Light front dynamics of one boson exchange models of the two-nucleon system

Michael G. Fuda and Yingfang Zhang
Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260
(Received 26 August 1994)

A general procedure for constructing front form, one boson exchange models for the two-nucleon system is developed. The procedure entails the construction of a mass-square operator which when used in conjunction with a free spin operator leads to an exactly Poincaré invariant model. An exchange model involving π, η, ρ, ω, δ, and σ mesons is constructed, and calculations of nucleon-nucleon phase shifts and deuteron properties are carried out. A comparison with the Bonn B potential, and a fit to a recent nucleon-nucleon phase shift analysis are presented.

PACS number(s): 13.75.Cs, 21.30.+y, 21.45.+v, 24.10.Jv

I. INTRODUCTION

Meson-exchange models for the nucleon-nucleon interaction originated with Yukawa's basic hypothesis that the nuclear force is mediated by the exchange of a massive particle [1]. Yukawa's idea of a scalar meson field interacting with a nucleon was extended to vector fields by Proca [2]; and to pseudoscalar, pseudovector, and tensor fields by Kemmer [3]. The combination of vector and pseudoscalar fields was considered in the early 1940's by Møller and Rosenfeld [4], and by Schwinger [5]. Thus the basic field theory that is needed to describe the interaction between mesons and nucleons has been in existence for some time now.

As is well known the pion was the first meson to be detected experimentally [6]. The early 1960s saw the discovery of the heavy mesons; in particular, the ρ meson [7] and the ω meson [8], both of which are vector mesons. Models based on the exchange of these and other mesons have been constructed and analyzed for some time now. The review article on the Bonn meson-exchange model by Machleidt *et al.* [9], as well as the general review by Machleidt [10], give extensive references on meson-exchange models for the nucleon-nucleon interaction. A somewhat more recent overview has been presented by Holinde [11].

The one-boson-exchange (OBE) model we will study here involves the same mesons as the OBE models given in Refs. [9,10], i.e. the $\pi(0^-,1)$, $\eta(0^-,0)$, $\rho(1^-,1)$, $\omega(1^-,0)$, $\delta(0^+,1)$, and $\sigma(0^+,0)$ mesons. The quantities in parentheses are (J^P,I); where J, P, and I are the corresponding meson's spin, parity, and isospin, respectively. The σ meson is viewed here as a fictitious meson which is introduced to take account of processes that are not treated explicitly in the OBE model. This includes, for example, contributions from 2π exchange which are not accounted for by the exchange of the ρ meson, as well as $\pi\rho$ exchange. It should be noted that the ρ and ω are two- and three-pion resonances, respectively.

The basic building blocks for a meson-exchange model are the various meson-baryon vertices. The specification of these vertices involves the form of the coupling, which is usually given in terms of an interaction Lagrangian density; the values of the coupling constants and meson masses; and some model for the meson-baryon form factors [9–11]. The form factor at a meson-baryon vertex takes into account the effects of the vertex's extension.

Various approaches exist for relating these vertices to the observables of a hadronic system. Manifestly covariant treatments are based on the use of the Bethe-Salpeter equation [12], or on one of its three-dimensional reductions. The three-dimensional reductions that are most widely used are due to Blankenbecler-Sugar [13], and to Gross [14]. Tjon and his collaborators [15] have employed both the Bethe-Salpeter equation and the Blankenbecler-Sugar equation. The most recent application of the Gross equation to the two-nucleon system is given in Ref. [16].

Holinde and his collaborators have made extensive use of time-ordered perturbation theory in developing the Bonn meson-exchange model for the nucleon-nucleon interaction [9–11]. Johnson's method of folded diagrams [17] has been used to eliminate the energy dependence of the amplitudes obtained from time-ordered perturbation theory. This leads to instantaneous interactions which can be conveniently used in calculating the properties of system with more than two nucleons.

The purpose of the present work is to introduce a framework for developing meson-exchange models within the context of the *front form* of relativistic quantum mechanics. Relativistic quantum mechanics arises when it is required that the state vectors of a quantum-mechanical system transform according to a unitary representation of the Poincaré group. The subgroup of continuous transformations, the so-called *proper* subgroup, involves 10 generators; the 4 components P_μ $(\mu = 0,1,2,3)$ of the four-momentum operator, and the 6 independent components $J_{\mu\nu} = -J_{\nu\mu}$ of the angular momentum tensor. These 10 operators must satisfy a set of commutation relations [see (2.5)], which is usually referred to as the *Poincaré algebra.* Several subsets of these generators have the property that they satisfy a *closed* subset of these commutation relations, and are therefore associated with a subgroup of the proper Poincaré transformations. Some of these subgroups are associated with three-dimensional hypersurfaces in Minkowski space that do not contain timelike directions. Each *form* of rela-

tivistic quantum mechanics is associated with such a hypersurface and its corresponding subgroup [18]. These subgroups are called kinematic subgroups [19] or stability groups [20]. The most obvious form, i.e., the *instant form*, is based on the hypersurface t =const, while the light front form is based on the *null plane* $ct + z = 0$. In each form of relativistic quantum mechanics the subset of generators associated with the form's hypersurface is chosen to be noninteracting, while the remaining generators contain interactions.

In the instant form the three-momentum $\mathbf{P}$ and the angular momentum $\mathbf{J}$ are noninteracting, while the Hamiltonian $H_{\rm inst}$ and $\mathbf{K}$, the generator of rotationless boosts, contain interactions. A rotationless boost is a Lorentz transformation which relates two inertial frames moving relative to each other with the corresponding spatial axes parallel. The instant form is like nonrelativistic quantum mechanics in that the state vectors are specified on the t =const hypersurfaces, however the nonrelativistic boost operators, which generate Galilean transformations, do not contain interactions.

In light front dynamics the operators $\bar{P} \equiv [(H_{\rm inst} + \mathbf{e}_3 \cdot \mathbf{P})/\sqrt{2}, \mathbf{P}_\perp]$, $J_3 \equiv \mathbf{e}_3 \cdot \mathbf{J}$, $K_3 \equiv \mathbf{e}_3 \cdot \mathbf{K}$, and $\mathbf{B} \equiv [\mathbf{K}_\perp - (\mathbf{e}_3 \times \mathbf{J})]/\sqrt{2}$ are noninteracting; while $H \equiv (H_{\rm inst} - \mathbf{e}_3 \cdot \mathbf{P})/\sqrt{2}$ and $\mathbf{S} \equiv [\mathbf{K}_\perp + (\mathbf{e}_3 \times \mathbf{J})]/\sqrt{2}$ are dynamical operators. Here $\mathbf{e}_3$ is a unit vector along the three-axis, and $\perp$ indicates spatial components transverse to this axis.

An important advantage of the front form over the instant form is that the three *noninteracting* operators K_3, B_1, and B_2 generate a subset of boosts, called the *front form boosts*, which form a *subgroup* of the stability group of the null plane [21]. As a consequence of this there is a clean separation of a system's variables into *internal* and *external* variables [20]. The dynamical content of a model is expressed in terms of the internal variables, while the external variables are associated with the kinematics.

A disadvantage of the light front form is that the transverse components of the angular momentum, i.e., $\mathbf{J}_\perp = [\mathbf{e}_3 \times (\mathbf{B} - \mathbf{S})]/\sqrt{2}$, contain interactions. This causes difficulties in the treatment of angular momentum and three-rotations. It is possible to express the interacting light front generators H and $\mathbf{S}$ in terms of a mass operator M, a relative angular momentum or spin operator $\boldsymbol{\mathcal{J}}$ [see (2.16) and (2.17)], and the kinematic generators [20]. It *does not* violate the commutation rules of the Poincaré algebra to choose M to be interacting and $\boldsymbol{\mathcal{J}}$ to be noninteracting [21–24]. Such a *free-spin model* circumvents the difficulty of dealing with an interacting angular-momentum operator, and is the type of model we will consider here.

In general light front models obtained from manifestly covariant models, such as the covariant harmonic oscillator [25], and from quantum field theory involve interacting spin operators $\boldsymbol{\mathcal{J}}$ [20]. Starting with the work of Karmanov [26,27], the author has developed a formalism [28–32] for dealing with interacting $\boldsymbol{\mathcal{J}}$'s. The standard picture of light front dynamics employs the null plane $ct + z = 0$; whereas the author's formalism, called *the new picture*, employs a set of null planes $\xi \cdot x = 0$, where ξ is a lightlike four-vector. Numerical calculations carried out within the framework of the new picture have shown how to extract from a two-particle interaction obtained from a quantum field theory, an interaction which leads to an exactly Poincaré invariant, front form, free-spin model [31,32]. Such a procedure is applied here to OBE models, and is used to calculate nucleon-nucleon phase shifts for nucleon laboratory kinetic energies below 300 MeV, as well as some deuteron properties.

The outline of this paper is as follows. Those aspects of relativistic quantum mechanics that are necessary for an understanding of the front form are reviewed and summarized in Sec. II. Section III discusses the Melosh rotation [33], and how it can be used to construct two-particle basis states which lead to a simple representation for the noninteracting, two-particle spin operator. The method for obtaining a front form, OBE model for the nucleon-nucleon interaction is presented in Sec. IV. Relative three-momentum variables are introduced, and a method for ensuring the Poincaré invariance of the OBE model is given. The partial-wave matrix elements of the OBE, nucleon-nucleon potential are derived in Sec. V. The numerical calculations are presented in Sec. VI, and a discussion of the results and suggestions for future work are given in Sec. VII. Throughout units in which $\hbar = c = 1$ are used.

II. GENERAL BACKGROUND

The elements (a, b) of the Poincaré group consist of the Lorentz transformations a and spacetime translations b that appear in the inhomogeneous Lorentz transformations $x' = ax + b$. In the passive interpretation x and x' refer to the spacetime coordinates of the same event in two different inertial frames, i.e., the x frame and x' frame, respectively.

The proper subgroup of the Poincaré group involves only continuous transformations. This subgroup is a ten-parameter group; four parameters are associated with translations in four-dimensional spacetime, while the other six are associated with "rotations" in spacetime. Three of these six can be associated with true rotations in three-dimensional space, while the other three can be associated with rotationless boosts. In what follows the expression *Poincaré group* refers only to the subgroup of continuous transformations.

In a relativistic quantum-mechanical model formulated on a Hilbert space there exists a set of operators $U(a, b)$ which form a unitary representation of the Poincaré group. If $|\Psi\rangle$ and $|\Psi'\rangle$ are states associated with the x frame and x' frame, respectively, they are related by $|\Psi'\rangle = e^{i\theta} U(a, b)|\Psi\rangle$, where the factor $e^{i\theta}$ appears because states are only defined to within a phase. If $|\Phi\rangle$ and $|\Psi\rangle$ are two different states, clearly $|\langle\Phi'|\Psi'\rangle|^2 = |\langle\Phi|\Psi\rangle|^2$, i.e., probabilities are invariant.

If the transformations $x' = ax + b$ and $x'' = a'x' + b'$ are combined, then $x'' = a''x + b''$ with $a'' = a'a$ and $b'' = a'b + b'$. Accordingly the law of combination for the Poincaré group is

$$(a', b') \cdot (a, b) = (a'a, a'b + b') = (a'', b'') \ . \qquad (2.1)$$

Clearly the group is non-Abelian. In order for the U's to represent the group, they must satisfy the rule

$$U(a',b')U(a,b) = U(a'',b'') \ . \tag{2.2}$$

For the infinitesimal transformation

$$a_{\mu\nu} = g_{\mu\nu} + \varepsilon_{\mu\nu} \ \ (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}), \ \ b_\mu = \varepsilon_\mu \ , \tag{2.3}$$

where $g_{\mu\nu}$ is the metric tensor, and the epsilons are the infinitesimals; the corresponding unitary operator can be written in the form

$$U(a,b) = 1 + i\varepsilon_\mu P^\mu - \frac{i}{2}\varepsilon_{\mu\nu}J^{\mu\nu} \ \ (J^{\mu\nu} = -J^{\nu\mu}) \ . \tag{2.4}$$

The multiplication rule (2.2) implies that the 10 Hermitian generators $\{P_\mu, J_{\mu\nu}\}$ satisfy the commutation rules

$$[P_\mu, P_\nu] = 0 \ , \tag{2.5a}$$
$$[J_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \ , \tag{2.5b}$$
$$[J_{\mu\nu}, J_{\rho\lambda}] = i(g_{\mu\lambda}J_{\nu\rho} + g_{\nu\rho}J_{\mu\lambda} - g_{\mu\rho}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\rho}) \ . \tag{2.5c}$$

These commutation rules, which define the so-called Poincaré algebra, are valid for any choice of the components $\{P_\mu, J_{\mu\nu}\}$ and the corresponding metric $(g_{\mu\nu})$.

In light front dynamics it is convenient to work with the metric

$$(g_{\mu\nu}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ , \tag{2.6}$$

which goes with the components

$$(x^\mu) = \left(x^0 = \frac{ct+z}{\sqrt{2}}, x^1 = x, x^2 = y, x^3 = \frac{ct-z}{\sqrt{2}}\right) = (x^0, \mathbf{x}_\perp, x^3) = (\bar{x}, x^3) \ . \tag{2.7}$$

A convenient notation for the light front components of the four-momentum and the angular-momentum tensor is given by [34]

$$\bar{P} = (P^0, \mathbf{P}_\perp), \ \ H = P^3,$$
$$\mathbf{B} = (B_1, B_2, 0) = (J_{13}, J_{23}, 0) \ , \tag{2.8}$$
$$K_3 = J_{03}, \ \ J_3 = J_{12}, \ \ \mathbf{S} = (S_1, S_2, 0) = (J_{10}, J_{20}, 0) \ .$$

The light front components P^μ are related to the more conventional components as in (2.7). The component P^3 is called H, since it is a Hamiltonian in that it generates translations in the light front "time" $x^0 = (ct+z)/\sqrt{2}$. The transverse components of the angular momentum $\mathbf{J}$ and rotationless boost operator $\mathbf{K}$ are related to the light front components $J_{\mu\nu}$ by

$$J_1 = (S_2 - B_2)/\sqrt{2}, \ \ J_2 = (B_1 - S_1)/\sqrt{2} \ , \tag{2.9a}$$
$$K_1 = (S_1 + B_1)/\sqrt{2}, \ \ K_2 = (S_2 + B_2)/\sqrt{2} \ . \tag{2.9b}$$

It follows from (2.3) and (2.4) that the generators

$$\bar{P}, \mathbf{B}, K_3, J_3 \ \ \text{(stability group generators)} \tag{2.10}$$

induce transformations on the light front hypersurface $ct + z = 0$. Accordingly these generators are taken to be noninteracting in light front dynamics. As H plays the role of a Hamiltonian in light front dynamics, it contains an interaction. It follows from (2.5b), (2.6) and (2.8) that

$$[S_r, P^s] = -i\delta_{rs}H, \ \ r,s = 1,2, \tag{2.11}$$

so S_1 and S_2 must also be interacting. Equation (2.9a) implies that the transverse components of $\mathbf{J}$ contain interactions. The *new picture* [28–32] is a way of dealing with this peculiar feature of light front dynamics.

An attractive feature of light front dynamics is that it allows for a clean separation of matrix elements into an *external* factor and an *internal* factor. It is always possible [20] to construct states $|\bar{p},\alpha\rangle$ such that $\bar{P}$, $\mathbf{B}$ and K_3 have the representations

$$\langle\bar{p},\alpha|\bar{P} = \bar{p}\langle\bar{p},\alpha|, \tag{2.12}$$

$$\langle\bar{p},\alpha|B_r = -ip^0\frac{\partial}{\partial p^r}\langle\bar{p},\alpha|, \tag{2.13a}$$

$$\langle\bar{p},\alpha|K_3 = -ip^0\frac{\partial}{\partial p^0}\langle\bar{p},\alpha|, \tag{2.13b}$$

where α is a set of quantum numbers that specifies the internal properties of the system. It is straightforward to show that if O is any operator that satisfies

$$[\bar{P}, O] = [\mathbf{B}, O] = [K_3, O] = 0, \tag{2.14}$$

then

$$\langle\bar{p},\alpha|O|\bar{p}',\alpha'\rangle = (2\pi)^3 2p^0\delta^3(\bar{p}-\bar{p}')O(\alpha,\alpha'), \tag{2.15}$$

where the factor $(2\pi)^3 2$ is arbitrary.

The most important operators associated with the internal structure of a system are the mass operator M and spin operator $\boldsymbol{\mathcal{J}}$. The mass operator is defined by

$$M^2 = P_\mu P^\mu = 2P^0H - \mathbf{P}_\perp^2 \ , \tag{2.16}$$

while the components of the spin operator are given by [20,23]

$$M\mathcal{J}_r = \varepsilon_{rs}(S_sP^0 - K_3P^s - B_sH) - \mathcal{J}_3P^r, \ \ r,s,= 1,2, \tag{2.17a}$$

$$\mathcal{J}_3 = \frac{B_1}{P^0}P^2 - \frac{B_2}{P^0}P^1 + J_3, \tag{2.17b}$$

where ε_{rs} is a two-dimensional Levi-Civita symbol. The operators M^2 and $\boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{J}}$ are Casimir operators, and as such, commute with all of the generators of the group; $\mathcal{J}_3$ is noninteracting and commutes with all of the stability group generators, while $\mathcal{J}_1$ and $\mathcal{J}_2$ commute with all of the stability group generators except J_3. The matrix elements of M^2 and $\boldsymbol{\mathcal{J}}$ have the structure (2.15), and the eigenvalues of M^2 and $\boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{J}}$ are invariants. The

commutation rules of M and the components of $\mathcal{J}$ are [20]

$$[M, \mathcal{J}_a] = 0 \ , \tag{2.18}$$

$$[\mathcal{J}_a, \mathcal{J}_b] = i\varepsilon_{abc}\mathcal{J}_c \ . \tag{2.19}$$

In *constructing* Poincaré invariant light front models the generators $\bar{P}$, K_3 and $\mathbf{B}$ are *defined* by (2.12) and (2.13). The matrix elements of M and $\mathcal{J}$ are assumed to be of the form (2.15); and their internal parts, i.e., $M(\alpha, \alpha')$ and $\mathcal{J}(\alpha, \alpha')$, are constructed so that M and $\mathcal{J}$ satisfy (2.18) and (2.19). The generators $H, \mathbf{S}$, and J_3 are obtained by solving (2.16) and (2.17). The resulting set of 10 generators will satisfy the commutation rules (2.5).

As pointed out in Sec. I, it is mathematically consistent to *choose* M to be interacting and $\mathcal{J}$ to be noninteracting, however when M and $\mathcal{J}$ are *derived* from a quantum field theory they both contain interactions. The new picture provides one way of dealing with the complications due to an interacting $\mathcal{J}$. In Sec. IV we will follow an alternative approach; we will use an approximation technique that leads to an M that satisfies (2.18) when $\mathcal{J}$ is noninteracting, i.e., we will construct a *free-spin model.*

III. THE SPIN OPERATOR FOR TWO FREE PARTICLES

The two-particle basis states that we will work with are of the form

$$|p_1h_1, p_2h_2\rangle_g = |p_1h_1\rangle_g \otimes |p_2h_2\rangle_g, \quad g = f, c, \tag{3.1}$$

where the subscript g distinguishes the *front form* (f) and *canonical form* (c) of the states [21]. Here p_1 and p_2 are the on-mass-shell four-momenta of the particles, and h_1 and h_2 are the z components of the spins of the particles. In this section we will suppress all other quantum numbers. The single-particle states that appear in (3.1) are defined in terms of rest frame states $|h_i\rangle$ by

$$|p_ih_i\rangle_g \equiv U[l_g(\lambda_i)]|h_i\rangle, \tag{3.2}$$

$$\lambda_i \equiv p_i/m_i, \quad (p_i^2 = m_i^2), \tag{3.3}$$

where $l_g(\lambda_i)$ is a Lorentz boost whose inverse takes us from the arbitrary frame in which we are working, the x frame, to a rest frame of particle i, according to

$$x_{g\lambda} = l_g^{-1}(\lambda)x. \tag{3.4}$$

For each choice of λ and g we have a different rest frame, which is the reason for the subscripts on the left-hand side of (3.4). The *canonical boost* ($g = c$) is given by

$$\begin{aligned} x^0 &= \lambda^0 x^0_{c\lambda} + \boldsymbol{\lambda}\cdot\mathbf{x}_{c\lambda}, \\ \mathbf{x} &= \mathbf{x}_{c\lambda} + \left(x^0_{c\lambda} + \frac{\boldsymbol{\lambda}\cdot\mathbf{x}_{c\lambda}}{\lambda^0+1}\right)\boldsymbol{\lambda}, \end{aligned} \tag{3.5}$$

where x^0, λ^0, and $x^0_{c\lambda}$ are the usual time components of the four-vectors, e.g., $x^0 = ct$. The *front form boost* is given by

$$x^0 = \sqrt{2}\lambda^0 x^0_{f\lambda}, \quad \mathbf{x}_\perp = \sqrt{2}\boldsymbol{\lambda}_\perp x^0_{f\lambda} + \mathbf{x}_{f\lambda\perp}, \tag{3.6}$$

where here x^0, λ^0 and $x^0_{f\lambda}$ are the $\mu = 0$ light front components defined by (2.7). It is worth noting that the first three light front components (x^0, x^1, x^2) transform among themselves under front form boosts.

We will assume that under a three-rotation r the rest frame states transform according to

$$U(r)|h_i\rangle = \sum_{h_i'} |h_i'\rangle D^{(s_i)}_{h_i'h_i}(r), \tag{3.7}$$

with $D^{(s_i)}$ the standard matrix representation of SU(2) for the spin s_i.

For each type of boost we define a *Wigner rotation* by

$$r_g(a, \lambda) \equiv l_g^{-1}(a\lambda)al_g(\lambda), \tag{3.8}$$

which when combined with (3.2) and (3.7) leads to the transformation law

$$U(a)|p_ih_i\rangle_g = \sum_{h_i'} |ap_i, h_i'\rangle_g D^{(s_i)}_{h_i'h_i}[r_g(a, \lambda_i)]. \tag{3.9}$$

It follows from (3.5) and (3.6) that the Wigner rotations have the following properties:

$$r_c(r, \lambda) = r, \tag{3.10}$$

$$r_f(r, \lambda) \neq r \text{ unless } r = r_3, \tag{3.11}$$

$$r_f(l_f, \lambda) = 1. \tag{3.12}$$

Equations (3.10) and (3.11) show that the canonical basis states transform more simply under three-rotations than the front form states, while (3.12) shows that the front form states transform simply under front form boosts.

In order to simplify the description of three-rotations in the front form of relativistic quantum mechanics it is convenient to introduce the Melosh rotation [33]

$$r_{fc}(\lambda) \equiv l_f^{-1}(\lambda)l_c(\lambda), \tag{3.13}$$

which from (3.4) relates the two rest frames according to

$$x_{f\lambda} = r_{fc}(\lambda)x_{c\lambda}. \tag{3.14}$$

Equations (3.2), (3.7), and (3.13) imply that the single-particle basis states are related by

$$|p_ih_i\rangle_c = \sum_{h_i'} |p_ih_i'\rangle_f D^{(s_i)}_{h_i'h_i}[r_{fc}(\lambda_i)]. \tag{3.15}$$

For the two-particle states (3.1) we introduce the four-vectors

$$p = p_1 + p_2 \ , \tag{3.16}$$

$$\Lambda = p/(p\cdot p)^{1/2} \ , \tag{3.17}$$

and define a two-particle c.m. frame by

$$x_\Lambda = x_{f\Lambda} = 1_f^{-1}(\Lambda)x\ . \tag{3.18}$$

Following Keister and Polyzou [21] we also define a Melosh rotated, two-particle, rest frame state by

$$|p_{1\Lambda}h_1, p_{2\Lambda}h_2\rangle_M \equiv \sum_{h_1'h_2'} |p_{1\Lambda}h_1', p_{2\Lambda}h_2'\rangle_f \times D^{(s_1)}_{h_1'h_1}[r_{fc}(\lambda_{1\Lambda})]D^{(s_2)}_{h_2'h_2}[r_{fc}(\lambda_{2\Lambda})]. \tag{3.19}$$

At this point it is convenient to introduce a relative three-momentum variable $\mathbf{q}$ through the relations

$$[\varepsilon_1(\mathbf{q}), \mathbf{q}] = (p^\mu_{1\Lambda}) = p_{1\Lambda} = l_f^{-1}(\Lambda)p_1, \tag{3.20a}$$

$$[\varepsilon_2(\mathbf{q}), -\mathbf{q}] = (p^\mu_{2\Lambda}) = p_{2\Lambda} = l_f^{-1}(\Lambda)p_2, \tag{3.20b}$$

where

$$\varepsilon_i(\mathbf{q}) = (\mathbf{q}^2 + m_i^2)^{1/2}. \tag{3.21}$$

Clearly $\mathbf{q}$ is the three-momentum of particle 1 in the c.m. frame defined by (3.18) and (3.6). We also introduce the notation

$$|\mathbf{q}h_1h_2\rangle = |p_{1\Lambda}h_1, p_{2\Lambda}h_2\rangle_M. \tag{3.22}$$

It follows from (3.9), (3.19), (3.20), and the identity

$$r_f(r,\lambda)r_{fc}(\lambda) = r_{fc}(r\lambda)r, \tag{3.23}$$

that the states (3.22) rotate according to

$$U(r)|\mathbf{q}h_1h_2\rangle = \sum_{h_1'h_2'} |r\mathbf{q}, h_1'h_2'\rangle D^{(s_1)}_{h_1'h_1}(r)D^{(s_2)}_{h_2'h_2}(r). \tag{3.24}$$

We see that our Melosh transformed, rest frame states behave simply under three-rotations.

The rest frame states can be boosted to define the states

$$|\bar{p}\mathbf{q}h_1h_2\rangle \equiv U[l_f(\Lambda)]|\mathbf{q}h_1h_2\rangle = \sum_{h_1'h_2'} |p_1h_1', p_2h_2'\rangle_f \times D^{(s_1)}_{h_1'h_1}[r_{fc}(\lambda_{1\Lambda})]D^{(s_2)}_{h_2'h_2}[r_{fc}(\lambda_{2\Lambda})], \tag{3.25}$$

where the second form follows from (3.22), (3.19), (3.1), (3.9), and (3.12). It should be noted that the unit four-vector Λ defined by (3.16) and (3.17) can be expressed in the form

$$\Lambda = \left(\bar{p}, \frac{\mathbf{p}_\perp^2 + W^2(\mathbf{q})}{2p^0}\right) \Big/ W(\mathbf{q}), \tag{3.26}$$

where here light front components [see (2.7)] have been given. The invariant mass of the two free particles, $W(\mathbf{q})$, is given by

$$W(\mathbf{q}) = \varepsilon_1(\mathbf{q}) + \varepsilon_2(\mathbf{q}). \tag{3.27}$$

We will now show that the spin operator for two free particles has a simple representation in the basis provided by the states (3.25). Since in terms of the light front components (2.7) the total c.m. four-momentum is given by

$$(p^\mu_\Lambda) = \frac{W(\mathbf{q})}{\sqrt{2}}(1, \mathbf{0}_\perp, 1), \tag{3.28}$$

it follows from (2.5a), (2.8), (2.9a), and (2.16)–(2.18) that

$$\mathcal{J}|\mathbf{q}h_1h_2\rangle = \mathbf{J}|\mathbf{q}h_1h_2\rangle. \tag{3.29}$$

This shows that the action of the spin operator $\mathcal{J}$ on a rest frame state is equivalent to the action of the angular momentum operator $\mathbf{J}$. The unitary boost operator $U[l_f(\Lambda)]$ is generated by the stability group generators $\mathbf{B}$ and K_3, so it commutes with $\mathcal{J}$, and therefore (3.25) and (3.29) imply that

$$\mathcal{J}|\bar{p}\mathbf{q}h_1h_2\rangle = U[l_f(\Lambda)]\mathbf{J}|\mathbf{q}h_1h_2\rangle\ . \tag{3.30}$$

Equation (3.24) shows that the rest frame states rotate just as they do in nonrelativistic quantum mechanics, therefore the two-particle angular-momentum operator $\mathbf{J}$ has the same representation as in nonrelativistic quantum mechanics, i.e.,

$$\langle\mathbf{q}h_1h_2|\mathbf{J} = \sum_{h_1'h_2'} \mathcal{J}_{h_1h_2,h_1'h_2'}(\mathbf{q})\langle\mathbf{q}h_1'h_2'|, \tag{3.31}$$

where

$$\mathcal{J}(\mathbf{q}) \equiv I_1 \otimes I_2(i\nabla_q \times \mathbf{q}) + \mathbf{S}_1 \otimes I_2 + I_1 \otimes \mathbf{S}_2, \tag{3.32}$$

with I_i and $\mathbf{S}_i$ the unit matrix and spin matrix vector for particle i, respectively. It now follows from (3.30) that the two-particle spin operator has the simple representation

$$\langle\bar{p}\mathbf{q}h_1h_2|\mathcal{J}|\bar{p}'\mathbf{q}'h_1'h_2'\rangle = (2\pi)^3 2p^0\delta^3(\bar{p}-\bar{p}')(2\pi)^3\frac{2\varepsilon_1(\mathbf{q})\varepsilon_2(\mathbf{q})}{W(\mathbf{q})}\mathcal{J}_{h_1h_2,h_1'h_2'}(\mathbf{q})\delta^3(\mathbf{q}-\mathbf{q}'). \tag{3.33}$$

In obtaining (3.33) we have assumed that our two-particle states are normalized according to

$$\begin{aligned}\langle\bar{p}\mathbf{q}h_1h_2|\bar{p}'\mathbf{q}'h_1'h_2'\rangle &= {}_f\langle p_1h_1, p_2h_2|p_1'h_1', p_2'h_2'\rangle_f \\ &= (2\pi)^3 2p_1^0\delta^3(\bar{p}_1-\bar{p}_1')\delta_{h_1h_1'}(2\pi)^3 2p_2^0\delta^3(\bar{p}_2-\bar{p}_2')\delta_{h_2h_2'} \\ &= (2\pi)^3 2p^0\delta^3(\bar{p}-\bar{p}')(2\pi)^3\frac{2\varepsilon_1(\mathbf{q})\varepsilon_2(\mathbf{q})}{W(\mathbf{q})}\delta^3(\mathbf{q}-\mathbf{q}')\delta_{h_1h_1'}\delta_{h_2h_2'}.\end{aligned} \tag{3.34}$$

It is worth noting in passing that the matrix elements (3.33) have the anticipated structure given by (2.15), with the set of internal quantum numbers $\alpha = \{\mathbf{q}, h_1, h_2\}$.

In Sec. IV we will construct an effective two-nucleon interaction that commutes with the spin operator defined by (3.32) and (3.33). This will guarantee that we have a Poincaré invariant model.

IV. THE TWO-NUCLEON INTERACTION

We will construct our effective two-particle interaction using the formalism described in Refs. [31,32]. This formalism is an extension to light front dynamics of procedures developed by Okubo [35], and Glöckle and Müller [36]. In this formalism the Fock space of a quantum field theory is divided into two orthogonal subspaces with projection operators η and Λ. The η subspace is spanned by a limited number of states, in our case only $|NN\rangle$ states. The formalism leads to an effective operator in the η subspace.

We can take for our η-subspace basis states the front form states denoted by [see (3.1)–(3.3)]

$$|p_1 i_1 h_1, p_2 i_2 h_2\rangle_f = |\bar{p}\mathbf{q} i_1 h_1 i_2 h_2\rangle_f = |\bar{p}\alpha\rangle_f \tag{4.1a}$$

with

$$\alpha = \{\mathbf{q}, i_1, h_1, i_2, h_2\}. \tag{4.1b}$$

Here we have added the nucleons' isospin quantum numbers, i_1 and i_2, to those considered in Sec. III. We recall that $\bar{p}$ stands for the first three light front components [see (2.7)] of the total four-momentum $p = p_1 + p_2$, and that $\mathbf{q}$ is a relative three-momentum variable defined by (3.20).

If we write the quantum field theory Hamiltonian for our meson exchange model in the form

$$H = H_0 + H_1, \tag{4.2}$$

where H_0 is the free Hamiltonian, then according to (2.16) our mass-square operator has the decomposition

$$M^2 = M_0^2 + V, \tag{4.3}$$

where

$$M_0^2 = 2P^0 H_0 - \mathbf{P}_\perp^2, \tag{4.4}$$

and

$$V = 2P^0 H_1 = 2H_1 P^0. \tag{4.5}$$

For a field theory quantized on the null plane ($ct+z=0$), whose interaction Lagrangian is of the type assumed in meson-exchange models [9–11], the interaction V has the structure

$$V = V_1 + V_2, \tag{4.6}$$

where V_1 and V_2 are first order and second order in the meson-nucleon coupling constants, respectively [37].

If we let $M_\eta^2 = \eta M_0^2 \eta + V_\eta$ be the effective mass-square operator in the η subspace, then we can write

$$\begin{aligned} &{}_f\langle \bar{p}\alpha | M_\eta^2 | \bar{p}'\alpha'\rangle_f \\ &\quad = (2\pi)^3 2p^0 \delta^3(\bar{p}-\bar{p}')[W^2(\mathbf{q})\delta_{\alpha\alpha'} + V_f(\alpha,\alpha')], \end{aligned} \tag{4.7}$$

where $W(\mathbf{q})$ is given by (3.27), and to second order in the coupling constants $V_f(\alpha,\alpha')$ is defined by [31,32]

$$\begin{aligned} {}_f\langle \bar{p}\alpha | V_\eta | \bar{p}'\alpha'\rangle_f &= (2\pi)^3 2p^0 \delta^3(\bar{p}-\bar{p}') V_f(\alpha,\alpha') \\ &= {}_f\langle \bar{p}\alpha | V + \frac{1}{2} V_1 \left[\frac{\Lambda}{W^2(\mathbf{q}) - M_0^2} + \frac{\Lambda}{W^2(\mathbf{q}') - M_0^2} \right] V_1 | \bar{p}'\alpha'\rangle_f. \end{aligned} \tag{4.8}$$

To evaluate the one-boson-exchange potentials directly from (4.8) is a rather tedious process, but fortunately it has been shown in Ref. [32] that they can be determined by using a slight variation of the standard Feynman diagram rules. The potentials can be obtained by first drawing the relevant one-boson-exchange Feynman diagrams, and then determining the four-momentum of the virtual meson in each diagram by assuming that the total four-momentum is conserved *either* at the vertex on the "right" *or* at the vertex on the "left", but not necessarily at both vertices. The potentials are obtained by adding together the two resulting Feynman amplitudes and dividing by 2. The result for the OBE potential involving the exchange of $b = \pi$, η, ρ, ω, δ, and σ mesons is given by

$$V_f(\alpha,\alpha') = \sum_b V_f^b(\alpha,\alpha'), \tag{4.9}$$

$$V_f^b(\alpha,\alpha') = g_b^2 \delta_b f_b[(p_1 - p_1')^2] f_b[(p_2 - p_2')^2] \frac{\bar{u}(p_1,h_1)u(p_1',h_1')\bar{u}(p_2,h_2)u(p_2',h_2')}{2[(p_1-p_1')^2 - m_b^2]} + (1 \leftrightarrow 2), \quad b = \delta, \sigma, \tag{4.10a}$$

$$\delta_\delta = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2, \quad \delta_\sigma = 1, \tag{4.10b}$$

$$\begin{aligned} V_f^b(\alpha,\alpha') = &-g_b^2 \delta_b f_b[(p_1 - p_1')^2] f_b[(p_2 - p_2')^2] \\ &\times \frac{\bar{u}(p_1,h_1)\Gamma_b(p_1-p_1')u(p_1',h_1')\bar{u}(p_2,h_2)\Gamma_b(p_1'-p_1)u(p_2',h_2')}{2[(p_1-p_1')^2 - m_b^2]} + (1 \leftrightarrow 2), \quad b = \pi, \eta, \end{aligned} \tag{4.11a}$$

$$\delta_\pi = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2, \quad \delta_\eta = 1, \tag{4.11b}$$

$$\Gamma_b(q) = \left[\lambda_b + \frac{\not{q}(1-\lambda_b)}{2m_N}\right]\gamma_5, \tag{4.11c}$$

$$V_f^b(\alpha,\alpha') = g_b^2 \delta_b f_b[(p_1-p_1')^2] f_b[(p_2-p_2')^2] \times \frac{\bar{u}(p_1,h_1)\Gamma_\mu^b(p_1-p_1')u(p_1',h_1')\Delta_b^{\mu\nu}(p_1-p_1')\bar{u}(p_2,h_2)\Gamma_\nu^b(p_1'-p_1)u(p_2',h_2')}{2[(p_1-p_1')^2-m_b^2]} + (1 \leftrightarrow 2), \quad b=\rho,\omega, \tag{4.12a}$$

$$\delta_\rho = \tau_1 \cdot \tau_2, \quad \delta_\omega = 1, \tag{4.12b}$$

$$\Gamma_\mu^b(q) = \gamma_\mu + (\kappa_b/2m_N) i\sigma_{\mu\nu}q^\nu, \tag{4.12c}$$

$$\Delta_b^{\mu\nu}(q) = -g^{\mu\nu} + (q^\mu q^\nu/m_b^2). \tag{4.12d}$$

The functions $f_b(q^2)$ are form factors which take into account the extensions of the vertices. The front form Dirac spinors $u(p,h)$ are normalized according to [32]

$$\bar{u}(p,h)u(p,h') = 2m_N\delta_{hh'}. \tag{4.13}$$

As (4.11c) shows, following Gross et al. [16], we have allowed for a mixture of pseudoscalar and pseudovector coupling for both the π and η mesons. It is straightforward to show that the second term on the right-hand side of (4.12d) does not contribute in (4.12a). It was included in (4.12d) so as not to create the impression that it was simply ignored.

We now subject our basis states (4.1) to the Melosh rotations given by (3.25), and denote the potential matrix elements in the rotated basis by

$$V(\alpha,\alpha') = \sum_b V^b(\alpha,\alpha'), \tag{4.14}$$

$$\langle\bar{p}\alpha|V_\eta^b|\bar{p}'\alpha'\rangle = (2\pi)^3 2p^0\delta^3(\bar{p}-\bar{p}')V^b(\alpha,\alpha'). \tag{4.15}$$

The Melosh rotated matrix elements are distinguished by the lack of the subscript f that appears in (4.9). The states $|\bar{p}\alpha\rangle = |\bar{p}\mathbf{q}i_1h_1i_2h_2\rangle$ are defined by (3.25) with the isospin quantum numbers added. According to (3.3) and (3.20) the $\lambda_{i\Lambda}$ that appear in (3.25) are given by

$$\lambda_{i\Lambda} = p_{i\Lambda}/m_N = l_f^{-1}(\Lambda)(p_i/m_N). \tag{4.16}$$

It is shown in Ref. [32] that

$$\sum_{h_i'} u(p_i,h_i')D^{(1/2)}_{h_i'h_i}[r_{fc}(\lambda_{i\Lambda})] = S[l_f(\Lambda)]u_c(p_{i\Lambda},h_i), \tag{4.17}$$

where the matrix $S[l_f(\Lambda)]$ satisfies

$$S^{-1}(a)\gamma^\mu S(a) = a^\mu{}_\nu\gamma^\nu, \tag{4.18}$$

with $a = l_f(\Lambda)$, the light front boost defined by (3.4) and (3.6). The canonical spinor u_c [32] in (4.17) is equal to the one given by Eq. (3.7) of Bjorken and Drell [38], multiplied by $\sqrt{2m_N}$ so as to give the normalization (4.13). With the help of (3.20), we find that

$$u_c(p_{i\Lambda},h_i) = [\varepsilon(\mathbf{q})+m_N]^{1/2}\begin{bmatrix}\chi_{h_i}\\ (-1)^{i+1}\frac{\boldsymbol{\sigma}_i\cdot\mathbf{q}}{\varepsilon(\mathbf{q})+m_N}\chi_{h_i}\end{bmatrix}, \quad i=1,2, \tag{4.19a}$$

with

$$\chi_{1/2} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \chi_{-1/2} = \begin{bmatrix}0\\1\end{bmatrix}, \tag{4.19b}$$

and

$$\varepsilon(\mathbf{q}) = (\mathbf{q}^2+m_N^2)^{1/2}. \tag{4.20}$$

It follows from (4.15) and (4.17) that the Melosh rotated potential matrix elements, $V^b(\alpha,\alpha')$, are obtained from the original matrix elements, $V_f^b(\alpha,\alpha')$, by simply making the replacements

$$\begin{aligned} u(p_i,h_i) &\to S[l_f(\Lambda)]u_c(p_{i\Lambda},h_i),\\ u(p_i',h_i') &\to S[l_f(\Lambda')]u_c(p_{i\Lambda'}',h_i'). \end{aligned} \tag{4.21}$$

When this is done we encounter various matrices such as the unit matrix or the Feynman slash quantity $\not{p}_1$ sandwiched between $S^{-1}[l_f(\Lambda)]$ and $S[l_f(\Lambda')]$. These products can be worked out by using the identities [32]

$$S^{-1}[l_f(\Lambda)]S[l_f(\Lambda')] = \exp\left(\tfrac{1}{2}\alpha_3\zeta\right), \tag{4.22}$$

$$S^{-1}[l_f(\Lambda)]\ \not{p}_1 S[l_f(\Lambda)] = \not{p}_{1\Lambda} = \varepsilon(\mathbf{q})\gamma^0 - \mathbf{q}\cdot\boldsymbol{\gamma}. \tag{4.23}$$

Here α_3 is a standard Dirac matrix, and ζ is defined by

$$\zeta = \ln[\varepsilon(\mathbf{q})/\varepsilon(\mathbf{q}')]. \tag{4.24}$$

The invariant denominators that appear in the potentials, i.e., $(p_i-p_i')^2-m_b^2$, can be worked out in any convenient frame, such as the c.m. frame associated with the final state, the $x_{\Lambda'}$ frame. According to (3.18) this frame is related to the c.m. frame of the initial state, the x_Λ frame by

$$x_\Lambda = l_f^{-1}(\Lambda)l_f(\Lambda')x_{\Lambda'}. \tag{4.25}$$

The front form boosts form a subgroup, and in particular

it can be shown that [32]

$$l_f^{-1}(\Lambda) l_f(\Lambda') = l_f(\Omega), \tag{4.26a}$$

$$\Omega = (\Omega^\mu) = (e^\zeta/\sqrt{2}, \mathbf{0}_\perp, e^{-\zeta}/\sqrt{2}), \tag{4.26b}$$

where in (4.26b) the components are light front components. The components of $p_{1\Lambda}$ and $p_{2\Lambda}$ are given by (3.20), the components of $p'_{1\Lambda'}$ and $p'_{2\Lambda'}$ are obtained from (3.20) by making the replacement $\mathbf{q} \to \mathbf{q}'$.

If the above procedures are applied to the scalar meson exchange potential (4.10), it is found that after Melosh rotation this potential is given in the spin space of the two nucleons by

$$\begin{aligned} V^b(\alpha,\alpha') = {} & \frac{g_b^2 \delta_b}{2} f_b[(p_1 - p'_1)^2] f_b[(p_2 - p'_2)^2] \left[\frac{1}{(p_1 - p'_1)^2 - m_b^2} + \frac{1}{(p_2 - p'_2)^2 - m_b^2} \right] \\ & \times (\varepsilon + m_N)(\varepsilon' + m_N) \Bigg\{ \frac{(\varepsilon + \varepsilon')^2}{4\varepsilon\varepsilon'} (1 - \boldsymbol{\sigma}_1 \cdot \mathbf{x} \boldsymbol{\sigma}_1 \cdot \mathbf{x}')(1 - \boldsymbol{\sigma}_2 \cdot \mathbf{x} \boldsymbol{\sigma}_2 \cdot \mathbf{x}') \\ & + \frac{(\varepsilon^2 - \varepsilon'^2)}{4\varepsilon\varepsilon'} [(\boldsymbol{\sigma}_1 \cdot \mathbf{u} \boldsymbol{\sigma}_1 \cdot \mathbf{x}' - \boldsymbol{\sigma}_1 \cdot \mathbf{x} \boldsymbol{\sigma}_1 \cdot \mathbf{u})(1 - \boldsymbol{\sigma}_2 \cdot \mathbf{x} \boldsymbol{\sigma}_2 \cdot \mathbf{x}') \\ & - (\boldsymbol{\sigma}_2 \cdot \boldsymbol{u} \boldsymbol{\sigma}_2 \cdot \mathbf{x}' - \boldsymbol{\sigma}_2 \cdot \mathbf{x} \boldsymbol{\sigma}_2 \cdot \mathbf{u})(1 - \boldsymbol{\sigma}_1 \cdot \mathbf{x} \boldsymbol{\sigma}_1 \cdot \mathbf{x}')] \\ & - \frac{(\varepsilon - \varepsilon')^2}{4\varepsilon\varepsilon'} (\boldsymbol{\sigma}_1 \cdot \mathbf{u} \boldsymbol{\sigma}_1 \cdot \mathbf{x}' - \boldsymbol{\sigma}_1 \cdot \mathbf{x} \boldsymbol{\sigma}_1 \cdot \mathbf{u})(\boldsymbol{\sigma}_2 \cdot \mathbf{u} \boldsymbol{\sigma}_2 \cdot \mathbf{x}' - \boldsymbol{\sigma}_2 \cdot \mathbf{x} \boldsymbol{\sigma}_2 \cdot \mathbf{u}) \Bigg\}. \end{aligned} \tag{4.27}$$

Here $\varepsilon = \varepsilon(\mathbf{q})$, $\varepsilon' = \varepsilon(\mathbf{q}')$, $\mathbf{u} = (0,0,1)$, and

$$\mathbf{x} = \frac{\mathbf{q}}{\varepsilon + m_N}, \quad \mathbf{x}' = \frac{\mathbf{q}'}{\varepsilon' + m_N}. \tag{4.28}$$

The momentum transfers are given by

$$\begin{aligned} (p_1 - p'_1)^2 = {} & -(\mathbf{q} - \mathbf{q}')^2 + (\varepsilon^2 - \varepsilon'^2) \left(\frac{\mathbf{q} \cdot \mathbf{u}}{\varepsilon} - \frac{\mathbf{q}' \cdot \mathbf{u}}{\varepsilon'} \right) \\ & + (\varepsilon - \varepsilon')^2 \frac{\mathbf{q} \cdot \mathbf{u}}{\varepsilon} \frac{\mathbf{q}' \cdot \mathbf{u}}{\varepsilon'}, \end{aligned} \tag{4.29a}$$

$$\begin{aligned} (p_2 - p'_2)^2 = {} & -(\mathbf{q} - \mathbf{q}')^2 - (\varepsilon^2 - \varepsilon'^2) \left(\frac{\mathbf{q} \cdot \mathbf{u}}{\varepsilon} - \frac{\mathbf{q}' \cdot \mathbf{u}}{\varepsilon'} \right) \\ & + (\varepsilon - \varepsilon')^2 \frac{\mathbf{q} \cdot \mathbf{u}}{\varepsilon} \frac{\mathbf{q}' \cdot \mathbf{u}}{\varepsilon'}. \end{aligned} \tag{4.29b}$$

We see that this potential *does not* commute with the free, two-particle spin operator defined by (3.32) and (3.33). In the context of light front dynamics, this is not surprising. Light front angular-momentum operators are expected to contain interactions. The *new picture* formalism [28–32] shows that the interaction in the angular-momentum operator can by rigorously taken into account by allowing the unit vector $\mathbf{u}$ that appears above to become a variable. We note that the above potential is a rotationally invariant function of $\mathbf{q}$, $\mathbf{q}'$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\mathbf{u}$. This is true in general; which makes it possible to carry out a partial-wave analysis of the momentum-space integral equations, and thereby obtain partial-wave S-matrix elements. Using results obtained in this way as a basis for comparison, a couple of different approximations which lead to potentials that commute with the free, two-particle spin operator have been investigated [31,32]. One approximation entails averaging over the direction of $\mathbf{u}$, while in the other approximation the $\mathbf{u}$-dependent terms are simply dropped. It has been found that slight readjustments of the parameters in the form factors bring the approximate results into good agreement with the new picture results. Here we will take the simplest approach, and drop the $\mathbf{u}$-dependent terms. It should be noted that these terms vanish on shell, i.e., when $\varepsilon(\mathbf{q}) = \varepsilon(\mathbf{q}')$, as well as in the nonrelativistic limit, i.e., when $|\mathbf{q}|/m_N \ll 1$, $|\mathbf{q}'|/m_N \ll 1$. We also make the approximation $(\varepsilon + \varepsilon')^2/(4\varepsilon\varepsilon') \approx 1$, since this is compensated for by the form factors.

The approximate potentials are given by

$$V^b(\mathbf{q},\mathbf{q}') = -g_b^2 \delta_b \frac{f_b^2[-(\mathbf{q} - \mathbf{q}')^2]}{m_b^2 + (\mathbf{q} - \mathbf{q}')^2} (\varepsilon + m_N)(\varepsilon' + m_N)[1 - A(\mathbf{x},\mathbf{x}') + B(\mathbf{x},\mathbf{x}')], \quad b = \delta, \sigma, \tag{4.30}$$

$$\begin{aligned} V^b(\mathbf{q},\mathbf{q}') = {} & -g_b^2 \delta_b \frac{f_b^2[-(\mathbf{q} - \mathbf{q}')^2]}{m_b^2 + (\mathbf{q} - \mathbf{q}')^2} (\varepsilon + m_N)(\varepsilon' + m_N) \\ & \times \left\{ \left[1 + (1 - \lambda_b) \frac{\varepsilon - \varepsilon'}{m_N} \right] C(\mathbf{x}) + \left[1 + (1 - \lambda_b) \frac{\varepsilon' - \varepsilon}{m_N} \right] C(\mathbf{x}') - D(\mathbf{x},\mathbf{x}') \right\}, \quad b = \pi, \eta, \end{aligned} \tag{4.31}$$

$$
\begin{aligned}
V^b(\mathbf{q},\mathbf{q}') = {} & g_b^2\delta_b \frac{f_b^2[-(\mathbf{q}-\mathbf{q}')^2]}{m_b^2+(\mathbf{q}-\mathbf{q}')^2}(\varepsilon+m_N)(\varepsilon'+m_N) \\
& \times\Bigg\{1+\frac{2\kappa_b}{m_N}(2m_N-\varepsilon-\varepsilon')+\left(\frac{\kappa_b}{2m_N}\right)^2[8(\varepsilon-m_N)(\varepsilon'-m_N)+m_b^2] \\
& +\left[3+4\kappa_b+\left(\frac{\kappa_b}{2m_N}\right)^2[8(m_N^2-\varepsilon\varepsilon')-m_b^2]\right]A(\mathbf{x},\mathbf{x}') \\
& +\left[1+\frac{2\kappa_b}{m_N}(\varepsilon+\varepsilon'+2m_N)+\left(\frac{\kappa_b}{2m_N}\right)^2[8(\varepsilon+m_N)(\varepsilon'+m_N)+m_b^2]\right]B(\mathbf{x},\mathbf{x}') \\
& +\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\Bigg\{\left[(1+\kappa_b)^2+\kappa_b(1+\kappa_b)\frac{(\varepsilon-\varepsilon')}{m_N}\right]C(\mathbf{x}) \\
& +\left[(1+\kappa_b)^2+\kappa_b(1+\kappa_b)\frac{(\varepsilon'-\varepsilon)}{m_N}\right]C(\mathbf{x}')-(1+k_b)^2D(\mathbf{x},\mathbf{x}')\Bigg\} \\
& -\left(\frac{\kappa_b}{2m_N}\right)^2[m_b^2+(\mathbf{q}-\mathbf{q}')^2][1-A(\mathbf{x},\mathbf{x}')+B(\mathbf{x},\mathbf{x}')]\Bigg\}, \quad b=\rho,\omega,
\end{aligned}
\tag{4.32}
$$

where

$$A(\mathbf{x},\mathbf{x}') = \boldsymbol{\sigma}_1\cdot\mathbf{x}\boldsymbol{\sigma}_1\cdot\mathbf{x}'+\boldsymbol{\sigma}_2\cdot\mathbf{x}\boldsymbol{\sigma}_2\cdot\mathbf{x}', \tag{4.33a}$$

$$B(\mathbf{x},\mathbf{x}') = \boldsymbol{\sigma}_1\cdot\mathbf{x}\boldsymbol{\sigma}_2\cdot\mathbf{x}\boldsymbol{\sigma}_1\cdot\mathbf{x}'\boldsymbol{\sigma}_2\cdot\mathbf{x}', \tag{4.33b}$$

$$C(\mathbf{x}) = \boldsymbol{\sigma}_1\cdot\mathbf{x}\boldsymbol{\sigma}_2\cdot\mathbf{x}, \tag{4.33c}$$

$$D(\mathbf{x},\mathbf{x}') = \boldsymbol{\sigma}_1\cdot\mathbf{x}\boldsymbol{\sigma}_2\cdot\mathbf{x}'+\boldsymbol{\sigma}_2\cdot\mathbf{x}\boldsymbol{\sigma}_1\cdot\mathbf{x}'. \tag{4.33d}$$

For our form factors we have chosen the same functions as those adopted by other workers [9–11], i.e.,

$$f_b[-(\mathbf{q}-\mathbf{q}')^2] \equiv \left[\frac{\Lambda_b^2-m_b^2}{\Lambda_b^2+(\mathbf{q}-\mathbf{q}')^2}\right]^{n_b}. \tag{4.34}$$

It is interesting to note that the scalar and pseudoscalar meson-exchange potentials, (4.30) and (4.31), are identical to those that have been used in the various OBE potentials developed in connection with the Bonn model [9,10]. The Bonn OBE potentials are given by Eqs. (E.32)–(E.34c) of Ref. [9]. The vector meson-exchange potential, (4.32), differs, however, in the κ_b^2 terms; the so-called *tensor-tensor* terms. We have

$$
\begin{aligned}
V^b_{\rm Bonn}(\mathbf{q},\mathbf{q}')-V^b(\mathbf{q},\mathbf{q}') = {} & g_b^2\delta_b\frac{f_b^2[-(\mathbf{q}-\mathbf{q}')^2]}{m_b^2+(\mathbf{q}-\mathbf{q}')^2}(\varepsilon+m_N)(\varepsilon'+m_N) \\
& \times\left[\frac{\kappa_b(\varepsilon-\varepsilon')}{2m_N}\right]^2\{1-3A(\mathbf{x},\mathbf{x}')+B(\mathbf{x},\mathbf{x}')+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2[C(\mathbf{x})+C(\mathbf{x}')+D(\mathbf{x},\mathbf{x}')]\}, \quad b=\rho,\omega.
\end{aligned}
\tag{4.35}
$$

This difference vanishes on shell, and is of order $(v/c)^4$ off shell. Numerical studies have indicated that this difference is not important, and can be compensated for by slight readjustments in the form factors [39].

V. PARTIAL-WAVE ANALYSIS

The partial-wave matrix elements of the potentials (4.30)–(4.32) are defined by

$$V^{sjt,b}_{ll'}(q,q') \equiv (-1)^{l+l'}\int d\Omega_q d\Omega_{q'}\mathcal{Y}^{m\dagger}_{lsj}(\hat{q})V^{t,b}(\mathbf{q},\mathbf{q}')\mathcal{Y}^m_{l'sj}(\hat{q}'), \tag{5.1}$$

where

$$\mathcal{Y}^m_{lsj}(\hat{q}) \equiv \sum_{m_l m_s} Y_l^{m_l}(\hat{q})|sm_s\rangle\langle lsm_lm_s|jm\rangle \equiv \langle\hat{q}|l,(s_1s_2)s;jm\rangle \quad (s_1=s_2=\tfrac{1}{2}), \tag{5.2}$$

and $V^{t,b}(\mathbf{q},\mathbf{q}')$ is the projection of $V^b(\mathbf{q},\mathbf{q}')$ onto the isospin channel t $(=0,1)$. Here, and in what follows, we employ

the notation of Messiah [40]. The arbitrary phase factor $(-1)^{l+l'}$ has been put in so that the partial-wave matrix elements encountered here have the same phases as those given by Eqs. (C.21), (E.43)–(E.45), and (E.50)–(E.52c) of Ref. [9].

According to (4.30)–(4.32) we need to evaluate the expressions

$$X^{sj,b}_{ll'}(q,q') = \int d\Omega_q d\Omega_{q'} \mathcal{Y}^{m\dagger}_{lsj}(\hat{q}) \frac{f_b^2[-(\mathbf{q}-\mathbf{q}')^2]}{m_b^2+(\mathbf{q}-\mathbf{q}')^2} X(\mathbf{x},\mathbf{x}') \mathcal{Y}^{m}_{l'sj}(\hat{q}'), \quad (X = I, A, B, C, D), \tag{5.3}$$

where $I(\mathbf{x},\mathbf{x}') = 1$ and $C(\mathbf{x},\mathbf{x}') = C(\mathbf{x})$, as well as

$$G^{sj,b}_{ll'}(q,q') = \int d\Omega_q d\Omega_{q'} \mathcal{Y}^{m\dagger}_{lsj}(\hat{q}) f_b^2[-(\mathbf{q}-\mathbf{q}')^2][1 - A(\mathbf{x},\mathbf{x}') + B(\mathbf{x},\mathbf{x}')] \mathcal{Y}^{m}_{l'sj}(\hat{q}'). \tag{5.4}$$

In working out these expressions we encounter, for example, the action of $\boldsymbol{\sigma}_1 \cdot \hat{q}$ on the state (5.2). It is convenient to first consider its action on the recoupled state

$$|(ls_1)g, s_2; jm\rangle = \sum_{m_g m_2} |(ls_1)gm_g\rangle |s_2 m_2\rangle \langle g s_2 m_g m_2 | jm\rangle. \tag{5.5}$$

The action of $\boldsymbol{\sigma}_1 \cdot \hat{q}$ on (5.5) is determined by the simple identity

$$\boldsymbol{\sigma}_1 \cdot \hat{q} \langle \hat{q} | (ls_1)gm_g\rangle = -\langle \hat{q} | (2g - l, s_1) g m_g\rangle. \tag{5.6}$$

The state (5.2) is related to the states (5.5) in the usual way, i.e.,

$$|l, (s_1 s_2)s; jm\rangle = \sum_g |(ls_1)g, s_2; jm\rangle \sqrt{(2s+1)(2g+1)} (-1)^{l+s_1+s_2+j} \begin{Bmatrix} l & s_1 & g \\ s_2 & j & s \end{Bmatrix}, \tag{5.7}$$

where $\{\,\}$ is a $6j$ symbol. For $s_1 = s_2 = \frac{1}{2}$ the $6j$ symbols are given by simple closed form expressions [40].

Using these basic results it is straightforward to show that the matrix elements (5.3) can be expressed in terms of the integrals

$$Q^b_l(q,q') = 2\pi \int_{-1}^{1} dx P_l(x) \frac{f_b^2[-(\mathbf{q}-\mathbf{q}')^2]}{m_b^2+(\mathbf{q}-\mathbf{q}')^2}, \tag{5.8}$$

where f_b is given by (4.34). We find

$$I^{sj,b}_{ll'}(q,q') = \delta_{ll'} Q^b_l(q,q'), \tag{5.9}$$

$$A^{0j,b}_{jj}(q,q') = \frac{2xx'}{2j+1}[jQ^b_{j-1}(q,q') + (j+1)Q^b_{j+1}(q,q')], \tag{5.10a}$$

$$A^{1j,b}_{ll'}(q,q') = \delta_{ll'} \frac{xx'}{2l+1}[(l+1-j)(l+2+j)Q^b_{l-1}(q,q') + (l+j)(j+1-l)Q^b_{l+1}(q,q')], \tag{5.10b}$$

$$B^{sj,b}_{jj}(q,q') = (xx')^2 Q^b_j(q,q'), \quad s = 0, 1, \tag{5.11a}$$

$$B^{1j,b}_{j-1,j-1}(q,q') = \frac{(xx')^2}{(2j+1)^2}[Q^b_{j-1}(q,q') + 4j(j+1)Q^b_{j+1}(q,q')], \tag{5.11b}$$

$$B^{1j,b}_{j+1,j+1}(q,q') = \frac{(xx')^2}{(2j+1)^2}[4j(j+1)Q^b_{j-1}(q,q') + Q^b_{j+1}(q,q')], \tag{5.11c}$$

$$B^{1j,b}_{j\mp1,j\pm1}(q,q') = -\frac{2(xx')^2\sqrt{j(j+1)}}{(2j+1)^2}[Q^b_{j-1}(q,q') - Q^b_{j+1}(q,q')], \tag{5.11d}$$

$$C^{0j,b}_{jj}(q,q') = -x^2 Q^b_j(q,q'), \quad C^{1j,b}_{jj}(q,q') = x^2 Q^b_j(q,q'), \tag{5.12a}$$

$$C^{1j,b}_{j\mp1,j\mp1}(q,q') = \pm\frac{x^2}{2j+1} Q^b_{j\mp1}(q,q'), \tag{5.12b}$$

$$C^{1j,b}_{j\mp1,j\pm1}(q,q') = -\frac{2x^2\sqrt{j(j+1)}}{(2j+1)} Q^b_{j\pm1}(q,q'),$$

$$D^{0j,b}_{jj}(q,q') = -\frac{2xx'}{2j+1}[jQ^b_{j-1}(q,q') + (j+1)Q^b_{j+1}(q,q')], \tag{5.13a}$$

$$D^{1j,b}_{jj}(q,q') = \frac{2xx'}{2j+1}[(j+1)Q^b_{j-1}(q,q') + jQ^b_{j+1}(q,q')], \tag{5.13b}$$

$$D^{1j,b}_{j\mp1,j\mp1}(q,q') = \pm\frac{2xx'}{2j+1}Q^b_j(q,q'), \tag{5.13c}$$

$$D^{1j,b}_{j\mp1,j\pm1}(q,q') = -\frac{4xx'\sqrt{j(j+1)}}{(2j+1)}Q^b_j(q,q').$$

In the above equations it is assumed that $Q^b_l(q,q') = 0$ if $l < 0$.

It follows from a comparison of (5.3) and (5.4) that $G^{sj,b}_{ll'}(q,q')$ can be obtained from the formulas for $I^{sj,b}_{ll'}(q,q')$, $A^{sj,b}_{ll'}(q,q')$, and $B^{sj,b}_{ll'}(q,q')$ by simply replacing the denominator, $m_b^2 - \Delta^2$, in (5.8) by 1. In Sec. VI we will use these partial-wave results to determine the nucleon-nucleon partial-wave amplitudes that arise from the potentials (4.30)–(4.32).

VI. NUMERICAL RESULTS

In order to obtain the nucleon-nucleon phase parameters we must solve the integral equations

$$T^{sjt}_{ll'}(q,k;s) = V^{sjt}_{ll'}(q,k) + \sum_{l''}\int_0^\infty \frac{dq''q''^2}{(2\pi)^3\varepsilon(q'')}\frac{V^{sjt}_{ll''}(q,q'')}{s - W^2(q'')}T^{sjt}_{l''l'}(q'',k;s), \tag{6.1}$$

where the invariant mass-square of the nucleon-nucleon scattering process is given by

$$s = W^2(\mathbf{k}) = 4\varepsilon^2(\mathbf{k}) = 4(\mathbf{k}^2 + m_N^2), \tag{6.2}$$

and the partial-wave potentials are given by

$$V^{sjt}_{ll'}(q,q') = \sum_b V^{sjt,b}_{ll'}(q,q'), \tag{6.3}$$

as well as (4.30)–(4.32) and (5.1). In order to solve the singular integral equations numerically, we have first converted them to nonsingular integral equations using Kowalski's method [41].

The partial-wave S-matrix elements are related to the on-shell ($q = k$) T-matrix elements by

$$S(k) = 1 - \frac{ik}{16\pi^2 W(k)}T(k). \tag{6.4}$$

For the uncoupled states

$$S^j = \exp(2i\delta_j), \tag{6.5}$$

while for the coupled states we have

$$S = \begin{bmatrix} \cos(2\varepsilon_j)\exp(2i\delta_{j-1}) & i\sin(2\varepsilon_j)\exp[i(\delta_{j-1}+\delta_{j+1})] \\ i\sin(2\varepsilon_j)\exp[i(\delta_{j-1}+\delta_{j+1})] & \cos(2\varepsilon_j)\exp(2i\delta_{j+1}) \end{bmatrix}, \tag{6.6}$$

which is the usual Stapp parametrization [42].

As pointed out at the end of Sec. IV, the only difference between our OBE potentials and those used in the OBE version of the Bonn model [9,10] occur in the so-called tensor-tensor term of the vector meson exchange potentials. The difference is given by (4.35). In order to see if this difference leads to dramatic differences in parameters obtained by fitting to the properties of the two-nucleon system, we have carried out a calculation using our version of Machleidt's *potential B*, given in Appendix A of Ref. [10]. The parameters given by Machleidt [10] for his potential are reproduced here in Table I, under the heading *Model MB*. Our potential differs only in the tensor-tensor part of the ρ-meson-exchange potential, since $\kappa_\omega = 0$. We have adjusted the parameters in boldface under the heading *Model I* in Table I so as to produce a least-squares fit of our phase parameters to Machleidt's. Machleidt's phase parameters and ours are labeled MB and I, respectively, in Table II. We see that the agreement between the two sets of phase parameters is quite good, and that there are no dramatic differences between Machleidt's potential parameters and those of our model I.

We have also adjusted our potential parameters so as to give a least-squares fit to the VZ40 phase-shift analysis of the two-nucleon data [43]. The resulting potential parameters are given under the heading *Model II* in Table I, and a comparison of the resulting phase parameters and those of VZ40 are given in Table III. We see that a reasonable fit

TABLE I. OBE parameters for three models. All masses are in MeV, and $n_b = 1$ except for $n_\rho = n_\omega = 2$.

Meson		Model MB	Model I	Model II
π	$g_\pi^2/4\pi$	14.4	14.4	**14.5**
	λ_π	1	1	1
	m_π	138.03	138.03	138.03
	Λ_π	1700	**1691.9**	**1680.2**
η	$g_\eta^2/4\pi$	3	**2.9075**	**3.2915**
	λ_η	1	1	1
	m_η	548.8	548.8	548.8
	Λ_η	1500.0	**1317.1**	**1602.4**
ρ	$g_\rho^2/4\pi$	0.9	**0.80595**	**0.96247**
	κ_ρ	6.1	6.1	**6.09**
	m_ρ	769	769	769
	Λ_ρ	1850	**2014.9**	**1898.8**
ω	$g_\omega^2/4\pi$	24.5	**24.423**	**24.611**
	κ_ω	0.0	0.0	0.0
	m_ω	782.6	782.6	782.6
	Λ_ω	1850	**1848.8**	**1868.5**
δ	$g_\delta^2/4\pi$	2.488	**2.6998**	**2.8079**
	m_δ	983	983	983
	Λ_δ	2000	**2105.9**	**2019.7**
$\sigma, t=0, t=1$	$g_\sigma^2/4\pi$	18.3773,8.9437	**18.225, 8.8810**	**18.168, 8.9381**
	m_σ	720,550	720,550	**713.3, 549.5**
	Λ_σ	2000,1900	**1919.9, 1891.1**	**1975.3, 1929.2**

to VZ40 is obtained with our model II version of Machleidt's potential B.

We have also calculated the deuteron's binding energy ($-\varepsilon_d$), percentage D state (P_D), and asymptotic normalization constants (A_S and A_D). The deuteron's wave function and binding energy are obtained by solving the equation

$$\Psi_l(q) = [(2m_N + \varepsilon_d)^2 - W^2(q)]^{-1} \sum_{l'=0,2} \int_0^\infty \frac{dq' q'^2}{(2\pi)^3 \varepsilon(q')} V_{ll'}^{1,1,0}(q,q') \Psi_{l'}(q'), \quad l = 0, 2. \tag{6.7}$$

TABLE II. Neutron-proton phase parameters, in degrees, at various laboratory kinetic energies in MeV, for two one-boson-exchange potentials.

State	Potential	25	50	100	150	200	300
$^1S_0$	I	50.72	39.99	25.18	14.32	5.55	−8.43
	MB	50.72	39.98	25.19	14.38	5.66	−8.18
$^3P_0$	I	9.36	12.26	9.79	4.50	−1.16	−11.78
	MB	9.34	12.24	9.80	4.57	−1.02	−11.48
$^1P_1$	I	−7.18	−11.13	−16.31	−20.27	−23.65	−29.19
	MB	−7.21	−11.15	−16.31	−20.21	−23.47	−28.70
$^3P_1$	I	−5.33	−8.78	−13.48	−17.19	−20.50	−26.40
	MB	−5.33	−8.77	−13.47	−17.18	−20.49	−26.38
$^3S_1$	I	80.32	62.16	41.97	28.88	18.91	3.79
	MB	80.32	62.16	41.99	28.94	19.04	4.07
$^3D_1$	I	−2.99	−6.88	−13.03	−17.36	−20.39	−23.78
	MB	−2.99	−6.86	−12.98	−17.28	−20.28	−23.72
ε_1	I	1.76	1.99	2.19	2.48	2.86	3.70
	MB	1.76	2.00	2.24	2.58	3.03	4.03
$^1D_2$	I	0.68	1.57	3.31	4.89	6.14	7.42
	MB	0.68	1.58	3.34	4.94	6.21	7.49
$^3D_2$	I	3.88	9.29	17.67	22.58	24.97	25.43
	MB	3.88	9.29	17.67	22.57	24.94	25.36
$^3P_2$	I	2.63	6.17	11.78	15.04	16.67	17.37
	MB	2.62	6.14	11.73	14.99	16.65	17.40
$^3F_2$	I	0.11	0.34	0.77	1.05	1.12	0.58
	MB	0.11	0.34	0.77	1.04	1.10	0.52
ε_2	I	−0.86	−1.82	−2.85	−3.06	−2.86	−2.00
	MB	−0.86	−1.82	−2.84	−3.05	−2.85	−2.02

TABLE III. Neutron-proton phase parameters, in degrees, at various laboratory kinetic energies in MeV, for potential II and the VZ40 solutions of Arndt and Roper [43].

State	Potential	25	50	100	150	200	300
$^1S_0$	II	50.38	39.49	24.45	13.42	4.50	−9.71
	VZ40	49.17	39.40	25.59	14.68	5.87	−7.06
$^3P_0$	II	9.18	11.84	9.06	3.58	−2.21	−12.98
	VZ40	8.18	11.11	9.55	4.67	−1.14	−12.68
$^1P_1$	II	−7.31	−11.41	−16.92	−21.20	−24.87	−30.98
	VZ40	−5.73	−9.00	−14.39	−18.90	−22.49	−27.46
$^3P_1$	II	−5.38	−8.88	−13.68	−17.47	−20.86	−26.90
	VZ40	−4.79	−7.98	−12.86	−17.27	−21.43	−28.82
$^3S_1$	II	80.03	61.76	41.45	28.29	18.29	3.16
	VZ40	79.94	62.84	43.43	30.60	20.80	4.84
$^3D_1$	II	−2.99	−6.85	−12.84	−16.90	−19.57	−22.00
	VZ40	−2.48	−6.04	−11.98	−16.19	−19.16	−22.84
ε_1	II	1.74	1.95	2.13	2.41	2.78	3.61
	VZ40	2.02	2.61	3.33	3.38	3.45	4.36
$^1D_2$	II	0.69	1.60	3.41	5.07	6.40	7.77
	VZ40	0.66	1.61	3.52	5.33	7.00	9.93
$^3D_2$	II	3.92	9.37	17.83	22.74	25.09	25.41
	VZ40	3.78	9.00	16.87	21.34	23.53	23.99
$^3P_2$	II	2.59	6.05	11.51	14.68	16.28	16.98
	VZ40	2.60	5.97	11.32	14.62	16.50	17.72
$^3F_2$	II	0.11	0.34	0.77	1.03	1.08	0.46
	VZ40	0.09	0.33	0.82	1.12	1.20	0.88
ε_2	II	−0.87	−1.83	−2.84	−3.03	−2.80	−1.90
	VZ40	−0.73	−1.56	−2.50	−2.79	−2.74	−2.17

The first factor on the right-hand side of this equation has a simple pole in q^2, at $q^2 = -\gamma^2$, where γ is determined by

$$W(i\gamma) = 2m_N + \varepsilon_d. \tag{6.8}$$

The residues of this pole are proportional to the asymptotic normalization parameters; more precisely we have

$$A_l = \frac{1}{16\pi[\varepsilon(i\gamma)]^{1/2}} \sum_{l'=0,2} \int_0^\infty \frac{dqq^2}{(2\pi)^3\varepsilon(q)} V_{ll'}^{1,1,0}(i\gamma, q)\Psi_{l'}(q), \quad l = 0, 2, \tag{6.9}$$

where we have assumed that the components of the wave function satisfy the normalization condition

$$\int_0^\infty \frac{dqq^2}{(2\pi)^3\varepsilon(q)}[\Psi_0^2(q) + \Psi_2^2(q)] = 1. \tag{6.10}$$

The deuteron and low-energy parameters for Machleidt's potential and our two potentials are given in Table IV. We see that the low energy properties of the two-nucleon system are properly accounted for by our models.

VII. DISCUSSION

It is worthwhile at this point to "rederive" the nucleon-nucleon potential given by (4.30)–(4.33) so as to summa-

TABLE IV. Deuteron and low-energy parameters. Scattering lengths and effective ranges are denoted by a and r, respectively, with the subscript referring to spin singlet (s) and spin triplet (t). The experimental values are from Table 4.2 of Ref. [10].

Parameter	Model MB	Model I	Model II	Experiment
$-\varepsilon_d$ (MeV)	2.225	2.220	2.221	2.224 575
$P_D(\%)$	4.99	4.97	4.81	
A_S (GeV$^{1/2}$)	0.3936	0.3931	0.3940	0.3930±0.0004
A_D/A_S	0.0264	0.0264	0.0264	0.0256±0.0004
a_s (fm)	−23.75	−23.64	−23.71	−23.748±0.010
r_s (fm)	2.71	2.66	2.68	2.75±0.05
a_t (fm)	5.424	5.429	5.438	5.419±0.007
r_t (fm)	1.761	1.769	1.782	1.754±0.008

rize the procedure that was followed. The first step is to "derive" the second-order, nucleon-nucleon potentials in the forms given by (4.10)–(4.12). These forms are obtained by drawing the relevant one-boson-exchange Feynman diagrams and determining the four-momentum of the virtual meson in each diagram by assuming that the total four-momentum is conserved *either* at the vertex on the "right" *or* at the vertex on the "left," but not necessarily at both vertices. The potentials (4.10)–(4.12) are obtained by adding together the two resulting Feynman amplitudes and dividing by 2.

The next step is to express the potentials in terms of the relative three-momentum variables $\mathbf{q}$ and $\mathbf{q}'$ defined by (3.20). This transformation involves the initial- and final-state light front spinors, $u(p_i', h_i')$ and $u(p_i, h_i)$, respectively; various matrices such as the unit matrix or Feynman slash quantities that appear between these spinors; and the invariant denominators $(p_i - p_i')^2 - m_b^2$.

The light front spinors are replaced by the canonical spinors (4.19) according to the rule (4.21). This leads to the various matrices sandwiched between $S^{-1}[l_f(\Lambda)]$ and $S[l_f(\Lambda')]$. These combinations can be expressed in terms of $\mathbf{q}$ and $\mathbf{q}'$ by using identities such as (4.22) and (4.23). The invariant denominators can be worked out in the c.m. frame of the final state, the x_Λ frame. In this frame $p_1 = [\varepsilon(\mathbf{q}), \mathbf{q}]$ and $p_2 = [\varepsilon(\mathbf{q}), -\mathbf{q}]$. The final-state c.m. frame is related to the initial-state c.m. frame by (4.25) and (4.26). In the initial-state c.m. frame $p_1' = [\varepsilon(\mathbf{q}'), \mathbf{q}']$ and $p_2' = [\varepsilon(\mathbf{q}'), -\mathbf{q}']$. After transforming these initial-state momenta to the final-state c.m. frame it is trivial to work out the invariant denominators.

Upon carrying out the above procedure it is found that the resulting potential is not *kinematically* rotationally invariant. It can be made so by simply dropping all of the $\mathbf{u}$-dependent terms. This leads to a Poincaré invariant, *free-spin model* of the two-nucleon system. Another procedure for obtaining free spin models, which is discussed in Refs. [31,32], is to average over the direction of $\mathbf{u}$. This procedure is a natural first approximation to a complete, *new picture* treatment of the interacting spin operator. It is easy to see from (4.27)–(4.29) that averaging over the direction of $\mathbf{u}$ does not lead to expressions for the OBE potentials that are as "simple" as (4.30)–(4.33). In spite of this, the averaging procedure is worth pursuing as it is important to determine the sensitivity of results to the approximation adopted.

It will be of interest to go beyond the one-boson-exchange framework so as to include processes such as 2π exchange and $\pi\rho$ exchange. This will entail going to higher order in the author's front form extension [31,32] of the Okubo-Glöckle-Müller formalism [35,36]. It will be of particular interest to see if the fourth-order contributions can be obtained from simple variations of the Feynman rules, as was found for the second-order contributions.

Another extension of the front form NN model presented here that will be worth considering is a treatment of coupled NN, $N\Delta$, and possibly $\Delta\Delta$, channels. There is no difficulty in principle in constructing exactly Poincaré invariant, front form, coupled-channel models [21]. Such a model is a relatively straightforward way to account for inelasticity in NN scattering. A coupled channel model has been constructed [44] using the method of folded diagrams [17], and has been found to give a good account of the nucleon-nucleon phase shifts up to laboratory kinetic energies of about 325 MeV.

An advantage of a front form model of the NN system is that there exists a consistent impulse approximation [21] for calculating the electromagnetic form factors of the deuteron. This approximation has been used to calculate the form factors that arise from purely phenomenological front form potentials [24]. It will be of interest to carry out such calculations using the meson exchange model developed here. All of the above possibilities are presently under investigation, and will be the subject of future publications.

ACKNOWLEDGMENT

This work was supported in part by National Science Foundation Grants PHY-9101331 and PHY-9403763.

[1] H. Yukawa, Proc. Phys. Math. Soc. Jpn. **17**, 48 (1935).
[2] A. Proca, J. Phys. Radium **7**, 347 (1936).
[3] N. Kemmer, Proc. R. Soc. London Ser. A **166**, 127 (1938).
[4] C. Møller and L. Rosenfeld, Kgl. Dan. Vid. Selskab, Math.-Fys. Medd. **17**, No. 8 (1940).
[5] J. Schwinger, Phys. Rev. **61**, 387 (1942).
[6] G. P. S. Occhialini, C. F. Powell, C. M. G. Lattes, and H. Muirhead, Nature **159**, 186 (1947); **159**, 694 (1947); C. M. G. Lattes, G. P. S. Occhialini, and C. F. Powell, *ibid.* **160**, 453 (1947); **160**, 486 (1947).
[7] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Lett. **6**, 628 (1961).
[8] B. C. Maglic, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Lett. **7**, 178 (1961).
[9] R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **149**, 1 (1987).
[10] R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989).
[11] K. Holinde, Nucl. Phys. **A543**, 143c (1992).
[12] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[13] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
[14] F. Gross, Phys. Rev. **186**, 1448 (1969).
[15] J. Fleischer and J. A. Tjon, Nucl. Phys. B **84**, 375 (1975); Phys. Rev. D **15**, 2537 (1977); **21**, 87 (1980); M. J. Zuilhof and J. A. Tjon, Phys. Rev. C **22**, 2369 (1980); **24**, 736 (1981); E. van Faasen and J. A. Tjon, *ibid.* **28**, 2354 (1983); **30**, 285 (1984); **33**, 2105 (1986).
[16] F. Gross, J. W. van Orden, and K. Holinde, Phys. Rev. C **41**, R1909 (1990); **45**, 2094 (1992).
[17] M. B. Johnson, Ann. Phys. (N.Y.) **97**, 400 (1976).
[18] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[19] F. Coester and W. N. Polyzou, Phys. Rev. D **26**, 1348

(1982).
[20] H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).
[21] B. D. Keister and W. N. Polyzou, in *Advances in Nuclear Physics,* edited by J. W. Negele and E. W. Vogt (Plenum, New York, 1991), Vol. 20, p. 225.
[22] B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terent'ev, Nucl. Phys. B **158**, 497 (1979).
[23] M. G. Fuda, Phys. Rev. C **36**, 1489 (1987).
[24] P. L. Chung, F. Coester, B. D. Keister, and W. N. Polyzou, Phys. Rev. C **37**, 2000 (1988).
[25] M. G. Fuda, Phys. Rev. D **42**, 2898 (1990).
[26] V. A. Karmanov, Fiz. Elem. Chastits At. Yad. **19**, 525 (1988) [Sov. J. Part. Nucl. **19**, 228 (1988)]. This article reviews Karmanov's earlier work on the use of invariant null planes.
[27] V. A. Karmanov and A. V. Smirnov, Nucl. Phys. A **546**, 691 (1992).
[28] M. G. Fuda, Ann. Phys. (N.Y.) **197**, 265 (1990).
[29] M. G. Fuda, Phys. Rev. D **41**, 534 (1990).
[30] M. G. Fuda, Phys. Rev. D **44**, 1880 (1991).
[31] M. G. Fuda, Nucl. Phys. A **543**, 111c (1992).
[32] M. G. Fuda, Ann. Phys. (N.Y.) **231**, 1 (1994).
[33] H. J. Melosh, Phys. Rev. D **9**, 1095 (1974).
[34] J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).
[35] S. Okubo, Prog. Theor. Phys. **12**, 603 (1954).
[36] W. Glöckle and L. Müller, Phys. Rev. C **23**, 1183 (1981).
[37] S.-J. Chang, R. G. Root, and T.-M. Yan, Phys. Rev. D **7**, 1133 (1973); S.-J. Chang and T. M. Yan, *ibid.* **7**, 1147 (1973).
[38] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[39] M. G. Fuda and Y. F. Zhang (unpublished).
[40] A. Messiah, *Quantum Mechanics* (Wiley, New York, 1965).
[41] K. L. Kowalski, Nucl. Phys. A **190**, 645 (1972).
[42] H. P. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).
[43] R. A. Arndt and L. D. Roper, Scattering Analysis and Interactive Dial-In (SAID) program, Virginia Polytechnic Institute and State University.
[44] J. Haidenbauer, K. Holinde, and M. B. Johnson, Phys. Rev. C **48**, 2190 (1993).