Berlin, 1982).
[2] A. Pascolini, G. Pisent, and F. Zardi, Nuovo Cimento Lett. **1**, 643 (1969); O. Mikoshiba, T. Teresawa, and M. Tanifugi, Nucl. Phys. **A168**, 417 (1971).
[3] P. Beregi and I. Lovas, Z. Phys. **241**, 410 (1971).
[4] G. Cattapan, L. Canton, and G. Pisent, Phys. Rev. C

43, 1395 (1991).
[5] J. P. Svenne, T. A. Osborn, G. Pisent, and D. Eyre, Phys. Rev. C **40**, 1136 (1989).
[6] G. Pisent and L. Canton, Nuovo Cimento A **91**, 33 (1986).
[7] L. Canton, G. Cattapan, and G. Pisent, Nuovo Cimento A **97**, 319 (1987); Nucl. Phys. **A487**, 333 (1988).
[8] J. Haidenbauer and W. Plessas, Phys. Rev. C **30**, 1822 (1984); C **32**, 1424 (1985); J. Haidenbauer, Y. Koike, and W. Plessas, *ibid.* **33**, 439 (1986).
[9] R. H. Lemmer, Rep. Prog. Phys. **29**, 131 (1966).
[10] G. Pisent and J. P. Svenne, Nuovo Cimento A **97**, 51 (1987).