Cranking Bohr-Mottelson Hamiltonian applied to normal bands of odd-A nuclei

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Normal bands of the well-deformed odd-A nuclei are studied with the cranking Bohr-Mottelson Hamiltonian proposed by us. The condition of validity of the model and comparison with the Harris expansion are discussed. The phenomena of identical bands are briefly discussed with the help of the present model.

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I. INTRODUCTION

The rotational spectrum of the well-deformed nuclei has been one of the most fascinating and fruitful fields in nuclear structure studies. The picture of a rotating nucleus is quite simple. The cranked shell model has been successfully applied to many interesting phenomena, such as backbending [1], intruder states [2], and superdeformed states [3]. In all such developments, emphasis is given to the single particle picture and the collectivities are taken into account by the cranking frequency and nuclear deformation. The Bohr-Mottelson Hamiltonian has long stood as the principal model for nuclear collectivities. It has been successfully applied to nuclear collective motions of low spin values [4]. Extending to higher spin states was not very successful [5].

In a recent publication [6], we have shown that a cranking Bohr-Mottelson Hamiltonian (BMH) can be derived from the cranked shell model and a simple formula for the rotational spectrum of the well-deformed nucleus was obtained. In this theory, collective motions are put in the foreground and the effects of the single particle motions are taken into account perturbatively. They are included in a few parameters which can be calculated from microscopic models. This simple model has been successfully applied to the low lying normal bands of the even-even nuclei [6] and the superdeformed bands in the $A \sim 190$ [7] and 150 [8] regions with precision of the order of 0.1% in the level energies.

It is to be noted that the success of the model relies on the condition that the nucleus rotates as a whole without abrupt changes in the single particle orbits. For low lying bands of even-even nuclei before band crossing, this condition is met by the joint action of deformation and pairing interaction. For the superdeformed bands, the large deformation hinders the change of the internal structure during the transitions. For an odd-A-nucleus, the situation is complicated due to the presence of the unpaired nucleon. For simplicity, the bands with clear indications of coupling between vibration and single particle motion or mixing of single particle states [9] will be excluded from our studies. Signature splitting for K=1/2 and 3/2 bands can be taken into consideration by perturbations. To avoid other complications, we shall define the levels of a band to be regular when the energy differences between successive levels increase monotonically and evenly with level spins. Signature splitting of $K = 1/2$ and 3/2 bands is considered by comparing average energies of successive pairs of levels. Our studies will be limited to regular levels. Fortunately, most of the low lying odd-A bands are partly or wholly regular, so that we are still able to make a general survey of the bands of odd-A nuclei with the present model.

In the next section, a brief sketch of the model will be given with emphasis on the modifications necessary for application to the odd-A nuclei. In the third section, all the rare-earth and actinide bands based on a single particle state, with four or more regular levels, are analyzed and the results discussed. As may be expected, important nuclear structure information is contained in the parameters of the model. Hence the validity of the model is discussed in the fourth section in comparison with the Harris expansion. Finally, as a topic of current interest, identical bands at normal deformations [10, 11] are discussed with the help of the present model and a long list of the identical bands is given in the last section.

II. THE CRANKING BMH

For quadrupole deformations, the cranking BMH can be written in the general form

$$H' = -\frac{\hbar^2}{2B_0}\frac{\partial^2}{\partial a_0^2} - \frac{\hbar^2}{4B_2}\frac{\partial^2}{\partial a_2^2} + V(\omega, a_0, a_2) \; , \qquad (1)$$

where $V(\omega, a_0, a_2)$ can be calculated as the eigenenergy of the cranked shell model Hamiltonian for fixed ω, a_0, a_2, and intrinsic states. B_0 and B_2 are the mass parameters for β and γ vibrations. For the well-deformed states, $V(\omega, a_0, a_2)$ has a deep minimum at $a_0 = \alpha_0(\omega)$, $a_2 =$

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$\alpha_2(\omega)$, and can be expanded around α_0 and α_2:

$$V(\omega,a_0,a_2) \cong V(\omega,\alpha_0,\alpha_2) + \tfrac{1}{2}V_{00}(a_0-\alpha_0)^2 + V_{02}(a_0-\alpha_0)(a_2-\alpha_2) + \tfrac{1}{2}V_{22}(a_2-\alpha_2)^2 \ , \qquad (2)$$

where V_{00}, V_{02}, and V_{22} are functions of ω, α_0, and α_2. With this expansion, the eigenenergy E' of H' can be easily evaluated as a function of ω, which is the Routhian of the system.

The projected angular momentum and energy of the system are given, respectively, by

$$\langle J\rangle = \sqrt{I(I+1)-K^2} = -\frac{1}{\hbar}\frac{\partial E'}{\partial\omega} \qquad (3)$$

and

$$E(\omega) = E' - \omega\frac{\partial E'}{\partial\omega} = E' + \hbar\omega\sqrt{I(I+1)-K^2} \ . \quad (4)$$

The above formulation is general enough to include all the cases in our former applications [6–8] and the application to the odd-A nuclear bands. In all cases, once V is obtained, simple algebra will complete the rest of the deductions.

For application to the normal bands with axisymmetrical deformation, it is sufficient to evaluate V by perturbation. As in Ref. [6], V can be put in the form

$$V = -\tfrac{1}{2}B_1\omega^2(3a_0^2+2a_2^2) + \tfrac{1}{2}C_0(a_0-\bar{a}_0)^2 + \tfrac{1}{2}C_2a_2^2 \ , \qquad (5)$$

where B_1 is the mass parameter for the rotational motion and $\bar{a}_0$ is the static deformation value ($\bar{a}_2 = 0$ for axisymmetrical deformation). Then E' is given by

$$E' = (n_\gamma+\tfrac{1}{2})\hbar\omega_\gamma\left(1-\frac{B_1\omega^2}{B_2\omega_\gamma^2}\right)^{1/2} + (n_\beta+\tfrac{1}{2})\hbar\omega_\beta\left(1-\frac{3B_1\omega^2}{B_0\omega_\beta^2}\right)^{1/2} - \tfrac{3}{2}B_1\bar{a}_0^2\omega^2\left(1-\frac{3B_1\omega^2}{B_0\omega_\beta^2}\right)^{-1} \ . \qquad (6)$$

For the present applications, excitation of the vibrational modes has been excluded, hence $n_\beta = n_\gamma = 0$. In this case, the ω-dependent parts of the first two terms in Eq. (6) amount to a few percent only. To reduce the free parameters, we shall further assume $\hbar\omega_\beta = \hbar\omega_\gamma = E_v =$ 1 MeV and $B_2 = B_0$ in the first two terms. For most of the neighboring even nuclei, the experimental values of $\hbar\omega_\beta$ and $\hbar\omega_\gamma$ vary from 0.8 to 1.2 MeV. The error involved in the above approximation is compensated by slight changes of the values of other parameters (within 1%) and the quality of fitting is unaffected. As $\omega \to 0$, the constant term $\frac{1}{2}(\hbar\omega_\beta + \hbar\omega_\gamma)$ depends obviously on the value of $\hbar\omega_\beta + \hbar\omega_\gamma$. But we are always fitting the level energies with respect to the bandhead, so this term cancels exactly. With these simplifications, E' can be put in the simple form

$$E' = \tfrac{1}{2}E_v\left(\sqrt{1-B\omega^2}+\sqrt{1-B\omega^2/3}\right) - \tfrac{1}{2}A\omega^2\left(1-B\omega^2\right)^{-1} \ , \qquad (7)$$

where

$$A = 3B_1\bar{a}_0^2 = 2\hbar^2\sum_{n\neq 0}|\langle n|j_x|0\rangle|^2/(E_n-E_0) \ , \qquad (8)$$

which is exactly the usual formula for the moment of inertia. Another free parameter in Eq. (7) is

$$B = \frac{3B_1}{B_0\omega_\beta^2} = \frac{A}{C_0\bar{a}_0^2} \ , \qquad (9)$$

where $C_0\bar{a}_0^2$ may be determined solely from the static potential energy surface of the nucleus.

Signature splitting for $K = 1/2$ and $3/2$ bands can be considered in the framework of the perturbation expansion. For the $K = 1/2$ bands, the first order perturbation of the Coriolis term does not vanish; hence a term of the form $-\langle\frac{1}{2}\alpha|\hbar\omega j_x|\frac{1}{2}\alpha\rangle$ must be added to the expansion of E', where $\alpha = \pm 1/2$ is the signature quantum number. Let

$$a = \langle\tfrac{1}{2}\tfrac{1}{2}\,|j_x|\,\tfrac{1}{2}\tfrac{1}{2}\rangle \ , \qquad (10)$$

which is connected to the usual decoupling parameter a_d by the relation

$$a = -\tfrac{1}{2}a_d \ . \qquad (11)$$

Then for $K = 1/2$ bands we have

$$E' = \tfrac{1}{2}E_v\left(\sqrt{1-B\omega^2}+\sqrt{1-B\omega^2/3}\right) - \tfrac{1}{2}A\omega^2(1-B\omega^2)^{-1} - (-1)^{I-1/2}a\hbar\omega \ . \qquad (12)$$

For $K = 3/2$ bands, the third order perturbation energy does not vanish and is signature dependent. In this case, the nonvanishing term is of the form

$$-\hbar^3\omega^3\sum_{n,m\neq 0}\frac{\langle 0\frac{3}{2}\alpha\,|j_x|\,m\frac{1}{2}\alpha\rangle\langle m\frac{1}{2}\alpha\,|j_x|\,n\frac{1}{2}\alpha\rangle\langle n\frac{1}{2}\alpha\,|j_x|\,0\frac{3}{2}\alpha\rangle}{(E_0-E_n)(E_0-E_m)} \doteq -(-1)^{I-1/2}F\omega^3a_0^3 \ . \qquad (13)$$

The expressions for V and E' become

$$V = -\tfrac{1}{2}B_1\omega^2(3a_0^2+2a_2^2) - (-1)^{I-1/2}F\omega^3a_0^3 + \tfrac{1}{2}C_0(a_0-\bar{a}_0)^2 + \tfrac{1}{2}C_2a_2^2 \qquad (14)$$

and

$$E' = \tfrac{1}{2}E_v\left[\sqrt{1-B\omega^2-2G\xi\omega^3}+\sqrt{1-B\omega^2/3}\right] - \tfrac{1}{2}A\omega^2\xi^2\left[1-\frac{(1-\xi)^2}{B\xi^2\omega^2}+\frac{2G\omega\xi}{3B}\right] \ , \qquad (15)$$

where $G = 3(-1)^{I-1/2}F\bar{a}_0/C_0$, $B = 3B_1/C_0$, and ξ is given by the equation

$$\xi = (1 - B\omega^2 - G\omega^3\xi)^{-1} . \quad (16)$$

Signature splitting for other K bands can be considered by higher order perturbations. However, it may be more useful to study large signature splitting with different approaches. Small splitting can be tolerated for the first few regular levels.

III. ANALYSIS OF THE ODD-A BANDS

Rotational bands for the odd-A nuclei in the $155 \leq A \leq 185$ and $A \geq 231$ regions are studied. For the nuclei in the range $155 \leq A \leq 185$, all the data given in Nuclear Data Sheets [14] are considered. For the actinides, data are mainly taken from Ref. [15] with a few supplements from Nuclear Data Sheets [16]. Only those bands with a definitely assigned single particle state and four or more consecutively measured regular levels are considered.

The method for fitting the parameters may be explained as follows. Let the number of levels of a given band be $m + 1$. This number is limited by the number of the measured levels or by the occurrence of irregularities in level spacing due to band crossing or any other causes. The experimental energy differences between the ith level and the bandhead

$$D_e(i) = E_e(i) - E_e(0) , \quad i = 1, 2, \ldots, m , \quad (17)$$

are compared with the calculated energy differences $D_c(i)$, and the relative root mean square (rms) deviation Δ is calculated for a given set of parameters,

$$\Delta = \left[\frac{1}{m} \sum_{i=1}^{m} \left(\frac{D_e(i) - D_c(i)}{D_e(i)} \right)^2 \right]^{1/2} . \quad (18)$$

The parameters are determined by the condition of the minimum Δ which is taken as the rms deviation of the fitting. Since the first several differences $D_e(i)$ are usually smaller than the later ones, such a fitting procedure gives more weight to the levels with small ω. This is reasonable, since we expect that the model is more accurate for small ω values. However, there are cases, particularly for $K = 1/2$ bands, where $D_e(1)$ or $D_e(2)$ becomes very small; then Eq. (18) may greatly exaggerate the error of the fitting. For the long $K = 1/2$ bands (more than seven levels), the first one or two levels may be omitted to improve the quality of fitting. In some cases, only $\alpha = 1/2$ or $-1/2$ of the $K = 1/2$ band is measured. Such bands are also used in the analysis.

In total 280 regular or partly regular odd-A bands are analyzed with rms deviation Δ less than 1%, with a few

TABLE I. Experimental ($D_e(i)$) and calculated ($D_c(i)$) energies (in MeV) of five long bands, with $D(i) = E(i) - E(0)$.

	^{171}Ta $\frac{1}{2}$[541][a]		^{171}Ta $\frac{7}{2}$[404]		^{233}U $\frac{5}{2}$[633]		^{235}U $\frac{7}{2}$[743]		^{239}Pu $\frac{1}{2}$[630][b]	
i	$D_e(i)$	$D_c(i)$	$D_e(i)$	$D_c(i)$	$D_e(i)$	$D_c(i)$	$D_e(i)$	$D_c(i)$	$D_e(i)$	$D_c(i)$
1	0.0946	0.0946	0.1310	0.1307	0.0432	0.0406	0.0462	0.0472	0.0184	0.0185
2	0.2917	0.2935	0.2843	0.2836	0.0922	0.0925	0.1030	0.1047	0.1065	0.1066
3	0.5898	0.5881	0.4579	0.4562	0.1553	0.1557	0.1707	0.1727	0.1355	0.1350
4	0.9787	0.9689	0.6484	0.6465	0.2297	0.2300	0.2491	0.2510	0.2608	0.2614
5	1.4436	1.4259	0.8540	0.8524	0.3149	0.3151	0.3385	0.3397	0.3008	0.2995
6	1.9682	1.9501	1.0723	1.0724	0.4107	0.4109	0.4386	0.4386	0.4619	0.4630
7	2.5382	2.5334	1.3023	1.3052	0.5176	0.5171	0.5504	0.5477	0.5128	0.5105
8	3.1467	3.1689	1.5431	1.5495	0.6352	0.6334	0.6709	0.6669	0.7074	0.7096
9	3.7967	3.8507	1.7944	1.8042	0.7612	0.7596	0.8051	0.7962	0.7696	0.7661
10	4.4957	4.5739	2.0567	2.0685	0.8988	0.8955	0.9448	0.9356	0.9956	0.9992
11	5.2489	5.3341	2.3285	2.3417	1.0431	1.0408	1.1004	1.0848	1.0693	1.0643
12	6.0574	6.1278	2.6126	2.6229	1.1994	1.1951	1.2578	1.2440	1.3244	1.3297
13	6.9200	6.9517	2.9020	2.9117	1.3607	1.3583	1.4345	1.4129	1.4093	1.4030
14	7.8351	7.8032	3.2082	3.2074	1.5347	1.5301	1.6062	1.5915	1.6917	1.6990
15	8.8010	8.6800	3.5172	3.5095	1.7114	1.7101	1.8028	1.7798	1.7881	1.7801
16	9.8170	9.5801	3.8354	3.8178	1.9022	1.8982	1.9870	1.9775	2.0953	2.1054
17			4.1460	4.1317	2.0931	2.0941	2.2021	2.1847	2.2038	2.1934
18			4.4910	4.4509	2.3000	2.2975	2.3962	2.4013	2.5330	2.5456
19					2.5029	2.5082	2.6285	2.6270	2.6547	2.6409
20					2.7232	2.7260	2.8309	2.8619	3.0032	3.0188
21					2.9344	2.9506	3.0786	3.1059	3.1417	3.1205
22					3.1719	3.1819	3.2874	3.3588	3.5011	3.5227

[a] $D(i) = E(2i + 2) - E(2)$; this band is observed with signature 1/2 only.
[b] The first two levels have been omitted, $D(i) = E(i + 2) - E(2)$. With the data from Ref. [15], large deviations occur for the five levels $i = 13, 15, 17, 19, 21$. These deviations disappear when more recent data from Ref. [16] are used, as listed in this table.

exceptions which are close to 1%. The rest are irregular bands, most of which fall into two categories; the major part of them are bands with intruding single particle states and large signature splitting, some of them are bands of nuclei near the transition region.

As an illustration for the quality of fitting, five long bands are listed in Table I with parameter values given in Table II. It is quite impressive that the simple formulas with two parameters can give a good representation of the spectrum up to very high spin values, such as $I = 28.5$ for the ground state band $\frac{7}{2}[743]$ of ^{235}U and $I = 34.5$ for the band $\frac{1}{2}[541]$ of ^{171}Ta.

IV. COMPARISON WITH HARRIS EXPANSION

Besides the fitting of the band spectra, comparison with the Harris expansion forms a crucial test of the validity of our model for odd-A nuclei. At the beginning of Sec. II, it was shown that the most general result of the cranking BMH is that the Routhian E' is a function of ω. Thus, the Harris expansion

$$E' = E_0 - \tfrac{1}{2}\beta_1\omega^2 - \beta_2\omega^4 \tag{19}$$

is a very general expression for E', which may be valid for small values of ω. With the present model, expanding

TABLE II. Determined values of the parameters A ($\hbar^2$ MeV^{-1}), B ($\hbar^2$ MeV^{-2}), and rms deviation Δ_c for some bands. The values of the parameters β_1 ($\hbar^2$ MeV^{-1}) and β_2 ($\hbar^4$ MeV^{-3}) and the rms deviation Δ_h fitted by Harris expansion are listed for comparison with β_{1c} and β_{2c} calculated from Eq. (20) with the A and B values given in the same lines. The first five bands are those listed in Table I.

Bands	No.[a]	A	B	β_{1c}	β_{2c}	Δ_c	β_1	β_2	Δ_h
^{171}Ta $\frac{1}{2}[541]$[b]	17	36.06	1.384	36.98	25.09	0.0116	36.44	34.37	0.0108
	10	35.84	1.444	36.80	26.02	0.0079	36.60	31.45	0.0104
^{171}Ta $\frac{7}{2}[404]$	19	31.00	3.116	33.08	48.97	0.0041	32.02	72.40	0.0052
	10	30.66	3.288	32.85	51.16	0.0016	32.18	68.10	0.0044
^{233}U $\frac{5}{2}[633]$	23	84.74	1.890	86.00	80.33	0.0027	86.00	86.67	0.0036
	10	85.46	1.426	86.41	61.07	0.0012	86.28	71.55	0.0019
^{235}U $\frac{7}{2}[743]$	23	95.06	0.440	95.35	20.93	0.0112	95.20	22.85	0.0113
^{239}Pu $\frac{1}{2}[630]$[c]	23	79.01	1.420	79.96	56.24	0.0045	79.96	59.06	0.0048
	10	79.33	1.210	80.14	48.10	0.0035	80.20	46.75	0.0036
^{175}Ta $\frac{7}{2}[404]$	18	31.32	2.664	32.10	42.21	0.0062	32.30	59.62	0.0108
	10	31.88	2.349	33.45	37.83	0.0047	33.20	47.525	0.0066
^{175}Ta $\frac{5}{2}[402]$	12	31.10	1.700	32.33	26.64	0.0033	32.00	32.20	0.0051
^{175}Ta $\frac{9}{2}[514]$	10	36.72	0.898	37.32	16.89	0.0039	37.08	19.35	0.0045
^{173}Yb $\frac{5}{2}[512]$	10	44.00	0.515	44.34	11.35	0.0005	44.42	11.00	0.0006
^{171}Yb $\frac{5}{2}[512]$	9	39.80	1.026	40.48	20.49	0.0014	40.14	26.90	0.0045
^{173}Ta $\frac{7}{2}[404]$	17	31.76	2.465	33.40	39.57	0.0024	32.80	53.85	0.0054
	10	31.84	2.425	33.46	39.01	0.0021	33.20	47.50	0.0040
^{169}Tm $\frac{7}{2}[404]$	8	37.40	0.837	37.96	15.70	0.0008	37.90	16.90	0.0011
^{167}Tm $\frac{7}{2}[404]$	13	36.08	2.093	37.48	38.06	0.0043	37.00	48.09	0.0061
^{169}Lu $\frac{7}{2}[404]$	12	33.96	2.156	35.40	36.93	0.0032	35.00	46.12	0.0053
^{167}Lu $\frac{7}{2}[404]$	11	29.78	2.015	31.12	30.29	0.0020	30.80	37.63	0.0039
^{165}Lu $\frac{7}{2}[404]$	14	24.60	2.830	26.49	35.37	0.0106	25.20	55.22	0.0165

[a]Number of fitted levels.
[b]Decoupling constant $a_d = 4.40$.
[c]Decoupling constant $a_d = -0.587$.

Eq. (7) in terms of ω^2, we have

$$\beta_1 = (A + \tfrac{2}{3}BE_v) \qquad \beta_2 = \tfrac{1}{2}B(A + \tfrac{5}{36}BE_v)\ . \qquad (20)$$

With $E_v = 1$ MeV, the parameters A and B can be obtained from β_1 and β_2. From the above formulas, one can see more clearly that a change of E_v can be compensated by a small change of the parameters A and B. In case of $K = 1/2$, Eq. (19) is changed to

$$E' = E_0 + \tfrac{1}{2}(-1)^{I-1/2} a_d \hbar\omega - \tfrac{1}{2}\beta_1\omega^2 - \beta_2\omega^4\ . \qquad (21)$$

To test the validity of the present model, we shall first compare the precision of fitting spectra with the present model and the Harris expansion. Since the difference between the two formulas begins at the ω^6 terms, comparison for a short band will not lead to definite conclusions. In Table II, we have listed all bands with number of regular levels greater than 12 and some shorter regular bands chosen at random. With the exception of the first line of entry of the table, the rms deviations of the Harris expansion are consistently larger than those of our formula. The first entry of Table II is a state with large decoupling constant, for which the rms deviation of fitting with our formula is 0.0116 while it is 0.0108 for the Harris expansion. Both are large in comparison with the average rms deviation $\Delta = 0.005$ for regular bands.

Secondly, the values of β_1 and β_2 determined by least squares fitting of the empirical spectrum with the Harris

TABLE III. Identical bands are listed in groups separated by the vertical stripes (g for ground band).

^{186}Os g	^{187}Ir $\frac{3}{2}$[402]	^{154}Gd g	^{152}Sm g	^{161}Tm $\frac{5}{2}$[402]	^{165}Lu $\frac{5}{2}$[402]
^{180}W g	^{181}Os $\frac{1}{2}$[521]	^{185}Os $\frac{7}{2}$[503]	^{179}Re $\frac{5}{2}$[402]	^{170}Hf g	^{171}Ta $\frac{5}{2}$[402]
^{178}Hf g	^{179}Re $\frac{9}{2}$[514]	^{156}Gd g	^{175}Ta $\frac{7}{2}$[404]	^{176}Hf g	^{177}Ta $\frac{7}{2}$[404]
^{176}Yb g	^{173}Ta $\frac{1}{2}$[541]	^{182}W g	^{183}Re $\frac{5}{2}$[402]	^{169}Yb $\frac{5}{2}$[512]	^{167}Tm $\frac{1}{2}$[411]
^{155}Sm $\frac{1}{2}$[521]	^{159}Gd $\frac{1}{2}$[521]	^{163}Tm $\frac{7}{2}$[523]	^{171}Tm $\frac{7}{2}$[523]	^{163}Ho $\frac{7}{2}$[523]	^{173}Yb $\frac{5}{2}$[512]
^{174}W g	^{176}W g	^{161}Tm $\frac{7}{2}$[404]	^{165}Lu $\frac{7}{2}$[404]	^{181}W $\frac{5}{2}$[512]	^{175}Re $\frac{5}{2}$[402]
^{160}Er g	^{168}Hf g	^{164}Yb g	^{162}Er g	^{166}Yb g	^{167}Lu $\frac{5}{2}$[402]
^{163}Tm $\frac{7}{2}$[404]	^{167}Lu $\frac{7}{2}$[404]	^{155}Dy $\frac{11}{2}$[505]	^{172}Hf g	^{173}Ta $\frac{5}{2}$[402]	^{179}W $\frac{1}{2}$[521]
^{155}Sm $\frac{3}{2}$[521]	^{163}Dy $\frac{3}{2}$[521]	^{165}Er $\frac{3}{2}$[521]	^{159}Gd $\frac{5}{2}$[523]	^{159}Dy $\frac{3}{2}$[521]	^{159}Tb $\frac{3}{2}$[411]
^{167}Yb $\frac{1}{2}$[521]	^{159}Ho $\frac{7}{2}$[523]	^{173}Hf $\frac{5}{2}$[512]	^{157}Eu $\frac{5}{2}$[413]	^{165}Dy $\frac{3}{2}$[512]	^{165}Er $\frac{5}{2}$[523]
^{173}Tm $\frac{7}{2}$[523]	^{163}Dy $\frac{5}{2}$[523]	^{155}Sm $\frac{5}{2}$[523]	^{169}Hf $\frac{5}{2}$[523]	^{171}Er $\frac{5}{2}$[512]	^{161}Tb $\frac{3}{2}$[411]
^{173}Hf $\frac{5}{2}$[512]	^{175}Yb $\frac{5}{2}$[512]	^{177}Os $\frac{5}{2}$[512]	^{167}Er $\frac{1}{2}$[510]	^{175}Yb $\frac{1}{2}$[510]	^{169}Er $\frac{1}{2}$[510]
^{158}Dy g	^{175}Re $\frac{9}{2}$[524]	^{164}Er g	^{174}Hf g	^{175}Ta $\frac{5}{2}$[402]	^{179}Ta $\frac{7}{2}$[404]
^{161}Er $\frac{5}{2}$[523]	^{181}Os $\frac{7}{2}$[514]	^{169}Lu $\frac{5}{2}$[402]	^{173}Ta $\frac{7}{2}$[404]	^{175}Re $\frac{7}{2}$[404]	^{177}Os $\frac{1}{2}$[521]
^{171}Lu $\frac{7}{2}$[404]	^{170}Yb g	^{153}Eu $\frac{3}{2}$[411]	^{179}Ta $\frac{9}{2}$[514]	^{183}W $\frac{5}{2}$[512]	^{165}Tm $\frac{1}{2}$[411]
^{165}Er $\frac{11}{2}$[505]	^{157}Tb $\frac{3}{2}$[411]	^{170}Er g	^{172}Yb g	^{167}Hf $\frac{5}{2}$[523]	^{169}Tm $\frac{7}{2}$[404]
^{154}Nd g	^{159}Gd $\frac{1}{2}$[521]	^{169}Yb $\frac{5}{2}$[523]	^{171}Lu $\frac{9}{2}$[514]	^{155}Eu $\frac{5}{2}$[413]	^{165}Ho $\frac{3}{2}$[411]
^{174}Yb g	^{175}Lu $\frac{7}{2}$[404]	^{175}Yu $\frac{5}{2}$[402]	^{177}Hf $\frac{7}{2}$[514]	^{177}Lu $\frac{9}{2}$[514]	
^{173}Lu $\frac{7}{2}$[404]	^{166}Er g	^{175}Hf $\frac{1}{2}$[521]	^{179}W $\frac{7}{2}$[514]	^{162}Dy g	
^{179}Hf $\frac{1}{2}$[510]	^{173}Ta $\frac{9}{2}$[514]	^{161}Ho $\frac{7}{2}$[404]	^{155}Tb $\frac{3}{2}$[411]	^{165}Er $\frac{5}{2}$[512]	^{173}Hf $\frac{1}{2}$[521]
^{154}Sm g	^{173}W $\frac{5}{2}$[512]	^{169}Lu $\frac{7}{2}$[404]	^{157}Dy $\frac{11}{2}$[505]	^{153}Sm $\frac{1}{2}$[521]	^{167}Yb $\frac{1}{2}$[521]
^{175}Yb $\frac{1}{2}$[521]					
^{177}Ta $\frac{9}{2}$[514]	^{175}Ta $\frac{9}{2}$[514]	^{166}Er g	^{173}Lu $\frac{5}{2}$[402]	^{158}Gd g	^{163}Er $\frac{11}{2}$[505]
^{175}W $\frac{5}{2}$[512]					
^{169}Er $\frac{5}{2}$[512]	^{161}Er $\frac{3}{2}$[521]	^{158}Sm g	^{157}Eu $\frac{3}{2}$[411]	^{153}Eu $\frac{5}{2}$[413]	^{171}Tm $\frac{1}{2}$[411]
^{173}Lu $\frac{9}{2}$[514]	^{161}Ho $\frac{1}{2}$[411]	^{175}Lu $\frac{9}{2}$[514]			
^{159}Dy $\frac{11}{2}$[505]	^{160}Gd g	^{164}Dy g	^{163}Ho $\frac{7}{2}$[404]	^{169}Er $\frac{5}{2}$[523]	^{169}Er $\frac{3}{2}$[512]
^{171}Hf $\frac{1}{2}$[521]	^{177}Yb $\frac{1}{2}$[510]	^{155}Eu $\frac{3}{2}$[411]			

TABLE IV. Examples of the identical band groups. For definition of $G(I)$, see Eqs. (22) and (23) in the text.

I	^{155}Tb $\frac{3}{2}[411]$ $G_1(\frac{3}{2}+I)$ (keV)	^{165}Er $\frac{5}{2}[512]$ $G_1(\frac{3}{2}+I)$ (keV)	^{161}Ho $\frac{7}{2}[404]$ $G_1(\frac{3}{2}+I)$ (keV)	^{173}Ta $\frac{9}{2}[514]$ $G_1(\frac{3}{2}+I)$ (keV)	^{173}Hf $\frac{1}{2}[521]$ $\bar{G}_2(I)$ (keV)	^{155}Tb $\frac{3}{2}[411]$ $\bar{G}_2(I)$ (keV)	^{161}Ho $\frac{7}{2}[404]$ $\bar{G}_2(I)$ (keV)	^{173}Ta $\frac{9}{2}[514]$ $\bar{G}_2(I)$ (keV)
0	65							
1	90	95			127			
2	118	111	118		177	182		
3	135	136	141	140	222	230		
4	168		163	166	270	277	282	
5			183	189	315	321	325	331
6			202	213	357	361	365	379
7			219	231	396	398	403	423
8			236	253	433	434	438	464
9			248	264		476	469	500

expansion can be compared with those calculated from Eq. (20). We can see from Table II that in every case the fitted β_1 agrees with the calculated value, while the fitted β_2 is greater than that given by Eq. (20). A simple explanation of this phenomenon is that the error involved by neglecting higher order terms in the Harris expansion is partly compensated by increasing β_2. The coefficient β_2 in the Harris expansion is a renormalized quantity, not a reliable coefficient of expansion.

V. IDENTICAL BANDS AT NORMAL DEFORMATION

In analogy with the superdeformed bands, the analysis of the transition energies in neighboring odd and even-even nuclei led Garrett *et al.* [10] to the suggestion of identical bands at normal deformation. Later on, it was discovered that the existence of the identical bands is a rather widespread phenomenon. Identical bands are discovered either between nearby even-even nuclei [11, 12] or between odd-A nuclei and their even-even neighbors [10–13]. It is further demonstrated that there are no sharp lines of demarcation between identical and nonidentical band pairs [11]. To promote research in this direction, it is necessary to have a clear and universal definition of the identical bands and a method to locate them. First of all, the identity between the bands is established by comparing the γ-ray energies of the corresponding transitions.

To compare the γ-ray energies of the different bands, the following conventions are usually adopted.

(1) Let I be the level's spin; then the γ-ray energies $G_1(I)$ and $G_2(I)$ are defined as

$$G_1(I) = E(I+1) - E(I) ,$$

$$G_2(I) = E(I+2) - E(I) . \tag{22}$$

(2) For two even-even nuclear bands, $G_2(I)$ can be compared. For two odd-A nuclear bands with $K \neq 1/2$, $G_1(I)$ can be directly compared.

TABLE V. Examples of the identical band pairs.

I	^{176}Yb g $G_2(I)$ (keV)	^{173}Ta $\frac{1}{2}[541]$ $G_2(\frac{5}{2}+I)$ (keV)	^{158}Dy g $G_2(I)$ (keV)	^{175}Re $\frac{9}{2}[514]$ $\bar{G}_2(I)$ (keV)	^{173}Yb $\frac{5}{2}[512]$ $G_1(\frac{5}{2}+I)$ (keV)	^{167}Ho $\frac{7}{2}[523]$ $G_1(\frac{5}{2}+I)$ (keV)	^{167}Tm $\frac{1}{2}[411]$ $\bar{G}_2(I)$ (keV)	^{169}Yb $\frac{5}{2}[512]$ $\bar{G}_2(I)$ (keV)
0	82	83	99		79			
1					101	100	125	
2	190	187	218		122	122	171	
3					144	144	219	221
4	293	291	321		165	165	262	266
5					186	188	307	308
6	389	390	406	421	206	205	345	347
7					225	230	385	383
8	477	482	476	484				
10	554	563	529	524				
12			563	561				
14			578	580				

(3) For comparison of an even-even band and an odd band, $G_2(I)$ is compared with the equivalent $\bar{G}_2(I)$ given by the expression

$$\bar{G}_2(I) = \tfrac{1}{2}[G_2(I-1/2) + G_2(I+1/2)] \, . \tag{23}$$

If the odd nuclear band is a $K = 1/2$ band with a large and positive decoupling constant (say, larger than 5), $G_2(I)$ can be directly compared with $G_2(I+5/2)$ of the odd nuclear band.

(4) For comparison of a $K = 1/2$ and another odd-A band, the $\bar{G}_2(I)$ constructed according to Eq. (23) are compared, with I taken as all positive integers.

With these rules, the criterion for a pair of identical bands can be taken as that the difference between the compared transition energies is less than 10 keV. For short bands with no more than five levels, this condition must be met for all the compared transitions. For long bands, this condition must also be met for the first four or five transitions. The condition of the identity may deteriorate slowly with increasing spin, but the difference must be kept to less than 30 keV for all regular levels. Compared with the the superdeformed bands, this is a rather loose criterion. Yet it is strict enough to discard some of the claimed identical pairs.

In principle, it is possible to make direct comparison between all regular bands with the above convention. It will be tedious work to select the identity pairs from hundreds of bands, since the objects to be compared are not just the experimentally determined level energies, but transition energies and averages of transition energies.

Comparison of the kinetic moment of inertia $\langle J \rangle/\omega$ has been proposed as a way of locating the identical pairs [13]. For our model, this is equivalent to the comparison of the parameters A and B for two bands. Only the bands with small difference in A and not very large difference in B can be candidates for an identical pair. Each possible pair is examined separately, and only those meeting the identity criterion are classified as identical bands.

All of the spotted identical bands of the rare-earth nuclei are listed in Table III and a few examples for comparing the transition energies are given in Tables IV and V. This investigation confirms decisively that the identical band is a common phenomenon for the normally deformed bands. It can be seen from Table III that nearly half of the bands in the rare-earth region are identically connected. Some others may be left out because our analysis is limited to regular bands. Tables III and IV also show that more than two bands can be connected by identity links to form a group of identical bands. However, the existence of the different identical bands may have different nuclear structure significance. Hence study of the different kinds of identical bands and their gradual loss of identities may form interesting systematics of the rotational spectra. In any case, the success in locating identical bands may be considered as a credit to the present model and an illustration of its usefulness in the analysis of the rotational spectra.

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