Analytical treatment of heavy-ion elastic scattering at intermediate energies

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The elastic scattering of heavy ions at energies where the eikonal approximation is valid is discussed in detail. The nuclear phase shift and deflection functions for a complex Woods-Saxon interaction are evaluated in closed form. The resulting functions represent extremely well the numerically generated ones, even in cases involving the scattering of loosely bound nuclei. The elastic amplitude is evaluated in closed form, and found to be a reasonable approximation to the optical model one. Applications to the scattering of ^{11}Li and ^{11}C are made.

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I. INTRODUCTION

Recently, guided by the suggestion of Satchler, McVoy, and Hussein [1], the scattering of neutron-rich nuclei, such as ^{11}Li, on light targets (^{12}C, ^{28}Si) have been measured [2,3], and theoretical analyses have been performed [3–5]. Conflicting conclusions have been reached concerning the degree of possible enhancement of refraction in the scattering of these "halo" nuclei [3–5]. In fact recently, Hussein and Satchler [5] have made a detailed analysis of the data of Kolata *et al.* [2] on the systems ^{11}C+^{12}C and ^{11}Li+^{12}C which combined an optical-model calculation with semiclassical arguments, and have reached the conclusion that contrary to what has been claimed [4], the ^{11}Li+^{12}C system is less refractive than ^{11}C+^{12}C [6].

It was essential in the analysis of Ref. [5] the use of analytical, albeit approximate expressions for the physical quantities that enter in the semiclassical theory of scattering at intermediate energies: the complex eikonal phase shift and deflection function, and that determine the general characteristics of the far-side amplitude that dominate the scattering at the energies and angles involved. To obtain these expressions, they relied on the treatment of Knoll and Schaeffer [7] and Pato and Hussein [8] which involves the use of the approximation, $\frac{a}{R} \ll 1$, where a is the diffuseness and R the radius of the optical potential.

It is clear that such an approximation is not generally valid for "halo" nuclei, where the excess neutrons extend far beyond the nucleus core. It is the purpose of this paper to develop a semianalytical semiclassical, theory of heavy-ion scattering valid even when $\frac{a}{R} \sim 1$. We apply the theory to the scattering of ^{11}Li on ^{12}C at 660 MeV.

II. ANALYTICAL FORMULAS

We start with the usual expression for the far-side amplitude in situations involving a nuclear rainbow (two stationary phase points)

$$f_F(\theta) = \sum_s \sqrt{\frac{b_s}{\sin\theta}\left|\frac{db_s}{d\theta}\right|} \times \exp\left\{i\left[2\delta(kb_s) + kb_s\theta - \tfrac{3}{4}\pi\right]\right\}. \quad (1)$$

The above expression is valid at angles less negative than the nuclear rainbow angle, b_s is the stationary impact parameter related to the stationary angular moment through $l_s + \frac{1}{2} = kb$, k is the wave number $k = \sqrt{\frac{2\mu E_{c.m.}}{\hbar^2}}$, μ is the reduced mass, and $E_{c.m.}$ is the center-of-mass energy. Finally $\delta(kb_s)$ is the total phase shift, which is nuclear dominated at intermediate energies. The nuclear part of δ, δ_N, is related to the complex optical potential, in the eikonal limit via

$$\delta_N = -\frac{k}{E}\int_{-\infty}^{\infty} V(b_s, z)\, dz \quad (2)$$

while the corresponding deflection function $\Theta_N = \frac{2d\delta_N}{kdb_s}$ is

$$\Theta_N = -\frac{1}{E}\int_{-\infty}^{\infty} \frac{dV(b_s, z)}{db_s} dz. \quad (3)$$

Knoll and Schaeffer determine from Eq. (3) a complex value of b_s by requiring

$$\mathrm{Im}\Theta_N(b_s) = 0. \quad (4)$$

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The solution of (4), when inserted in (2) and (1), determines completely the amplitude. For the usual Woods-Saxon form of V, an analytical solution of (4) is not possible. Knoll and Schaeffer [7] used $V = -(V_0 + iW_0) f(b,z)$ with $f(b,z) = \left[1 + \exp\left(\frac{r-R}{a}\right)\right]^{-1}$, $r = \sqrt{b^2+z^2}$. They then write for $\delta(kb_s)$ the following:

$$\delta(kb_s) = -\frac{k}{2E}(V_0 + iW_0)\sqrt{2aR}\, g(x,y),$$

$$g(x,y) = \int_0^\infty \frac{(1+xy)^{\frac{1}{2}}\, dv}{1 + \exp\left(\frac{2v^2}{1+\sqrt{1+\frac{2yv^2}{1+xy}}} + x\right)}, \quad (5)$$

where $x = \frac{b_s - R}{a}$, $y = \frac{a}{R}$.

They further set $y = 0$, to obtain

$$g(x,0) = \int_0^\infty \frac{dv}{1 + \exp(v^2 + x)}. \quad (6)$$

This function, called $g(x)$ by K-S has the following limiting form valid for the near amplitude

$$g(x,0) \sim \tfrac{1}{2}\pi^{\frac{1}{2}} e^{-x}, \quad x \gg 1. \quad (7)$$

The other limiting form obtained by Pato and Hussein [8] which is valid near the outer b_s in the far amplitude is

$$g(x,0) \sim \frac{1}{2}\sinh^{-1}\frac{x}{2}. \quad (8)$$

These limits were employed by several authors in the semiclassical analysis of the elastic scattering data. It is important to relax the condition $\frac{a}{R} \ll 1$, in order to extend the theory for the scattering of exotic nuclei. For this purpose we would like to mention that $g(x,0)$ is one particular function of a set of functions introduced by Dingle [9], and defined by

$$\mathcal{F}_p(x) = \frac{1}{p!}\int_0^\infty \frac{\epsilon^p d\epsilon}{1 + \exp(\epsilon - x)}. \quad (9)$$

These functions extensively studied by Dingle have well-defined properties. The K-S function, $g(x,0)$ is just $\frac{\sqrt{\pi}}{2}\mathcal{F}_{-\frac{1}{2}}(-x)$. Expanding now $g(x,y)$ in y and keeping first order we can write for the nuclear phase shift

$$\delta_N = \frac{k}{E}(V_0 + iW_0)\sqrt{\frac{\pi a R}{2}}\left[\mathcal{F}_{-\frac{1}{2}}(-x) + y\left(\frac{x}{2}\mathcal{F}_{-\frac{1}{2}}(-x) + \frac{3}{8}\mathcal{F}_{\frac{1}{2}}(-x)\right)\right], \quad (10)$$

from the property $\mathcal{F}'_p = \mathcal{F}_{p-1}$ we can write an expression for Θ_N

$$\Theta_N = -\frac{1}{E}(V_0 + iW_0)\sqrt{\frac{\pi R}{2}}\left[\mathcal{F}_{-\frac{3}{2}}(-x) + \frac{3y}{8}\mathcal{F}_{\frac{1}{2}}(-x) + y\left(\frac{x}{2}\mathcal{F}_{-\frac{3}{2}}(-x) + \frac{1}{2}\mathcal{F}_{-\frac{1}{2}}(-x)\right)\right]. \quad (11)$$

Analytical expressions can be generated by using the following relations derived by Dingle [9] using stationary-point methods

$$p!\mathcal{F}_{p-1}(x) = \frac{(1-\tau^2)\left(\frac{p}{\tau}\right)^p}{2}\left[\frac{\pi}{1-\tau^2\left(1-\frac{2}{p}\right)}\right]^{\frac{1}{2}}\sum_{r=0}^{\infty} Q_{2r}, \quad (12)$$

where the first two terms in the series are

$$Q_0 = 1 \quad \text{and} \quad Q_2 = \frac{1}{24F_2^3}\left(5F_3^2 - 3F_2F_4\right) \quad (13)$$

with

$$F_2 = \frac{1}{2}\left[1 - \tau^2\left(1 - \frac{2}{p}\right)\right], \quad (14)$$

$$F_3 = -\frac{\tau}{2}\left[1 - \tau^2\left(1 - \frac{4}{(p)^2}\right)\right], \quad (15)$$

$$F_4 = -\frac{1}{4}\left[1 - 4\tau^2 + 3\tau^4\left(1 - \frac{8}{(p)^3}\right)\right], \quad (16)$$

and τ is the root of the transcendental equation

$$\tau = \frac{p}{x + 2\arctan\tau}. \quad (17)$$

These relations are valid for $p > 0$. For smaller values of p, i.e., for $\mathcal{F}$ functions of order less than -1 we need a generalization of them. To do this we introduce the functions

$$I_\alpha^p = \int_0^\infty \frac{\epsilon^p \exp(\epsilon - x) d\epsilon}{[1 + \exp(\epsilon - x)]^\alpha} \quad (18)$$

which satisfies the recurrence relation

$$\frac{dI_\alpha^p}{dx} = (1-\alpha)\, I_\alpha^p + \alpha I_{(\alpha+1)}^p. \quad (19)$$

It is easy to derive, using the relation above and the recurrence relation of the $\mathcal{F}$ functions, the relations

$$p!\mathcal{F}_{p-1} = I_2^p, \qquad p!\mathcal{F}_{p-2} = -I_2^p + 2I_3^p \cdots. \quad (20)$$

The asymptotic approximations of the I functions are given by

$$\tau = \frac{p}{x + \ln\left(\frac{\alpha-1+\tau}{1-\tau}\right)} \quad (21)$$

and

$$I_{\alpha}^{p}(x) = \left(\frac{p}{\tau}\right)^{p} \frac{(1-\tau)(\alpha-1+\tau)^{\alpha-1}}{\alpha^{\alpha}} \sqrt{\frac{2\pi}{F_2}} \sum_{r=0}^{\infty} Q_{2r}, \tag{22}$$

where Q_0 and Q_2 are given by Eqs. (13) and

$$F_2 = \frac{(\alpha-1+\tau)(1-\tau)}{\alpha} + \frac{\tau^2}{p}, \tag{23}$$

$$F_3 = \frac{(\alpha-1+\tau)(1-\tau)}{\alpha^2}(-\alpha+2-2\tau) - \frac{2\tau^3}{p^2}, \tag{24}$$

$$F_4 = \frac{(\alpha-1+\tau)(1-\tau)}{\alpha^3}[\alpha^2 + 6(\alpha-1+\tau)(\tau-1)] + \frac{6\tau^4}{p^3}. \tag{25}$$

The relations (20) together with the above approximations extend the asymptotic calculation of the Fermi-Dirac functions, as they are also known, to any negative order.

In the next section we present numerical results to test the goodness of our approximate formulas above. To be specific, we consider recently studied systems $^{11}C+^{12}C$ and $^{11}Li+^{12}C$, where the comparison is made between the scattering of a halo ^{11}L with a normal nucleus ^{11}C.

III. NUMERICAL RESULTS

In this section we calculate the elastic scattering differential cross section for the systems $^{11}C+^{12}C$ and $^{11}Li+^{12}C$, recently measured by Kolata *et al.* [2]. We start showing, in Fig. 1, that the inclusion of a linear term in y gives a very good agreement to the exact function $g(x,y)$ which is, apart from a constant factor, the nuclear phase shift, even for diffusivities as large as one half of the radius. Next we compare the nuclear phase shift and the deflection function, in Figs. 2 and 3, for the the two systems considered. Finally, we generate from our formulas the total elastic optical-model cross-section and calculate the far-side cross section for ^{11}Li (Fig. 4) and for ^{11}C (Fig. 5). The calculations were done using the optical potentials of [5].

We turn our attention to the cross section. We consider angles at which the far-side amplitude is dominant. To calculate the cross section we have to modify Eq. (1) in order to take into account the fact that we are in the presence of a rainbow. So, we have to use the uniform semiclassical approximation [10]. The results are in Figs. 4 and 5.

The reason for the difference between the semiclassical approximation shown in the figures is in the inclusion of higher-order terms of the uniform approximation. We recall that this approximation is obtained introducing in the integral

$$I_{\rm Far}(\theta) = \int_{\frac{1}{2}}^{\infty} d\lambda |S(\lambda)| \lambda^{\frac{1}{2}} \exp i(2\delta - \lambda\theta) \tag{26}$$

the mapping of the phase $2\delta - \lambda\theta$ on to the cubic function,

$$2\delta(\lambda) - \lambda\theta = \frac{\mu^3}{3} + x(\theta)\mu + A, \tag{27}$$

where from the two stationary points λ_1 and λ_2 we derive

$$x(\theta) = \left\{\frac{3}{4i}[2\delta(\lambda_1) - \lambda_1\theta - 2\delta(\lambda_2) + \lambda_2\theta]\right\}^{\frac{2}{3}} \tag{28}$$

and

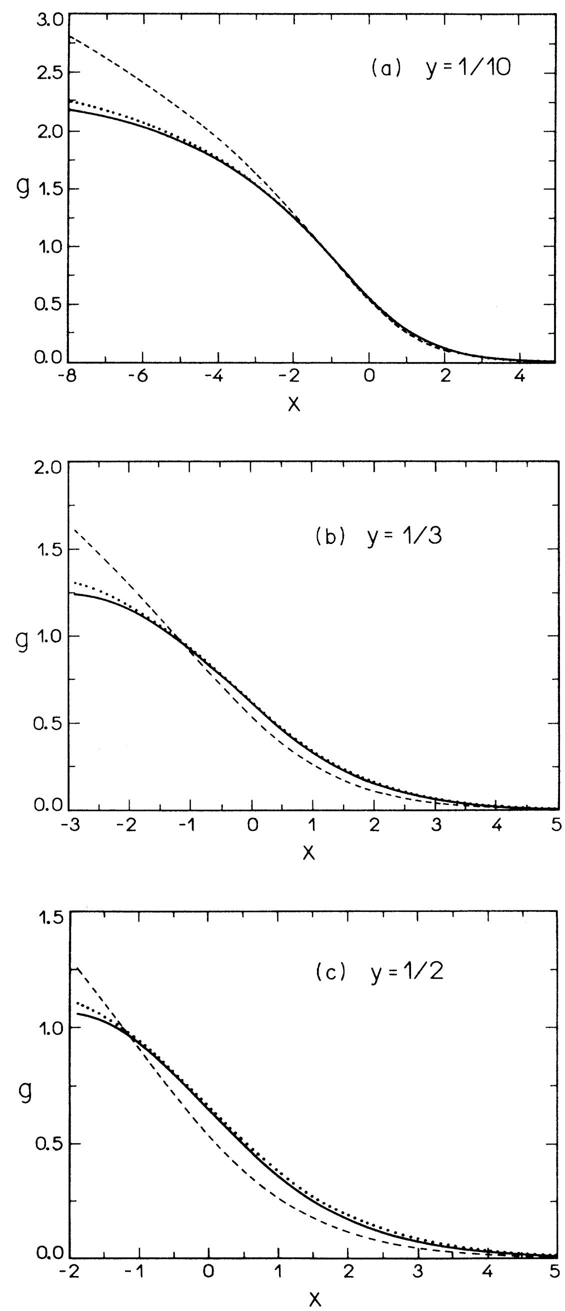

FIG. 1. The function $g(x,y)$ proportional to the nuclear phase shift calculated exactly (solid line), zero order (dashed line) and up to first order, (dotted line), for three values of $y = \frac{a}{R}$: (a) $y = 0.1$, (b) $y = \frac{1}{3}$, and (c) $y = 0.5$.

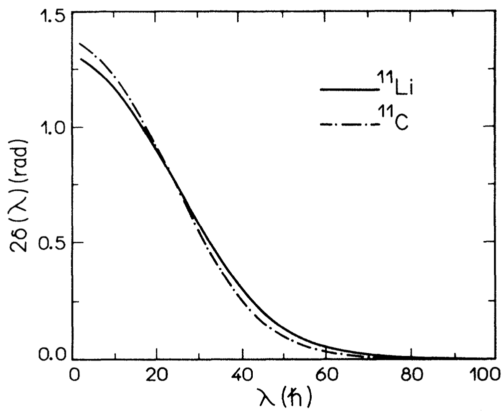

FIG. 2. The nuclear phase shift as a function of the angular momentum for ^{11}Li (solid line) and ^{11}C (dashed-dotted) line.

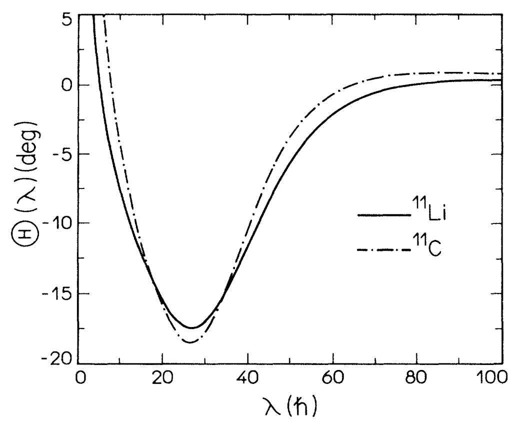

FIG. 3. The same as in Fig. 2 for the deflection function.

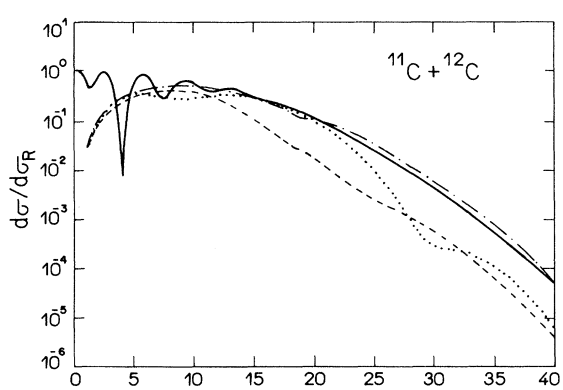

FIG. 4. The elastic cross section (solid line) of the system ^{11}C+^{12}C compared with the semiclassical far-side component calculated with the standard uniform approximation (dashed line) including first-order corrections (dotted line) and second-order terms (dashed-dotted line).

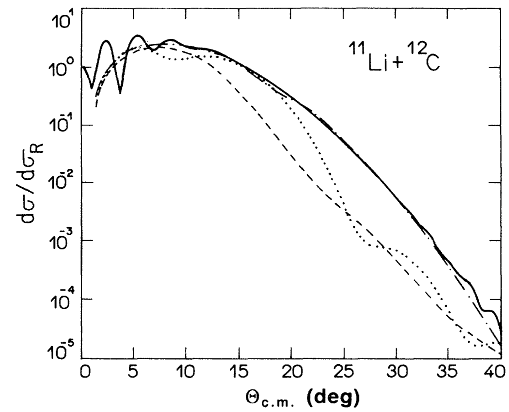

FIG. 5. The same as in Fig. 4 for the system ^{11}Li+^{12}C.

$$A(\theta) = \frac{2\delta(\lambda_1) - \lambda_1\theta + 2\delta(\lambda_2) - \lambda_2\theta}{2} \tag{29}$$

that leads to

$$I_{\rm Far}(\theta) = \exp(iA) \int_{\frac{1}{2}}^{\infty} d\mu \frac{d\lambda}{d\mu} |S(\lambda)| \lambda^{\frac{1}{2}} \exp i \left(\frac{\mu^3}{3} + x\mu\right). \tag{30}$$

Expanding the function $|S(\lambda)|\lambda^{\frac{1}{2}} \frac{d\lambda}{d\mu}$ as

$$|S(\lambda)|\lambda^{\frac{1}{2}} \frac{d\lambda}{d\mu} = \sum_{m=0}^{\infty} (p_m + \mu q_m)\left(\mu^2 + x\right)^m \tag{31}$$

and keeping up to second-order terms we find

$$I_{\rm Far}(\theta) = 2\pi \exp(iA)[(p_0 + iq_1 - 2iq_2 x) \times Ai(x) - i(q_0 + 2ip_2)Ai'(x)], \tag{32}$$

where $Ai(x)$ is the Airy function and we have used the relation $Ai'' = xAi$ to obtain the above equation. The usual uniform approximation consists in neglecting q_1, p_2, and q_2 in the above expression. The effect of the inclusion of these higher-order terms can be seen in Figs. 4 and 5. It is clear that in the present problem they have an important role.

IV. CONCLUSIONS

We have demonstrated in this paper that once the phase shifts and deflection functions are accurately calculated using, e.g., the method based on Dingle's functions developed in Sec. II, then the appropriate semiclassical calculation reproduces very well the optical model results, even in cases of loosely bounds nuclei. Our analytical formulas are quite good in describing the "data," represented here by the fit optical model calculation. One may ask here how does the halo in ^{11}Li manifest itself? Our findings in this paper support the conclusion reached by Hussein and Satchler [5], in that there is little difference between the elastic scattering of normal nucleus, ^{11}C, and a halo one, ^{11}Li [11]. The effect of the halo on the scattering is therefore more subtle and should be looked for in other observables, such as spin polarization and quasielastic (transfer) process.

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APPENDIX: HIGHER-ORDER TERMS OF THE UNIFORM APPROXIMATION

In this appendix we calculate the terms p_i and q_i for $i = 0, 1, 2$ that appear in Eq. (32). These terms are the coefficients of the expansion (31) and they are obtained taking the derivative of this equation at the stationary-phase points. Thus, for p_0 and q_0, we readily get without deriving

$$p_0 = \tfrac{1}{2}\left(G_1 F_1' + G_2 F_2'\right) \tag{A1}$$

and

$$q_0 = \frac{1}{\sqrt{-x}}\left(G_1 F_1' - G_2 F_2'\right), \tag{A2}$$

where the functions $G = |S|\lambda^{\frac{1}{2}}$ and $F = \lambda(\mu)$ were introduced and, from now on, primes denote a derivative with respect to the variable μ and subscripts refer to the stationary-phase points. By taking the first derivative

$$p_1 = \frac{1}{4\sqrt{-x}}\left(G_1' F_1' + G_1 F_1'' - G_2' F_2' - G_2 F_2''\right) \tag{A3}$$

and

$$q_1 = \frac{1}{-4x}\left(G_1' F_1' + G_1 F_1'' + G_2' F_2' + G_2 F_2'' - 2q_0\right). \tag{A4}$$

Finally the second derivative yields

$$p_2 = \frac{1}{-16x}(G_1'' F_1' + 2G_1' F_1'' + G_1 F_1''' + G_2'' F_2' + 2G_2' F_2'' + G_2 F_2''' - 4p_1) \tag{A5}$$

and

$$q_2 = \frac{1}{16\,(-x)^{\frac{3}{2}}}(G_1'' F_1' + 2G_1' F_1'' + G_1 F_1''' - G_2'' F_2' - 2G_2' F_2'' - G_2 F_2''' - 12\sqrt{-x}\,q_1). \tag{A6}$$

The first and the second derivatives of the function G are given by

$$G' = \left(\frac{d|S|}{d\lambda}\lambda^{\frac{1}{2}} + \frac{|S|}{2\lambda^{\frac{1}{2}}}\right) F' \tag{A7}$$

and

$$G'' = \left(\frac{d^2|S|}{d\lambda^2}\lambda^{\frac{1}{2}} + \frac{1}{\lambda^{\frac{1}{2}}}\frac{d|S|}{d\lambda} - \frac{|S|}{4\lambda^{3/2}}\right) F'^2 + \frac{G' F''}{F'}. \tag{A8}$$

The derivatives of the function $F(\mu)$ are calculated deriving Eq. (27) which defines the mapping. We have

$$F_{1,2}' = \sqrt{\frac{2\sqrt{-x}}{\mp\Theta'(\lambda_{1,2})}}, \tag{A9}$$

$$F_{1,2}'' = \frac{2 - \left(F_{1,2}'\right)^3 \Theta''(\lambda_{1,2})}{3F_{1,2}'\Theta'(\lambda_{1,2})}, \tag{A10}$$

and

$$F_{1,2}''' = -\frac{3\left(F_{1,2}''\right)^2 \Theta''(\lambda_{1,2}) + 6\left(F_{1,2}'\right)^2 F_{1,2}''\Theta''(\lambda_{1,2}) + \left(F_{1,2}'\right)^4 \Theta'''(\lambda_{1,2})}{4F_{1,2}'\Theta'(\lambda_{1,2})}. \tag{A11}$$

The above equations are valid in the bright side of the rainbow, the corresponding formulas for the dark side are obtained by analytical continuation.

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