Benchmark solutions for n-d breakup amplitudes

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Benchmark solutions for three-nucleon breakup amplitudes in n-d scattering have been produced by two groups separately using momentum space and configuration space techniques. These results have been obtained at two energies for both spin doublet and quartet configurations for an s-wave potential model and are in excellent agreement.

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Benchmark comparisons or "homework problems" have become increasingly common and useful in few-nucleon physics in recent years [1–5]. The remarkable successes in this field in solving seminal problems originally identified decades ago have been followed in some cases by the successful application of new techniques, which promise to have wider and easier applicability than some of the original methods. These efforts can be greatly aided by having benchmarked results for selected problems for purposes of comparison and analysis, and for establishing levels of accuracy.

By now the three-nucleon bound-state problem has yielded to a wide variety of methods[6, 7]: Faddeev calculations in momentum and configuration space[1],

TABLE I. Breakup amplitudes for n-d scattering in units of $\mathrm{fm}^{-\frac{3}{2}}$ presented in the format x.xx[n] $\equiv$ x.xx 10^n.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
				14.1 MeV doublet						
LA/Iowa Re ($^1S_0$)	8.79[-2]	8.59[-2]	8.03[-2]	7.28[-2]	6.65[-2]	6.41[-2]	6.84[-2]	8.43[-2]	1.11[-1]	-3.47[-1]
LA/Iowa Im ($^1S_0$)	1.84[-1]	1.82[-1]	1.72[-1]	1.50[-1]	1.14[-1]	7.19[-2]	2.60[-2]	-3.49[-2]	-1.78[-1]	-4.69[-1]
Bochum Re ($^1S_0$)	8.80[-2]	8.60[-2]	8.04[-2]	7.29[-2]	6.65[-2]	6.41[-2]	6.84[-2]	8.42[-2]	1.11[-1]	-3.48[-1]
Bochum Im ($^1S_0$)	1.84[-1]	1.82[-1]	1.72[-1]	1.50[-1]	1.14[-1]	7.18[-2]	2.60[-2]	-3.49[-2]	-1.78[-1]	-4.68[-1]
LA/Iowa Re ($^3S_1$)	-2.43[-2]	-2.21[-2]	-1.60[-2]	-7.89[-3]	-4.11[-4]	4.68[-3]	5.10[-3]	-2.40[-3]	-1.82[-2]	-2.95[-2]
LA/Iowa Im ($^3S_1$)	8.01[-2]	8.45[-2]	9.80[-2]	1.20[-1]	1.48[-1]	1.76[-1]	1.99[-1]	2.14[-1]	2.09[-1]	1.69[-1]
Bochum Re ($^3S_1$)	-2.44[-2]	-2.22[-2]	-1.60[-2]	-7.62[-3]	-4.52[-4]	4.65[-3]	5.14[-3]	-2.43[-3]	-1.82[-2]	-2.95[-2]
Bochum Im ($^3S_1$)	8.00[-2]	8.44[-2]	9.79[-2]	1.20[-1]	1.48[-1]	1.76[-1]	1.99[-1]	2.14[-1]	2.09[-1]	1.69[-1]
				14.1 MeV quartet						
LA/Iowa Re ($^3S_1$)	-1.92[-1]	-1.93[-1]	-1.94[-1]	-1.89[-1]	-1.75[-1]	-1.58[-1]	-1.47[-1]	-1.51[-1]	-1.78[-1]	-2.20[-1]
LA/Iowa Im ($^3S_1$)	3.65[-1]	3.67[-1]	3.70[-1]	3.72[-1]	3.73[-1]	3.81[-1]	4.00[-1]	4.31[-1]	4.62[-1]	4.68[-1]
Bochum Re ($^3S_1$)	-1.92[-1]	-1.93[-1]	-1.94[-1]	-1.89[-1]	-1.75[-1]	-1.58[-1]	-1.47[-1]	-1.51[-1]	-1.78[-1]	-2.20[-1]
Bochum Im ($^3S_1$)	3.65[-1]	3.67[-1]	3.70[-1]	3.72[-1]	3.73[-1]	3.81[-1]	4.00[-1]	4.31[-1]	4.62[-1]	4.68[-1]
				42.0 MeV doublet						
LA/Iowa Re ($^1S_0$)	5.01[-1]	4.94[-1]	4.59[-1]	3.62[-1]	2.19[-1]	8.78[-2]	-3.50[-2]	-2.10[-1]	-7.05[-1]	-4.46[0]
LA/Iowa Im ($^1S_0$)	5.56[-1]	5.91[-1]	6.70[-1]	6.66[-1]	4.63[-1]	2.09[-1]	-2.57[-2]	-2.99[-1]	-8.14[-1]	1.63[0]
Bochum Re ($^1S_0$)	4.99[-1]	4.92[-1]	4.58[-1]	3.63[-1]	2.18[-1]	8.73[-2]	-3.50[-2]	-2.10[-1]	-7.05[-1]	-4.45[0]
Bochum Im ($^1S_0$)	5.56[-1]	5.91[-1]	6.70[-1]	6.67[-1]	4.63[-1]	2.09[-1]	-2.53[-2]	-2.98[-1]	-8.14[-1]	1.63[0]
LA/Iowa Re ($^3S_1$)	-1.30[-2]	1.33[-2]	1.00[-1]	2.42[-1]	3.85[-1]	5.07[-1]	6.20[-1]	7.00[-1]	5.69[-1]	-8.52[-2]
LA/Iowa Im ($^3S_1$)	2.63[-1]	2.66[-1]	2.85[-1]	3.70[-1]	5.39[-1]	7.23[-1]	9.34[-1]	1.25[0]	1.70[0]	1.83[0]
Bochum Re ($^3S_1$)	-1.17[-2]	1.48[-2]	1.00[-1]	2.37[-1]	3.85[-1]	5.06[-1]	6.19[-1]	7.00[-1]	5.67[-1]	-8.54[-2]
Bochum Im ($^3S_1$)	2.64[-1]	2.66[-1]	2.85[-1]	3.67[-1]	5.39[-1]	7.23[-1]	9.33[-1]	1.25[0]	1.70[0]	1.83[0]
				42.0 MeV quartet						
LA/Iowa Re ($^3S_1$)	1.48[-2]	9.22[-4]	-3.21[-2]	-3.09[-2]	7.70[-2]	2.52[-1]	4.51[-1]	6.53[-1]	6.93[-1]	1.05[-1]
LA/Iowa Im ($^3S_1$)	1.69[0]	1.74[0]	1.87[0]	1.92[0]	1.80[0]	1.68[0]	1.70[0]	1.95[0]	2.52[0]	3.06[0]
Bochum Re ($^3S_1$)	1.48[-2]	1.21[-3]	-3.20[-2]	-3.16[-2]	7.69[-2]	2.52[-1]	4.51[-1]	6.53[-1]	6.92[-1]	1.05[-1]
Bochum Im ($^3S_1$)	1.69[0]	1.74[0]	1.87[0]	1.92[0]	1.80[0]	1.67[0]	1.70[0]	1.95[0]	2.52[0]	3.06[0]

Green's function Monte Carlo (GFMC) techniques, variational methods, and hyperspherical harmonics expansions (HHE). A benchmark for zero-energy n-d and p-d scattering compared Faddeev and HHE calculations[2]. Neutron-deuteron scattering below breakup threshold has also been calculated by these methods and compared[3]. A benchmark calculation[4] of scattering phase shifts δ and inelasticities η exists for n-d scattering above breakup threshold. Spin observables in elastic neutron-deuteron scattering calculated using high-rank separable approximations to realistic nucleon-nucleon forces have been compared[5] in two approaches, one relying on the finite-rank representation of the NN forces[8] and the other, which is general[9] and is being applied herein. In this work we complement the elastic phase-shift parameters of Ref. [4] with breakup amplitudes for the reaction $n + d \to n + n + p$ at two initial neutron (laboratory) energies, 14.1 MeV and 42 MeV, corresponding to two of the three energies used in that reference. The third of those energies was so low that there was virtually no breakup.

We solve the nonrelativistic Schrödinger equation containing the revised (and definitely not unique) Malfliet-Tjon I-III model s-wave potential[4]. The (spin) triplet and singlet potentials and nucleon mass are given by

$$V_t(r) = \frac{1}{r}\left(-626.885\, e^{-1.55r} + 1438.72\, e^{-3.11r}\right) , \qquad (1)$$

$$V_s(r) = \frac{1}{r}\left(-513.968\, e^{-1.55r} + 1438.72\, e^{-3.11r}\right) , \qquad (2)$$

and

$$\hbar^2/m = 41.47\,\mathrm{MeV\,fm}^2 , \qquad (3)$$

which leads to a deuteron binding energy of $E_d = 2.2307$ MeV. The breakup amplitude is defined by the asymp-

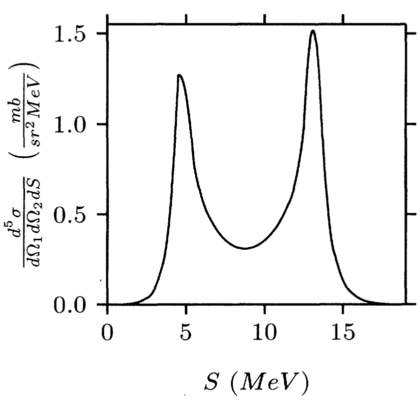

FIG. 1. The breakup cross section for n-d scattering at 14.1 MeV (lab) versus the arclength S. Two neutrons are detected at (lab angles) $\theta_1 = 45.0°$, $\theta_2 = 53.56°$, and $\Delta\phi = 180.0°$. These kinematic conditions include one final-state interaction peak and a peak where the S curve comes very close to a final-state interaction configuration. The solid and dashed lines for the two approaches completely overlap and cannot be distinguished.

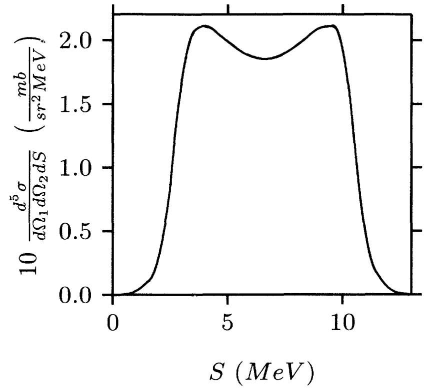

FIG. 2. The breakup cross section for n-d scattering at 14.1 MeV (lab) versus the arclength S. Two neutrons are detected at (lab angles) $\theta_1 = 51.02°$, $\theta_2 = 51.02°$, and $\Delta\phi = 120.0°$. These kinematic conditions include a space-star configuration, where the three nucleons have equal energies and interparticle angles of 120° in the center-of-mass system. The solid and dashed lines for the two approaches completely overlap and cannot be distinguished.

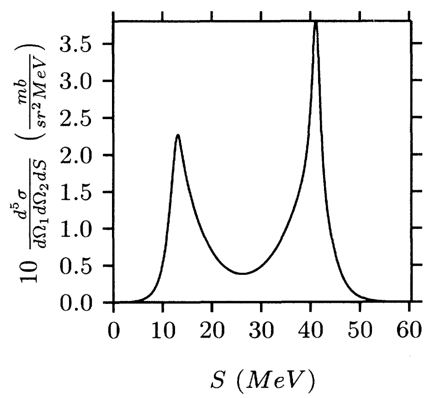

FIG. 3. The same as in Fig. 1, but for 42.0 MeV. Two neutrons are detected at $\theta_1 = 45.0°$, $\theta_2 = 60.54°$, and $\Delta\phi = 180.0°$.

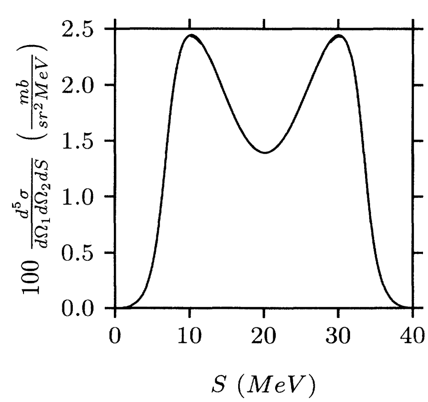

FIG. 4. The same as in Fig. 2, but for 42.0 MeV. Two neutrons are detected at $\theta_1 = 53.61°$, $\theta_2 = 53.61°$, and $\Delta\phi = 120.0°$.

totic three-nucleon Faddeev wave function in the center-of-mass frame, ψ_1, which has the form[10]:

$$\psi_1(\boldsymbol{x},\boldsymbol{y}) \sim \frac{\sin(\boldsymbol{p}\boldsymbol{y})u(x)}{yx} + \frac{\mathcal{A}(\theta)\, e^{iK\rho}}{(K\rho)^{\frac{5}{2}}} + \cdots \;, \qquad (4)$$

where scattered elastic waves have been ignored, and the Schrödinger (total) wave function is given by ψ_1 and its Faddeev permutations: $\Psi \equiv \psi_1 + \psi_2 + \psi_3$. The standard Jacobi coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$, as well as the usual hyperspherical coordinates $\rho = (x^2 + \frac{4}{3}y^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}\left[\frac{2y}{\sqrt{3}x}\right]$ determine the asymptotic form in terms of the initial neutron (laboratory) momentum $\boldsymbol{p}$ and breakup momentum K ($E = \hbar^2 K^2/M$). The relationship between the standard (reduced) breakup amplitude A and the T matrix, T, is discussed in Ref. [10] and is given by

$$A(\theta) = \frac{-K}{2} e^{i\pi/4} \sqrt{2\pi} \sin(\theta)\, e^{i\delta(k)}\, T(k) \;, \qquad (5)$$

where $k = K\cos(\theta)$ and $q = 2K\sin(\theta)/\sqrt{3}$ are the usual pair and spectator wave numbers. The *unreduced* breakup amplitude $\mathcal{A}$ used in Eq. (4) is defined in terms of the *reduced* one, A, by

$$\mathcal{A}(\theta) = \frac{A(\theta)\, K^2}{\frac{\sqrt{3}}{2}\sin(\theta)\cos(\theta)} \;. \qquad (6)$$

The Los Alamos/Iowa results were calculated in configuration space, where the Faddeev approach leads to a set of coupled integro-differential equations, whose unique solution is guaranteed by the boundary conditions specified in Eq. (4). The Bochum calculations were performed in momentum space, where the Faddeev approach leads to integral equations formulated directly for the off-shell breakup amplitudes [9]. We emphasize the totally different mathematical structure of the two formulations. Details of the configuration-space approach will be presented elsewhere [11], while those of the momentum-space approach are contained in Ref. [9]. Calculated breakup amplitudes $\mathcal{A}(\theta)$ are presented in Table I. The results obtained in the two approaches are in excellent agreement. Except for the triplet channel of the 42 MeV doublet amplitude at one angle, the level of agreement is always better than 1% of the absolute magnitude. Even for this case the agreement is only slightly worse than 1%.

Representative breakup cross sections formed from these amplitudes are depicted in Figs. 1–4. The agreement between the two approaches is again better than 1% and the corresponding curves in the figures are not distinguishable.

For completeness we compare in Table II the elastic phase shifts δ and inelasticities η for the various energies and cases. Agreement is excellent.

TABLE II. *n-d* elastic-scattering phase shifts and inelasticities.

	LA/Iowa	Bochum
	14.1 MeV doublet	
Re(δ)	105.48°	105.50°
η	0.4648	0.4649
	14.1 MeV quartet	
Re(δ)	68.95°	68.96°
η	0.9782	0.9782
	42.0 MeV doublet	
Re(δ)	41.34°	41.37°
η	0.5024	0.5022
	42.0 MeV quartet	
Re(δ)	37.71°	37.71°
η	0.9035	0.9033

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