Center of mass of a few-body quantum system in an effective central field

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The center of mass (CM) of a few-body quantum system with a central field is discussed. If the particles are in the designative eigenstates, the CM coordinates of the system can be well defined. In the CM bag model as well as in other models with central fields, the CM-freedom separation rule and effective nucleon electro magnetic currents can be presented without any undetermined parameter.

It is well known that the static properties of hadrons can be explained by the nonrelativistic constituent quark models [1], in which a baryon consists of three confined valence, constituent, quarks. The pattern of the hadron spectrum fits well in the symmetry group $SU(3)_{\text{flavor}} \otimes SU(2)_{\text{spin}}$, together with the $O(3)$-nonrelativistic oscillator spatial wave functions. In this case, the center-of-mass (CM) motion can be well separated from the internal relative motion. However, the motion of the light quarks, e.g., up and down quarks in the nucleon, is highly relativistic because the kinetic energy of a quark is almost the same as its constituent mass. Hadron models, which can account for relativistic motion of the quarks, lost the advantage of the nonrelativistic quark model due to the intrinsic nonseparability of the center-of-mass motion for a relativistic three-body system. For the MIT bag model [2], the CM is at rest and the bag is static, hence the CM degree of freedom has been completely disregarded. This does not matter for describing the mass spectroscopy of the hadrons, but does matter for the hadron decay and scattering processes where the recoil effect, hence the CM motion, cannot be neglected. One of the consequences of neglecting the CM degree of freedom is that the translational invariance and four-momentum conservation are lost [3].

To recover the four-momentum conservation, Barnhill III has made a proposal [4] that the wave function of a three-quark system has a collective plane-wave factor:

$$\phi_P(y) = \frac{1}{(2\pi)^{3/2}} e^{-iP \cdot y}, \qquad (1)$$

where $y^\mu = (T, \mathbf{y})$ is the center-of-mass (CM) coordinate and P^μ is the total four-momentum of the system. In [5] we have modified and generalized this assumption and developed a formalism for the calculation of hadron structure functions and electromagnetic form factors [6,7]. We also provided a field-theory basis of this idea and proposed the so-called CM bag model [8]. Some Feynman rules and their applications, which includes a possible explanation of the nuclear EMC effect, were presented [9,10]. Recently, considering the symmetry-breaking ef-

fects coming from the spin-dependent quark-quark interactions, the magnetic moments of baryons, elastic form factors, and deep inelastic structure functions of the nucleon are calculated [11]. We note that there is another approach in the literature to avoid the disadvantage of the original MIT bag model, e.g., see Ref. [12]. In this approach, the Peirels-Yoccoz projection [13] is used to obtain an eigenstate of zero momentum.

In Ref. [8], we gave a brief discussion of how we introduce the plane-wave function for the CM degrees of freedom. The CM freedom separation rule is (see Eq. (20b) in Ref. [8]):

$$\Psi(x_1, x_2, x_3) = \sum_n \int d^3 P \lambda^{3/2} a_P \frac{e^{-iP \cdot y}}{(2\pi)^{3/2}} b_n \, q_{np}(\xi_1, \xi_2, \xi_3),$$

$$(2)$$

where λ is an undetermined factor of dimension $[\lambda] = L$ (length).

In this paper, we will discuss in more detail about our basic assumptions on the four-potential V_μ, the restrictions on the states of particles, the definition, and the *revised* separation rule of the CM coordinates y^μ for a nonrelativistic or relativistic N-particle quantum system. The parameter λ does not appear in our revised separation rule, nor in the revised effective electromagnetic current of the nucleon. It appears only when we use the free-quark approximation for outgoing quarks in deep inelastic collisions. Our method can be applied to other hadron or nuclear models with central fields.

I. THE FOUR-POTENTIAL AND THE FOUR-MOMENTA

Our main assumption is that there are N particles moving in an effective four-potential: $V_a^\mu \equiv V^\mu(x_a)$, which is reduced to a static central field in a CM-rest reference frame (CMRF), i.e., where the total three-momentum of the particle system vanishes. The center of the field is always located at the CM position $\mathbf{y}$. There-

fore, in a CMRF, we can write

$$V_a^\mu \equiv V^\mu(r_a) = [V(r_a), \mathbf{0}] \;, \tag{3}$$

$$r_a = |x_a - y|_{t_a=T} = |\mathbf{x}_a - \mathbf{y}| \equiv |\xi_a| \;. \tag{4}$$

Here we put $t_a = T$, because r_a are proper lengths and the events x_a and y always have to be simultaneously measured in the CMRF. This implies that, whatever the definition of the CM four-vector $y^\mu = (T, \mathbf{y})$ is, $T = T(t_1, ..., t_N)$ has to contain the following special solution:

$$t_1 = t_2 = \cdots = t_N = T \quad \text{in the CMRF} \;. \tag{5}$$

If it is a nonrelativistic N-body system, this of course must be true. Moreover, when we say there is a central field centered at the CM position, we mean that the CM position is fixed, i.e., $\mathbf{y} = \langle \mathbf{y} \rangle$. Since y^μ is a four-vector, we have the second requirement on CM:

$$y^\mu = \langle y^\mu \rangle \equiv Y^\mu \;. \tag{6}$$

To be consistent with our assumption on V^μ, we have to put some restrictions on the states of particles. For example, to make the effective interaction field static in a CMRF, we should only consider the stationary states or energy eigenstates. Moreover, because the effective field is isotropic, it is reasonable to restrict to states with zero expectation value of $x_a - y$, i.e., to eigenstates of parity. We call such states *designative states*.

In quantum mechanics, when we use the coordinate representation (or, the Schrödinger picture), the four-momentum becomes an operator, $\hat{p}_\mu \equiv i\partial_\mu$, and we have the commutators: $[x_a^\mu, \hat{p}_{b\,\nu}] = -i\delta_\nu^\mu \delta_{ab}$. In a given state $\psi(x)$, we have the expectation values of the momentum:

$$\bar{p}_\mu = \int d^3x\, \psi^*(x) i\partial_\mu \psi(x) \;.$$

The eigenvalue of total four-momentum $\hat{P}_\nu$ of the system is denoted by

$$P^\mu = \left(\sum_a \epsilon_a, \sum_a \mathbf{p}_a \right) = (E, \mathbf{P}) \;. \tag{7}$$

$\hat{P}_\nu$ and y^μ must be mutually canonically conjugate, or

$$\left[y^\mu, \hat{P}_\nu \right] = -i\delta_\nu^\mu \;. \tag{8}$$

In the Schrödinger picture, $\hat{P}^0 = \sum_a \hat{p}_a^0 = \sum_a i\partial_{t_a}$, we have the third requirement on the definition of CM time T, i.e., for any function $F(T)$,

$$\partial_T F(T) = \sum_a \partial_{t_a} F(T(t_1, \dots, t_N)) \;. \tag{9}$$

Now let $F(T) = T$, then Eq. (9) shows that T must be linear in t_a, with $y^0 = T = \sum_a \eta_a t_a$ where $\sum_a \eta_a = 1$. Now that y^μ is a four-vector, we have

$$y^\mu = \sum_a \eta_a x_a^\mu, \qquad \sum_a \eta_a = 1 \;, \tag{10}$$

where η_a have to be Lorentz scalars. Equation (10) is

also consistent with Eq. (5).

One might want to define $\eta_a = m_a / \sum_a m_a$, where m_a is the rest mass of the ath particle. Such a definition has two disadvantages here. First, it does not work for massless particles, like the quarks in the MIT bag model. Second, in the case of a central field with its center located at the CM of the system, it is impossible to solve for the wave functions exactly. In next two sections, we will see that it is more natural to define $\eta_a = \epsilon_a / \sum \epsilon_a$, where ϵ_a is the energy eigenvalue of the ath particle.

In a CMRF, P^μ and $P \cdot y$, the most important scalar to describe the motion of the CM due to Eq. (1), are reduced to

$$P^\mu = \left(\sum_a \epsilon_a^{(0)}, \mathbf{0} \right) = \left(\sum_a \omega_a, \mathbf{0} \right) = (M, \mathbf{0}), \tag{11}$$

$$P \cdot y = \left(\sum_a \omega_a \right) T = MT \;. \tag{12}$$

Here M should be taken as the rest mass of the system. Now let CMVF be a new frame, moving in the $-\mathbf{x}_\parallel$ direction at a velocity, $\mathbf{v} = (-v, \mathbf{0}_\perp)$ with respect to the CMRF, then we have the Lorentz transformation for $\tilde{x} = \Lambda x$, with

$$\Lambda = \begin{pmatrix} \gamma & v\gamma & 0 \\ v\gamma & \gamma & 0 \\ 0 & 0 & I_\perp \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} \gamma & -v\gamma & 0 \\ -v\gamma & \gamma & 0 \\ 0 & 0 & I_\perp \end{pmatrix},$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} \;. \tag{13}$$

In the CMVF, we find $P^\mu \to \tilde{P}^\mu = (\gamma M, \gamma M \mathbf{v})$, and

$$P \cdot y = \tilde{P} \cdot \tilde{y} = \gamma M \tilde{T} - \gamma M \mathbf{v} \cdot \tilde{\mathbf{y}} \;. \tag{14}$$

II. THE CM FOUR-VECTOR OF A CLASSICAL STEADY FEW-BODY SYSTEM

Before we go to the quantum system, let us consider a classical relativistic system first. According to our restrictions on the motion of particles, the closest classical cases are so-called steady or rigid-body solutions [14], when particles moving periodically around the center of mass in fixed orbits with fixed speed and energy in a CMRF:

$$\omega_a = \text{const} \;, \qquad \mathbf{x}_a(\tau + S) = \mathbf{x}_a(\tau) \;. \tag{15}$$

Similar to the expectation values of momentum in quantum mechanics, here we consider, instead of the instantaneous momentum, the time-averaged four-momentum of each particle,

$$\bar{p}_a^\mu = \frac{1}{S} \int_0^S p_a^\mu(\tau) \, d\tau \;. \tag{16}$$

Note that $\bar{p}_a^\mu$ is also a four-vector by definition, and in a

CMRF it is reduced to

$$\bar{p}_a^\mu = (\omega_a, \mathbf{0}) \ . \tag{17}$$

We see that the rest energy $\omega_a = \sqrt{\bar{p}^2}$ has to be considered as a Lorentz-invariant constant. The time-averaged total momentum of the system in a CMRF is the same as the instantaneous total momentum:

$$\bar{P}^\mu = \left(\sum \omega_a, \mathbf{0}\right) = (M, \mathbf{0}) = P^\mu \ . \tag{18}$$

Note, as a four-vector equation, $\bar{P}^\mu = P^\mu$ is true in any CMVF, and M is also a Lorentz invariant constant. This enables us to define the CM four-vector by

$$y^\mu = \frac{\sum \omega_a x_a^\mu}{\sum \omega_a} = \frac{1}{M} \sum_a \omega_a x_a^\mu = \sum_a \eta_a x_a^\mu \ . \tag{19}$$

By our definition, y^μ is a Lorentz four-vector. We see that $\bar{\mathbf{p}}_a = \mathbf{0}$ is crucial to our definition. It is easy to verify that this is consistent with Eqs. (5) and (10), and the Poisson brackets between y^μ and P_ν are satisfied. Besides, in a CMRF, $\mathbf{y} = \bar{\mathbf{y}} = \mathbf{Y}$, because we have chosen steady periodic solutions. Also, $T = \bar{T}$ because T and t_a's are linear in τ. Therefore, the requirement of Eq. (6) is valid.

In a CMRF, we have $P \cdot y = MT = \sum \omega_a t_a$, as given by Eq. (12). In a CMVF, $P \cdot y$ as a scalar is given by Eq. (14). But we can also obtain it by using the inverse Lorentz transformation $t_a = (\Lambda^{-1})^0_\mu \tilde{x}^\mu$, taking into account that ω_a are Lorentz-invariants and that by definition

$$\sum_a \omega_a \tilde{x}_{a\parallel} = \left(\sum_a \omega_a\right) \tilde{y}_\parallel \ .$$

Because $\epsilon_a = \gamma \omega_a$ for each a in a CMVF, our definition (19) is equivalent to

$$y^\mu = \frac{\sum \epsilon_a x_a^\mu}{\sum \epsilon_a} \ , \tag{20}$$

which reflects the fact that y^μ is a four-vector and we have used this notation in Ref. [8].

III. THE CM FOUR-VECTOR OF A NONRELATIVISTIC QUANTUM SYSTEM

Now let us turn to nonrelativistic quantum systems. We will see that many features of the CM degrees of freedom can be revealed in such cases, and they may have important applications in atomic and nuclear physics. We will still use our four-vector notation, though Lorentz covariance is not a requirement.

Under our assumption on the four-potential, in a CMRF each particle has the following Schrödinger equation:

$$i\partial_{t_a}\psi(x_a) = H(x_a)\psi(x_a)$$
$$= [m_a + \frac{1}{2m_a}\hat{\mathbf{p}}_a^2 + V(r_a)]\psi(x_a) \ ,$$

where we have temporally used the notation t_a, which has to satisfy Eq. (5) in the nonrelativistic limit. We also

have assumed that $\mathbf{y}_a = \mathbf{0}$ in the CMRF. We consider only *designative* states, or bounded energy eigenstate with definite parity. They can be written as $\psi_n(x_a) = \exp(-i\omega_a t_a)q_n(\xi_a)$ and normalized as

$$\int d^3 x_a \psi_n^*(x_a)\psi_n(x_a) = 1 \ .$$

The expectation values of momentum and coordinates are

$$\langle \hat{p}_a^\mu \rangle = \bar{p}_a^\mu = \int d^3 x_a \psi_n^*(x_a)i\frac{\partial}{\partial x_a^\mu}\psi_n(x_a) = (\omega_a, \mathbf{0})$$

$$\langle x_a^\mu \rangle = \bar{x}_a^\mu = (t_a, \mathbf{0}) \ .$$

Now we can define the CM four-vector as in Eq. (19). Here again, the definition of T is consistent with Eqs. (5) and (9), which are, of course, true in nonrelativistic theories. We also have $y^\mu = \langle y^\mu \rangle = Y^\mu$ as in Eq. (6). Moreover, from our definition of y^μ, one can easily find that y^μ and P_ν are canonically conjugate and for any function $F(y)$, we have

$$\sum_a \frac{\partial}{\partial x_a^\mu} F[y(x_1, ..., x_N)] = \frac{\partial}{\partial y^\mu} F(y) \ . \tag{21}$$

In a stable central field, we can find states $\psi_{\check{n}}(x)$ of definite energy and angular momentum $\mathbf{L}^2$, L_z, and S_z, with corresponding quantum numbers $\check{n} \equiv (n, l, m, m_s)$. In these states, we have the required expectation values of momentum and coordinates to use our CM definition. The product of N single particle wave functions leads to

$$\prod_a \psi_{\check{n}_a}(x_a) \equiv e^{-i\sum_a \omega_a t_a}\prod_a q_{\check{n}_a}(\xi_a) = e^{-iMT}\prod_a q_{\check{n}_a}(\xi_a)$$
$$\equiv e^{-iMT}q_{\check{n}}(\xi_1, ... \xi_N) \ .$$

The spatial part of the plane wave function $\exp(i\mathbf{P} \cdot \mathbf{y})$ disappears because $\mathbf{P} = \mathbf{0}$. Now let us check if dY^i/dT represents the motion of the CM in a CMVR, where $x_a^i = \xi_a^i + v^i T$ and there is no change in $H(x_a^i) = H(\xi_a^i)$. We find

$$\frac{dY^i}{dT} = \sum_a \eta_a \frac{d\langle x_a^i \rangle}{dT} = \sum_a \eta_a \left\langle \frac{\partial x_a^i}{\partial T} + i[H, x_a^i] \right\rangle$$
$$= \sum_a \eta_a v^i = v^i \ .$$

So the system does have a total three-momentum $P^i = MdY^i/dT = Mv^i$ and a total energy $E = M + Mv^2/2$ as expected. To recover $\exp(-iP \cdot Y)$ in a CMVF, we first apply Lorentz transformation to $MT = P \cdot Y$ and then take the nonrelativistic limit $v \ll 1$, to obtain the wave function in a CMVR (this could be called a *semirelativistic* treatment):

$$\Phi_{\check{n}P}(\xi_1, ..., \xi_N; Y) \equiv \frac{1}{(2\pi)^{3/2}}e^{-iP \cdot Y}q_{\check{n}}(\xi_1, ... \xi_N) \ , \tag{22}$$

$$\int d^3 Y d^{3N}\xi \, \Phi_{\check{n}P}^* \Phi_{\check{n}'P'} = \delta_{\check{n},\check{n}'}\delta^3(\mathbf{P} - \mathbf{P}') \ , \tag{23}$$

where $P^0 = E = \gamma M \approx M + Mv^2/2 + \cdots$, $P^i = \gamma Mv^i \approx Mv^i + \cdots$, as we expected, and $d^{3N}\xi = d^3\xi_1 \cdots d^3\xi_N$.

How can we consider Y^i as independent variables? It is important to note that, because $y^\mu = Y^\mu$ in the CMRF, we have the following three functional restrictions on the states:

$$y^\mu \equiv \sum_a \omega_a x_a^i = \sum_a \omega_a \langle x_a^i \rangle \equiv Y^\mu . \qquad (24)$$

These are very strong restrictions, implied by our assumption: the center of mass is the center of the potential. We see that the designative system states are not simply the product of any individual particle states. For example:

$$[c_a\psi_a(x_1) + c_b\psi_b(x_1)]\psi_{n_2}(x_2)\cdots\psi_{n_N}(x_N)$$

is a solution of the N-particle Schrödinger equation, but not an energy eigenstate of the system if $\omega_a \neq \omega_b$. Thus, in choosing our designative states, we have already reduced the degrees of freedom.

Now we want to expand any $\Psi(x_1,\ldots,x_N) = \langle x_1,\ldots,x_N|\Psi\rangle$ of the system by using our orthogonal function set (22), while keeping $\langle\Psi|\Psi\rangle = 1$. Since $\mathbf{P} = M\mathbf{v}$ and

$$\langle\mathbf{P}|\mathbf{P}'\rangle = \delta^3(\mathbf{P} - \mathbf{P}') = \delta^3(\mathbf{v} - \mathbf{v}')/M^3 ,$$

we introduce

$$|\mathbf{v}\rangle = (M)^{3/2}|\mathbf{P}\rangle , \qquad (25)$$

$$\phi_v(Y) = (M)^{3/2}\phi_P(Y) = \frac{(M)^{3/2}}{(2\pi)^{3/2}}e^{-iP\cdot Y} , \qquad (26)$$

with $P^\mu = (M, M\mathbf{v})$ in nonrelativistic limit. Thus we have the following *semirelativistic CM-freedom separation rule*:

$$\Psi(x_1,..,x_N) \equiv \Psi(\xi_1,..,\xi_N;Y)$$

$$= \sum_{\tilde{n}} \int d^3v\, a_v\phi_v(Y)b_{\tilde{n}}q_{\tilde{n}}(\xi_1,..,\xi_N) \qquad (27)$$

$$= \sum_{\tilde{n}} \int d^3P\, a_P\phi_P(Y)b_{\tilde{n}}q_{\tilde{n}}(\xi_1,\ldots,\xi_N) \qquad (28)$$

$$\equiv \sum_{\tilde{n}} \int d^3P\, a_P b_{\tilde{n}}\Psi_{\tilde{n}P}(\xi_1,\ldots;Y) , \qquad (29)$$

where we have used the facts that $d^3v = d^3P/M^3$ and $a_v = M^{3/2}a_P$. This separation rule has several advantages: it has no undetermined parameter λ, as in Eq. (2); it has the right normalization: $\int d^3Y\, d^{3N}\xi\,\Psi^*\Psi = 1$, if $\int d^3P|a_P|^2 = 1$; and it has right dimension $[|\Psi|^2] = L^{-3N-3}$ to fit the requirement from the field theory later. We should keep in mind the dimensional relation:

$$[\psi(x_1)\cdots\psi(x_N)] = L^{-3N/2} = L^{3/2}[\Psi(x_1,\ldots,x_N)] . \qquad (30)$$

In single-particle problems, when the perturbation Hamiltonian is $H'(x)$, initial state is $|\psi_n\rangle$, and final state

is $|\psi_{n'}\rangle$, the S-matrix element is $\langle\psi_{n'}|H'|\psi_n\rangle$. This can be readily extended to our confined N-body system:

$$\langle\Psi_{n'P'}|H'|\Psi_{nP}\rangle . \qquad (31)$$

For example, let us look at a simple example: a polarized (in x direction) γ ray traveling in $\mathbf{k}$ direction, with electric field $E_x = E_0 Re(e^{-ik\cdot x}) = E_0(e^{-ik\cdot x} + e^{ik\cdot x})/2$, is interacting with the N_p protons in a nucleus of N nucleons at initial state $\psi_{n_i}(x_i) = \exp(-i\omega_i t_i)q_{n_i}(\xi_i)$ in an isotropic harmonic oscillator potential $V(r)$ (not necessarily being relativistic solutions). The perturbation then is

$$H' = \sum_{i=1}^{N_p} -e(\mathbf{x}_i - \mathbf{y})\cdot\mathbf{E}\,Re(e^{-ik\cdot x_i})$$

$$= \sum_{i=1}^{N_p} -eE_0\,\xi_{ix}\,Re(e^{-ik\cdot y}e^{ik\cdot\xi_i}) ,$$

where we have used $\mathbf{x}_i = \mathbf{y}_i + \xi_i$ and $t_i = T$. The S-matrix element is

$$\langle\Psi_{n'P'}|H'|\Psi_{nP}\rangle = \sum_{i=1}^{N_p} \int d^4Y d^3\vec{\xi}_i \frac{e^{-i(P-P')\cdot Y}}{(2\pi)^3}$$

$$\times q_{n_i'}^*(\xi_i)H'(x_1)q_{n_i}(\xi_i)\prod_{j\neq i}^{N} q_{n_j}^* q_{n_j} ,$$

which leads to

$$\frac{1}{2}(2\pi)eE_0\delta^4(P + k - P')\sum_{i=1}^{N_p}\int d^3\vec{\xi}_i e^{ik\cdot\vec{\xi}_i}q_{n_i'}^*(\xi_i)\xi_{ix}q_{n_i}(\xi_i)$$

[there is another term with a factor $\delta^4(P - k - P')$, which is always zero]. From this equation we can easily find the recoil of the nucleus and the allowed change of states. For, example, let $P^\mu = (M, \mathbf{0})$, then $\epsilon_{n_i'} - \epsilon_{n_i} = k - k^2/(2M)$, which is not that when the recoil is neglected. One can also see that there is no elastic scattering, when $n_i' = n_i$ and $P'^2 = P^2 = M^2$, as is well known in the Compton effect.

IV. THE CM FOUR-VECTOR OF A RELATIVISTIC QUANTUM SYSTEM

Now we are ready to discuss the CM four-vector of a relativistic system with N spin-1/2 fermions, confined in a central field. We want to use the expectation values of p^μ and x^μ, thus $|\psi(x)|^2$ should be still interpreted as the probability distribution or particle density in the space. This is true if we only concentrate on particles (quarks or nucleons) and avoid the particle-antiparticle creation or annihilation. We use the Schrödinger picture where the states of particle are spinor functions of coordinates x_a^μ, satisfying the following Dirac equations:

$$\gamma^\mu(i\partial_{a\,\mu} - G_\mu(x_a))\psi(x_a) = m_a\psi(x_a) . \qquad (32)$$

In a CMRF, the four-potential is reduced to a time-independent central field, the designative states can be chosen as $\psi_{n_a}(x_a)$ with definite energy $E_{n_a} = \omega_a$, satisfying

$$i\partial_{a\,0}\psi_{n_a}(x_a) = \omega_a\,\psi_{n_a}(x_a) \; , \tag{33}$$

$$\psi_{n_a}(x_a) = e^{-i\omega_a t_a}\,q_{n_a}(\xi_a) \; . \tag{34}$$

The wave functions can be normalized as

$$\int d^3x_a \psi_{n_a}^\dagger(x_a)\psi_{n_a'}(x_a) = \delta_{n_a\,n_a'} \; .$$

When $n_a = n_a'$, it is the conserved total "charge." Now we have the expectation values of p_a^μ and x_a^μ in the designative states:

$$\bar{p}_a^\mu = \int d^3x_a \psi_{n_a}^\dagger i\partial_a^\mu \psi_{n_a} = \langle \hat{p}_a^\mu \rangle = (\epsilon_a, \mathbf{0}) \; , \tag{35}$$

$$\bar{x}_a^\mu = \int d^3x_a \psi_{n_a}^\dagger i x^\mu \psi_{n_a} = \langle x_a^\mu \rangle = (t_a, \mathbf{0}) \; ; \tag{36}$$

again, these are four-vectors by definition. The expectation value of total four-momentum is $(M, \mathbf{0})$. These relations enable us to use Eq. (19) as the definition of y^μ and obtain the product of designative states of the N-particle system:

$$\prod_a^N \psi_{n_a}(x_a) = e^{-i\sum \omega_a t_a} \prod_a^N q_{n_a}(\xi_a) \tag{37}$$

$$= e^{-iMT} \prod_a^N q_{n_a}(\xi_a)$$

$$= e^{-iMT} q_n(\xi_1, \ldots, \xi_N) \; . \tag{38}$$

Transformed to a CMVF, MT becomes $P \cdot Y$ through Lorentz transformation, as we did before, and the product of wave functions takes the form (with a normalization coefficient)

$$\psi_{nP}(x_1, \ldots, x_N) = \frac{1}{(2\pi)^{3/2}} e^{-iP\cdot Y} \prod_a^N S(\Lambda_v) q_{n_a}(\xi_a') \tag{39}$$

$$\equiv \phi_P(Y) \prod_a^N q_{n_a v}(\xi_a)$$

$$\equiv \phi_P(Y) q_{nv}(\xi_1, \ldots, \xi_N) \; , \tag{40}$$

where $\xi' = \Lambda^{-1}\xi = (\gamma\xi_\parallel, \xi_\perp)$, $E_p = \sqrt{M^2 + \mathbf{P}^2}$, and $S(\Lambda)$ is the Lorentz transformation matrix for a Dirac spinor [15]. We clearly see how we obtain the plane wave function $e^{-iP\cdot Y}$, which describes the motion of the CM of the isolated system. This equation also gives us the Lorentz transformation rule for the $q(\xi)$'s. Again, there are no internal time variables t_a in our formula, which follows from our restriction on energy eigenstates. The normalization of $q_n(x)$ is Lorentz invariant,

$$\int d^3x' q_n'^\dagger(x')q_{n'}'(x') = \int \frac{d^3x'}{\gamma} q_n^\dagger(x)q_{n'}(x)\gamma$$

$$= \int d^3x q_n^\dagger(x)q_{n'}(x) = \delta_{n,n'} \; . \tag{41}$$

Also we have the invariant normalization for $q_{nv}(x)$:

$$\int d^3x q_{nv}^\dagger(x)q_{n'v}(x) = \delta_{n,n'} \; . \tag{42}$$

To check, we let $x' = \Lambda x$ and note that $d^3x' = d^3x/\gamma$, as the measure of a proper volume in moving frame, and $\psi^\dagger\psi = \bar\psi\gamma^0\psi$, which is the zeroth component of a four-vector. Hence,

$$\int d^3x' \, q_n'^\dagger(x')q_{n'}'(x') = \int \frac{d^3x}{\gamma} q_n^\dagger(x)q_{n'}(x)\gamma = \delta_{n,n'},$$

$$\int d^3x \, q_{nv}^\dagger(x)q_{n'v}(x) \equiv \int d^3x \, q_n'^\dagger(x)q_{n'}'(x)$$

$$\equiv \int d^3x \, \gamma q_n^\dagger(\Lambda^{-1}x)q_{n'}(\Lambda^{-1}x)$$

$$= \int d^3x' \, \gamma q_n^\dagger(x)q_{n'}(x) = \delta_{n,n'} \; .$$

The normalization of $\psi_{nP}(x_1, \ldots, x_N)$ is similar as in Eq. (23):

$$\int d^3Y d^{3N}\xi \, \psi_{nP}^\dagger \psi_{n'P'} = \delta_{n,n'}\delta^3(\mathbf{P} - \mathbf{P}'). \tag{43}$$

Now we want to expand any function $\Psi(x_1, \ldots, x_N)$ which represents a moving confined system. Using $\mathbf{P} = \gamma M \mathbf{v}$, we define $|\mathbf{v}\rangle = (\gamma M)^{3/2}|\mathbf{P}\rangle$, or $\phi_P(Y) = (\gamma M)^{-3/2}\phi_v(Y)$. Thus we have

$$\langle \mathbf{P}|\mathbf{P}'\rangle = \delta^3(\mathbf{P} - \mathbf{P}') = \delta^3(\mathbf{v} - \mathbf{v}')/(\gamma M)^3 \; ,$$

and we obtain *the revised CM-freedom separation rule*:

$$\Psi(x_1, \ldots, x_N) = \int d^3v \, a_v \sum_n \phi_v(Y) b_n q_{nv}(\xi_1, \ldots \xi_N) \tag{44}$$

$$= \int d^3P \, a_P \sum_n \phi_P(Y) b_n q_{nv}(\xi_1, \ldots \xi_N) \; , \tag{45}$$

where we have used $a_v = a_P(\gamma M)^{3/2}$ and

$$d^3v = d^3P \det(\partial v^i/\partial P^j) = d^3P/(\gamma M)^3 \; .$$

Note that the expansion in (44) is to find all system states with $\mathbf{v} = \mathbf{0}$ and then boost each of them with all possible $\mathbf{v}$, with $P = (\gamma M, \gamma M \mathbf{v})$ and $P^2 = M^2$. Thus the expansion is the same either in CMRF or in CMVR. It has the same advantages as we have mentioned in the nonrelativistic study.

V. THE γ-N EFFECTIVE INTERACTION LAGRANGIAN

In Refs. [5–9] using the old separation rule of Eq. (2), we have introduced and used an effective γ-N interaction Lagrangian:

$$
\begin{aligned}
L^{\gamma-N}(Y) &= J_\mu(Y)A^\mu(Y) \\
&= \sum_{1\to 2,3}\int \frac{d^9\xi}{\lambda^3}\Psi^\dagger(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1 A^\mu(x_1) \\
&\quad \times \Psi(x_1,x_2,x_3) .
\end{aligned}
\tag{46}
$$

Now we use our revised one, Eq. (45), and we find that we do not need to introduce the parameter λ to make the action $S = \int d^4Y\, L^{\gamma-N}(Y)$ dimensionless. The effective Lagrangian now can be written without any undetermined parameter, namely:

$$
\begin{aligned}
L^{\gamma-N}(Y) &= J_\mu(Y)A^\mu(Y) \\
&= \sum_{1\to 2,3}\int d^9\xi\,\Psi^\dagger(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1 A^\mu(x_1) \\
&\quad \times \Psi(x_1,x_2,x_3) .
\end{aligned}
\tag{47}
$$

When Eq. (47) is applied to calculation of nucleon EM form factors [7], with

$$
\begin{aligned}
A^\mu(x_1) &= A^\mu(q)e^{-iq\cdot(Y+\xi_1)} = A^\mu(q)e^{-iq\cdot Y}e^{i\mathbf{q}\cdot\xi_1} \\
&= A^\mu(Y)e^{i\mathbf{q}\cdot\xi_1}
\end{aligned}
$$

and the normalization $\langle P'|P\rangle = \delta^3(\mathbf{P}'-\mathbf{P})$, the effective γ-N vertex is derived as

$$
\int d^4Y e^{-iq\cdot Y}\langle P'|J_\mu(Y)|P\rangle = (2\pi)\delta^4(P+q-P') \\
\times \langle P'|J_\mu(0)|P\rangle ,
$$

where

$$
\begin{aligned}
\langle P'|J_\mu(0)|P\rangle &= \sum_{1\to 2,3}\int d^9\xi e^{i\mathbf{q}\cdot\xi_1}q^\dagger_{v'n'}(\xi_1,\xi_2,\xi_3) \\
&\quad \times (\hat{e}\gamma_0\gamma_\mu)_1 q_{vn}(\xi_1,\xi_2,\xi_3) .
\end{aligned}
\tag{48}
$$

This vertex is exactly the right-hand side of Eq. (4) or Eq. (6) in Ref. [7], which leads to proton form factors in quite good agreement with the data. Hence, we can write our *revised effective current* as

$$
\begin{aligned}
J_\mu(Y) &= \sum_{1\to 2,3}\int d^9\xi e^{i\mathbf{q}\cdot\xi_1}\Psi^\dagger(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1 \\
&\quad \times \Psi(x_1,x_2,x_3) ,
\end{aligned}
\tag{49}
$$

which has no undetermined parameter. We note that unlike the ordinary current expression, where only *one* volume element d^3x times the zeroth component of a vector, which together make a Lorentz scalar, we now have a product of *three* volume integrals of a zeroth component of a four-vector of one struck quark, multiplied by an invariant (in Breit frame) from other two spectator quarks. Hence we have an extra factor $1/(ch\Omega)^2 =$

$1/(1+Q^2/4M^2)$ left uncanceled. This is the factor which appears in front of the electric form factor $G_E(Q^2)$ in Eq. (11a) in Ref. [7]. The current (49) can be easily extended to more general currents. For example, we can replace the U(1) generator $\hat{e}$ by λ^a of the flavor SU(3) generator to obtain the SU(3) current $J^a_\mu(Y)$.

VI. THE DEEP INELASTIC γ-N COLLISION AND FREE-QUARK APPROXIMATION

To find nucleon structure functions, we have to begin with the following tensor:

$$
\begin{aligned}
W_{\mu\nu} &= \frac{1}{4\pi}\int d^4Y d^3P'\sum_{S'}e^{iq\cdot Y}\langle P,S|J_\mu(Y)|P',S'\rangle \\
&\quad \times \langle P',S'|J_\nu(0)|P,S\rangle \\
&= \frac{1}{4\pi}\int d^4Y e^{iq\cdot y}\langle P,S|[J_\mu(Y),J_\nu(0)]|P,S\rangle ,
\end{aligned}
$$

where $\langle P,S|P',S'\rangle = (2\pi)^3 2E\delta^3(P-P')\delta_{S,S'}$. When using free-quark approximation for the "intermediate" states above, i.e., assuming that after the scattering all quarks go freely, we will use free-quark anticommutators like $\{\psi(x_i),\bar{\psi}(x_i)\}$ [5,10]. So we can not use our $J_\mu(Y)$ as defined in Eq. (49), where both incoming and outgoing quarks are in designative states according to the revised expansion rule of Ψ. We should do the following replacement in Eq.(49):

$$
\sum_{1\to 2,3}\int d^9\xi e^{i\mathbf{q}\cdot\xi_1}\Psi^\dagger(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1\Psi(x_1,x_2,x_3)
$$

$$
\to \sum_{1\to 2,3}\int d^9\xi e^{i\mathbf{q}\cdot\xi_1}\Psi(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1
$$

$$
\times \psi(x_1)\psi(x_2)\psi(x_3) .
$$

Therefore, according to Eq. (30), we must introduce a parameter λ with dimension L, and the effective current in the free-quark approximation becomes

$$
\begin{aligned}
\tilde{J}_\mu(Y) &= \sum_{1\to 2,3}\int d^9\xi e^{i\mathbf{q}\cdot\xi_1}\Psi^\dagger(x_1,x_2,x_3)(\hat{e}\gamma_0\gamma_\mu)_1 \\
&\quad \times \frac{1}{\lambda^{3/2}}\psi(x_1)\psi(x_2)\psi(x_3) .
\end{aligned}
\tag{50}
$$

In the bag model, the only reasonable choice for λ is

$$
\lambda_i = C_i(Q^2)R .
\tag{51}
$$

Here R is the bag radius (a Lorentz invariant constant!), and C_i is possibly Q^2 dependent [5], because of the factor $\exp(i\mathbf{q}\cdot\xi)$ in $\tilde{J}$ and it may also be flavor dependent [11]. We have compared our result in the nonrelativistic limit with Jaffi's result in Ref. [3], we find that $\lambda = R$ [6]. In Ref. [11], λ's for a proton are fixed through the rms radius of the neutron and proton, with the results $\lambda_u = R$ and $\lambda_d = 0.85R$.

In Ref. [10], some Feynman rules for the CM-bag model are given. We see that parameter λ does not appear in

the γ-N vertex anyway. Only in the last two graphs, where we have used free-quarks for outgoing states, λ is not canceled out. So our Feynman rules remain unchanged. In the same paper, we also mentioned that the bag radius R might be Q^2 dependent to explain the EMC effect.

For other models, we can introduce similar currents. When we use free-particle approximation for outgoing particles, we need to introduce λ as in Eq. (50), with λ proportional to the length scale given by the model. For example, if the potential is an isotropic harmonic oscillator potential $V(r) = kr^2/2 = m\Omega^2 r^2/2$, then in our semirelativistic approach, we have only one length scale (through three constants $m, \Omega, \hbar = 1$), namely

$$\lambda_i = \frac{C(Q^2)}{\sqrt{m_i \Omega}} \ . \tag{52}$$

For a relativistic oscillator [16,17], the way to get a length scale is not unique (note, e.g., that $[c\Omega] = [\sqrt{\hbar/(m\Omega)}] = L$), but if we look at the parameter $\lambda_{Nj\pm}$ used to define the dimensionless coordinate $r' = \lambda r$ in Ref. [17], we find Eq. (52) is still true.

VII. SUMMARY AND DISCUSSION

We find that if we assume a central field, and restrict to designative states, our CM four-vector is well defined. We have revised our CM-freedom separation rule and the effective γ-N current, neither of them now have undetermined parameters. The length scale λ comes in only when we use the free-particle approximation for deep inelastic scatterings.

Our method can be applied to the CM bag model and any other model with central field. A very interesting case will be an isotropic harmonic oscillator potential. This can be used for a (non)relativistic nuclear shell model or an alternative hadron model with quarks(antiquarks) of nonzero rest mass [so the SU(3) flavor asymmetry can be easily introduced]. We will discuss the nucleon structure functions given by such a hadron model in our future work.

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