Symmetry constraints on the classical Skyrmion

Mohammad Samiullah and Peter Rolnick
Division of Science, Northeast Missouri State University, Kirksville, Missouri 63501
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We derive the constraints on the solutions of the classical SU(2) Skyrme model imposed by requiring that angular momentum (**J**) and isospin (**I**) be well defined under the general symmetry $(aI_3 + bJ_3)$. We show that for all nontrivial solutions $(aI_3 + bJ_3)$ must be 0, a/b must be an integer, and for $b \neq 0$ the profile function must be of the form $\mathbf{F}(\mathbf{r}) = F\ ((1-A^2)^{1/2}\hat{\mathbf{z}} + A\{\hat{\mathbf{x}}\cos[(a/b)\phi + B] + \hat{\mathbf{y}}\sin[(a/b)\phi + B]\})(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are Cartesian unit vectors; r, θ, ϕ are the usual spherical polar coordinates; F, A, B are undetermined functions of r, θ).

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I. INTRODUCTION

The Skyrme model, in which topological solitary waves in a classical pion field are interpreted as baryons, has had some success in describing the low-energy realm of quantum chromodynamics (QCD) [1]. Skyrme made the initial observation that the nonlinear σ model, a classical chiral field theory of pions, can give rise to solitary wave solutions which could be interpreted as baryons, and added a term which stabilized these solitary waves, henceforth referred to as *Skyrmions* [2]. A topological invariant is associated with the Skyrme field, which is identified as the baryon number (B) of the Skyrmion. Skyrme obtained a classical, radially symmetric solution, called the *hedgehog* Skyrmion, in which the third (or z) components of isospin and spin are of equal magnitude. The possible fermionic character of the hedgehog Skyrmion was later confirmed [3], and a connection was made between the Skyrme model and the large-N_c (number of colors) limit of QCD [4]. The success of the Skyrme model stems from the observation that the $B = 1$ hedgehog solution predicts the properties of the nucleon to within about 30% [5]. Since then, there have been applications of the Skyrme model to multi-baryon systems [6], including descriptions of light nuclei as systems of Skyrmions [7,8], and investigations of the nucleon-nucleon force using the two-Skyrmion system [9,10]. General axial symmetry has been previously discussed [11], and axial symmetry for the case of two $B = 1$ Skyrmions combining to form a deuteron was argued [12] before being discovered numerically in some of the above investigations.

Though the hedgehog has been used almost exclusively in applications of the Skyrme model, there is some reference to "exotic textures [1]." The choice of the hedgehog is usually justified by its simplicity [1,13]. In this note we investigate the relationship between allowable textures and corresponding spin and isospin; we find that the requirement of well-defined values of spin and isospin puts severe constraints on the solutions of the Skyrme model. The rest of the report is organized as follows: Sec. II contains the derivation of the constraints, and conclusions are in Sec. III.

II. DERIVATION OF THE CONSTRAINTS

A. Skyrme model notation

The *Skyrme model* consists of the following Lagrangian for the matrix-valued unitary field $U(t,\mathbf{r})$:

$$L = \int d\mathbf{r}\left(\frac{1}{4}f_\pi^2 \mathrm{tr}[\partial_\mu U \partial^\mu U^\dagger]\right) + \int d\mathbf{r}\left(\frac{1}{32}e^{-2}\mathrm{tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2\right) . \tag{1}$$

f_π is the pion decay constant and e^2 is proportional to the nucleon axial form factor in the limit $N_c \to \infty$ [5,1,11]; summation is implied for repeated indices; Latin indices run from 1 to 3, Greek indices run from 0 to 3; the signature of the metric is $(+---)$; and tr indicates a trace over the matrices contained in the U field. We shall limit our discussion to the SU(2) Skyrme model, and to the most commonly used ansatz, in which case $U(t,\mathbf{r})$ can be written as

$$\begin{aligned} U(t,\mathbf{r}) &= \exp\left(i[\boldsymbol{\tau}\cdot\hat{\mathbf{n}}(t,\mathbf{r})]\,F(t,\mathbf{r})\right) \\ &= \cos F(t,\mathbf{r}) + i[\boldsymbol{\tau}\cdot\hat{\mathbf{n}}(t,\mathbf{r})]\sin F(t,\mathbf{r}) , \end{aligned} \tag{2}$$

where $\boldsymbol{\tau}$ represents the Pauli matrices in isospin space; $\hat{\mathbf{n}}(t,\mathbf{r}) \equiv n_1(t,\mathbf{r})\,\hat{\mathbf{x}} + n_2(t,\mathbf{r})\,\hat{\mathbf{y}} + n_3(t,\mathbf{r})\,\hat{\mathbf{z}}$ is the field (of unit length) representing the isospin direction at $(t,\mathbf{r})$; and $F(t,\mathbf{r})$ is called the profile function [$\cos F(t,\mathbf{r})$ and $\hat{\mathbf{n}}(t,\mathbf{r})\sin F(t,\mathbf{r})$ could be identified with meson fields σ and $\boldsymbol{\pi}$].

The energy of the system is conserved and can be written down in terms of the field configuration $U(\mathbf{r})$ at any fixed time:

$$E = \int d\mathbf{r}\left\{\frac{1}{4}f_\pi^2 \mathrm{tr}[(\partial_i U)(\partial_i U)^\dagger] + \frac{1}{32}e^{-2}\mathrm{tr}[(\partial_i U)U^\dagger, (\partial_j U)U^\dagger]^2\right\} . \tag{3}$$

The boundary conditon $U(r \to \infty) = 1$, where 1 is the 2×2 identity matrix, is needed to ensure that the energy remains finite. With this boundary condition, $U(\mathbf{r})$ is seen to map S^3 (the real three-dimensional space with all points at ∞ identified) $\to$ S^3 [the group manifold of SU(2)]. The integral over all space of the topological charge associated with this map must be an integer, which in turn is identified as the baryon number, which can be expressed in terms of the profile function as

$$B = -\int d\mathbf{r}\{(\sin^2 F(\mathbf{r})/12\pi^2)\varepsilon_{ijk}(\nabla n_i(\mathbf{r}) \times \nabla n_j(\mathbf{r})) \cdot (3n_k(\mathbf{r})\nabla F(\mathbf{r}) + \sin F(\mathbf{r})\cos F(\mathbf{r})\nabla n_k(\mathbf{r}))\} \quad (4)$$

(ε_{ijk} is the totally antisymmetric Levi-Civita tensor).

Thus, for all solutions of the Skyrme model, we require that E be finite, and that B be an integer.

B. Isospin and angular momentum constraints

Skyrme further constrained the Skyrme model by investigating the isospin and angular momentum properties of U. Although the quantum field theory based on quantization of U in the Skyrme Lagrangian is nonrenormalizable, we can still study the quantum numbers associated with the field operator U. Skyrme assumed maximal symmetry: $F = F(r \text{ only})$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ (hedgehog assumption). Though neither the z component of isospin (I_3) nor of angular momentum (J_3) are independently well defined for the hedgehog Skyrmion if they are nonzero, the quantity $K_3 \equiv (I_3 + J_3)$ is well defined and is identically zero. Thus the hedgehog configuration can describe only those nuclei which have $|I_3| = |J_3|$. However, for many nuclei (such as the deuteron) $I \neq J$. In the following we study the possible solutions to the Skyrme model under the more general symmetry $K_3(a,b) \equiv (aI_3 + bJ_3)$.

Here we look at constraints on the solution imposed by requiring that for a fixed time it have a well-defined J_3 and a well-defined I_3. From elementary quantum mechanics, it is known that a quantum field Φ will have a good quantum number q associated with an infinitesimal generator Q if $[Q,\Phi] = q\Phi$. Similarly, to study the solutions of the Skyrme model, we treat U as a quantum field operator. Thus, the requirement that J_3 and I_3 be well defined with eigenvalues j_3 and i_3, respectively, for a field configuration $U(\mathbf{r})$ at a fixed time implies the following:

$$[J_3, U(\mathbf{r})](= [-i(\mathbf{r} \times \nabla)_3, U(\mathbf{r})]) = j_3 U(\mathbf{r}) , \quad (5)$$

$$[I_3, U(\mathbf{r})](= [(\tau_3/2), U(\mathbf{r})]) = i_3 U(\mathbf{r}) . \quad (6)$$

Substituting (2) into (5) and (6), we get:

$$(\mathbf{r} \times \nabla F(\mathbf{r}))_3[i \sin F(\mathbf{r}) + \boldsymbol{\tau} \cdot \hat{\mathbf{n}} \cos F(\mathbf{r})] + \sin F(\mathbf{r})[\mathbf{r} \times \nabla(\boldsymbol{\tau} \cdot \hat{\mathbf{n}})] = j_3 U(\mathbf{r}) , \quad (7)$$

$$\sin F(\mathbf{r})(\boldsymbol{\tau} \times \hat{\mathbf{n}})_3 = i_3 U(\mathbf{r}) . \quad (8)$$

Treating $\mathbf{I}$ and $\mathbf{J}$ independently, the only solution to (7) and (8) has $j_3 = i_3 = 0$, thus there is no solution to the Skyrme model for which both isospin and angular momentum are independently well defined and have nonzero projection. However, consider $K_3(a,b)$:

$$[K_3(a,b), U(\mathbf{r})] = [(a(\tau_3/2) + -ib(\mathbf{r} \times \nabla)_3), U(\mathbf{r})] = (ai_3 + bj_3)U(\mathbf{r}) , \quad (9)$$

or, substituting (2) into (9):

$$a \sin F(\mathbf{r})(\boldsymbol{\tau} \times \hat{\mathbf{n}})_3 + b\{(\mathbf{r} \times \nabla F(\mathbf{r}))_3[i \sin F(\mathbf{r}) + (\boldsymbol{\tau} \cdot \hat{\mathbf{n}}) \cos F(\mathbf{r})] + \sin F(\mathbf{r})(\mathbf{r} \times \nabla(\boldsymbol{\tau} \cdot \hat{\mathbf{n}}))_3\} = (ai_3 + bj_3)U(\mathbf{r}) . \quad (10)$$

Note that if $a = b = 1$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ (the hedgehog assumption), then all terms on the left-hand side of (10) cancel, leaving $bj_3 = -ai_3$ as a nontrivial solution. In order to explore other solutions of the Skyrme model which admit nontrivial angular dependence, we must look at the most general form of (10), which, unless $b = 0$ (in which case a $i_3 = 0$) or one of $\sin(F)$ or $\cos(F) = 0$, reduces to the following conditions:

$$ai_3 + bj_3 = 0 , \quad (11)$$

$$\partial_\phi F(\mathbf{r}) = 0 , \quad (12)$$

$$\partial_\phi n_3(\mathbf{r}) = 0 , \quad (13)$$

$$\partial_\phi n_1(\mathbf{r}) = -(a/b)n_2(\mathbf{r}) , \quad (14)$$

$$\partial_\phi n_2(\mathbf{r}) = (a/b)n_1(\mathbf{r}) . \quad (15)$$

(∂_ϕ indicates differentiation with respect to the azimuthal angle ϕ.) Thus, for $b \neq 0, \sin(F) \neq 0$ and $\cos(F) \neq 0, \mathbf{F}(\mathbf{r})$ must be of the form:

$$\mathbf{F}(\mathbf{r}) = F(r,\theta)\{(1 - A^2(r,\theta))^{(1/2)}\hat{\mathbf{z}} + A(r,\theta)[\cos((a/b)\phi + B(r,\theta))\hat{\mathbf{x}} + \sin((a/b)\phi + B(r,\theta))\hat{\mathbf{y}}]\} \quad (16)$$

(where A, B, and F are undetermined functions of the radial variable r and the polar angle θ). The magnitude of $\mathbf{F}$ is independent of ϕ, as is the magnitude of the projection of $\hat{\mathbf{n}}$ onto the z axis and onto the x, y plane. But for a given value of r and θ, the projection of $\mathbf{F}$ in the x, y plane rotates a/b times as ϕ goes from 0 to 2π. For $\mathbf{F}$ to remain single-valued, a/b must be an integer.

III. CONCLUSION

In this note, we have explicitly shown the necessary symmetries of the solutions of the Skyrme model, using ansatz (2), which have well-defined nonzero z components of spin and isospin under the general symmetry $K_3(a,b) = (aI_3 + bJ_3)$. We find that all solutions must have $a|i_3| = b|j_3|$. Most uses of the Skyrme model to describe multi-baryon systems start with multiple hedgehog Skyrmions and quantize spin and isospin degrees of freedom separately. This allows one to construct matrix-valued functions that have $|i_3| \neq |j_3|$. It may prove valuable to explore the nonspherical solutions to the classical Skyrme model. In particular, $B = 2$ solutions may be used to study the Skyrmion-Skyrmion interaction potential by providing a small separation limit as two widely separated hedgehog Skyrmions are brought closer together [9,10].

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