PHYSICAL REVIEW C VOLUME 51, NUMBER 3 MARCH 1995

Isospin dependence of the oscillator spacing

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(Received 18 October 1994)

New improved expressions for the oscillator spacing $\hbar\omega$ are derived. These expressions have the advantage of being isospin dependent. They are obtained by employing new expressions for the mean square radius of nuclei, which fit the experimental mean square radii and the isotopic shifts of even-even nuclei much better than other frequently used relations. The trend of the variation of $\hbar\omega$ with the neutron excess is studied. A formula for $\hbar\omega$ calculated in the present approach as a function of the mass number A agrees well with $\hbar\omega$ coming from a completely different input, i.e., the separation energies of the last nucleon. Very accurate approximate asymptotic formulas for $\hbar\omega$ are also derived, which are suitable for practical use.

PACS number(s): 21.60.Ev, 21.60.Cs, 21.10.Pc, 21.10.Dr

I. INTRODUCTION

This paper addresses an old question of nuclear physics: the determination of the harmonic-oscillator energy level spacing $\hbar\omega$ as a function of N and Z. This quantity is useful in various nuclear studies.

It approximates the lowest single-particle energy level spacing, thus giving an estimate for this quantity as well as for its variation with the number of neutrons and protons. It represents also the average trend in the variation of the shape of the self-consistent nucleon-nucleus potential as a function of N and Z.

There are various expressions in the literature for $\hbar\omega$ as a function of A. The most well known expression [1,2] is

$$\hbar\omega = f A^{-1/3} , \tag{1}$$

where $f = \frac{5}{4}(\frac{\hbar}{mr_0^2})(\frac{3}{2})^{1/3} \simeq 41$ MeV ($r_0 = 1.2$ fm), which holds for large A. Other improved expressions [3–10] have been proposed with the aim of obtaining more satisfactory expressions for lighter nuclei. The most sophisticated approaches so far are those of [7] and [9,10]. In [7] certain approximations made in other approaches were removed and a Fermi density distribution for the nucleons was used with parameters ρ_0 and a determined by fitting to experimental values of the charge mean square (MS) radii $\langle r^2\rangle_{\rm ch}$. In [9,10] we used the rather recently proposed semiphenomenological density distribution [11–13] of Gambhir and Patil based on the separation energies of the last neutron or proton.

The aim of the present paper is to determine $\hbar\omega$ as a function of N and Z introducing an isospin dependence by exploiting very accurate recent experimental data for the isotopic shifts. The paper is organized as follows.

In Sec. II the usual method to determine $\hbar\omega$ is briefly outlined and a new formula for $\hbar\omega$ as a function of N and Z is derived, using a very recently proposed expression for the nuclear charge radius [14]. This expression is isospin dependent and is based on a uniform density distribution.

In Sec. III the procedure of Sec. II is repeated. Instead of a uniform distribution, the symmetrized Fermi (SF) density distribution [15,16] (see also [17],[18]) is used together with a new parametrization of the radius parameter R.

In Sec. IV approximate asymptotic expressions for $\hbar\omega$ are given.

In Sec. V $\hbar\omega$ is determined again, as in Secs. II and III, but the usual corrections due to the center of mass and finite size of the nucleons are taken into account together with the valence nucleons. Shell effects are observed at the closed shells.

Finally, Sec. VI contains our main conclusions.

II. DETERMINATION OF $\hbar\omega$ WITH A UNIFORM DENSITY DISTRIBUTION

The average harmonic oscillator shell model square radius for nucleons is

$$\overline{\langle r^2\rangle}_K = \frac{3}{4}\frac{\hbar}{m\omega}(K+1) , \tag{2}$$

where K is the number of the highest filled shell. Then using a uniform distribution of radius R one finds for the MS radius

$$\langle r^2\rangle = \frac{3}{5}R^2 . \tag{3}$$

For K the following relation holds:

$$K(K+1)(K+2) = \frac{3}{2}A , \tag{4}$$

which for $A \gg K$ gives the following approximate solution:

$$K+1 \simeq (\tfrac{3}{2}A)^{1/3} . \tag{5}$$

Equating the MS radii from (2) and (3) using (5), one may obtain (as is well known) relation (1) for $\hbar\omega$. Here

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one uses $R = R_{00} = r_0 A^{1/3}$, which is the simplest known liquid drop model formula for the nuclear radius. This formula, as pointed out in [14], does not lead to the experimentally known MS radii $\langle r^2 \rangle_{\rm exp}$ or isotopic shifts $\delta\langle r^2 \rangle_{\rm exp}$ and it does not describe properly the change of the charge radius with N when the proton number Z is kept constant.

Very recently, a new formula for the nuclear charge radius was proposed [14], dependent on the mass number A and neutron excess $N - Z$ in the nucleus:

$$R_{00} = 1.240 A^{1/3} \left(1 + \frac{1.646}{A} - 0.191 \frac{(N-Z)}{A}\right) . \quad (6)$$

In contrast to the simple expression $R = r_0 A^{1/3}$, the above formula reproduces well all the experimentally available MS radii and the isotopic shifts of even-even nuclei, much better than other frequently used relations. This should be expected as the isotopic shifts, which are obtained from high precision laser spectroscopy [19], provide an extra very accurate input. In addition, they give us the opportunity to study the effect of the isovector component on the nuclear radius and consequently on $\hbar\omega$.

As $\langle r^2 \rangle$ is directly connected to $\hbar\omega$, it is interesting to estimate the effect that the improved expression (6) may have on $\hbar\omega$. Thus, using (6) instead of $R = R_{00} = r_0 A^{1/3}$ and following the procedure described previously, we obtain the isospin dependent formula in a straightforward way:

$$\hbar\omega = \frac{38.6}{A^{1/3}[1 + 1.646/A - 0.191(N-Z)/A]^2} . \quad (7)$$

This expression could be compared with another isospin dependent expression existing in the literature, namely, the formula suggested in [20],

$$\hbar\omega = 41 A^{-1/3} \left(1 + 2t \frac{N-Z}{3A}\right), \quad (8)$$

where $t = 1/2(-1/2)$ for a neutron(proton). The first term of (8) (isoscalar term) originates from the condition that the radius of the nucleus should be given roughly by $1.2\,A^{1/3}$ fm. The second term (isovector term) comes from the requirement that $\langle r^2 \rangle$ should have roughly the same value for protons and neutrons. However, it was shown in [21] that this choice of $\hbar\omega$ does not correspond to the right values of the MS radii and the isotopic shifts.

As a test of the new formula and for the sake of comparison with other known formulas, which however depend only on A, we remove from (7) the isospin dependence putting N=Z and plot the resulting formula for $\hbar\omega$ for A up to about 60 (where N=Z is meaningful),

$$\hbar\omega = \frac{38.6}{A^{1/3}(1 + 1.646/A)^2} , \quad (9)$$

as a function of A in Fig. 1 (solid line). In the same figure the old formula (1) is also plotted (long-dashed line) together with the corresponding curve (short dashed line) obtained using the radii obtained with the distribution of Gambhir and Patil [11]. We recall that this semiphenomenological algebraic form for the nuclear densities has

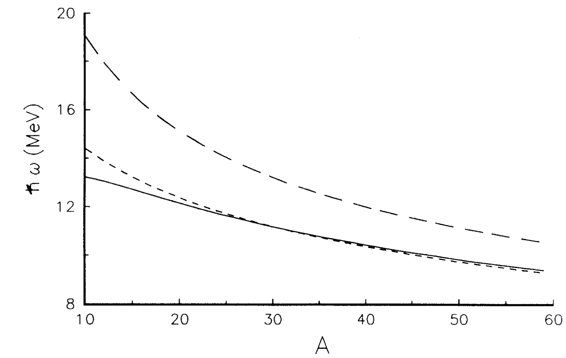

FIG. 1. Oscillator spacing $\hbar\omega$ as a function of A (without corrections) for three cases: (1) the simple liquid drop formula (long dashed curve), (2) formula (6) (solid curve) and (3) using the distribution of Gambhir and Patil (short dashed curve).

no free parameter and the only experimental input is the separation energies of the last proton or neutron. It is seen in Fig. 1 that the new formula gives values for $\hbar\omega$ significantly lower than the old one and also it agrees very well with the curve coming from the distribution of Gambhir and Patil. It is noted that the three curves of Fig. 1 were derived without taking into account any corrections as described below.

III. DETERMINATION OF $\hbar\omega$ WITH THE SYMMETRIZED FERMI DENSITY DISTRIBUTION

An isospin dependence of the charge radius can also be derived using the symmetrized Fermi density distribution [15,16]

$$\rho_{\rm SF} = \rho_0 \frac{\sinh(R/a)}{\cosh(r/a) + \cosh(R/a)} \quad (10)$$

with

$$\rho_0 = \frac{3}{4\pi R^3} \left[1 + \left(\frac{\pi a}{R}\right)^2\right]^{-1} . \quad (11)$$

The advantage over the usual Fermi distribution is that it is more suitable for light nuclei because it has zero slope at the origin. In addition the expression for MS radius,

$$\langle r^2 \rangle_{\rm SF} = \frac{3}{5} R^2 \left[1 + \frac{7}{3}\left(\frac{\pi a}{R}\right)^2\right] , \quad (12)$$

is exact and not a transcendental function of the radius R, as is the case with the Fermi distribution. We parametrize the radius R as follows:

$$R = c_1 A^{1/3} + c_2 A^{-1/3} + c_3 (N-Z) A^{-1} . \quad (13)$$

The parameters are determined by a least squares fitting of (12) to the experimental radii and isotopic shifts of 142 even-even isotopes, in the spirit of [14]. The best

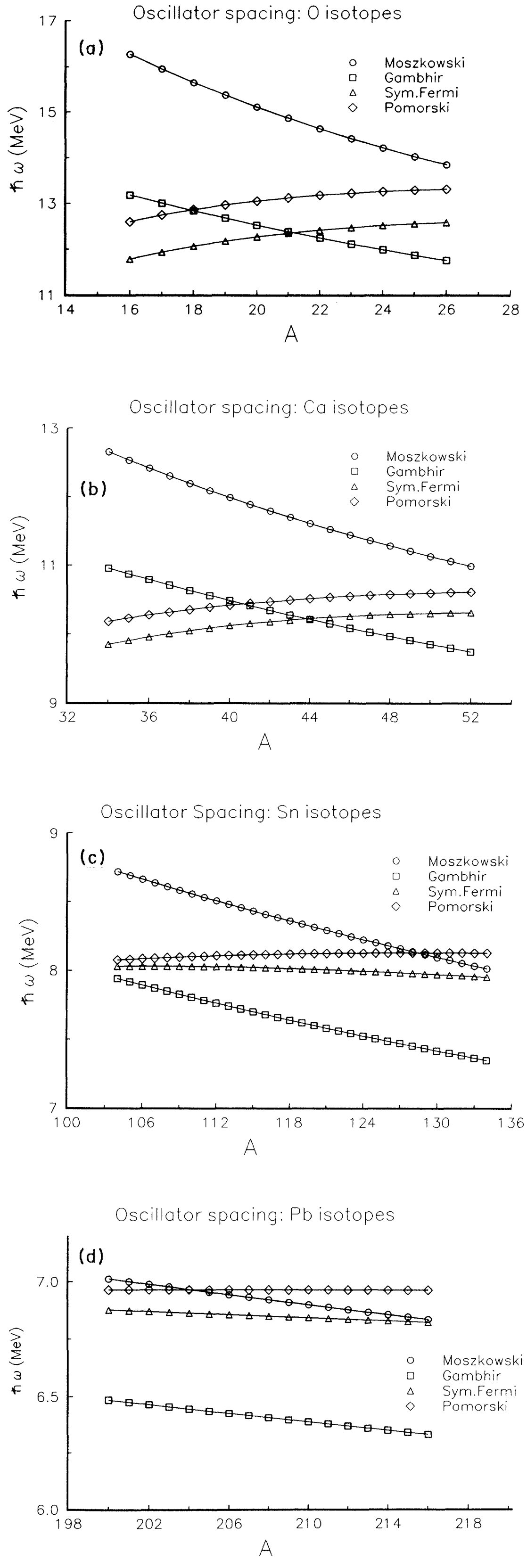

FIG. 2. Oscillator spacing for a number of isotopes for O, Ca, Sn, and Pb nuclei. For explanation see text.

fit values are $c_1 = 1.217$, $c_2 = -2.783$, $c_3 = -1.047$, and $a = 0.620$. We note that we have also tested different powers of A in the second and third term of the right-hand side of (13) and the best fit was obtained with the powers of A $(-1/3)$ and (-1), respectively.

Using the MS radius (12) with R from (13) and following the simple procedure described in Sec. II the following expression for $\hbar\omega$ is obtained:

$$\hbar\omega = \frac{35.6A^{1/3}}{0.6R^2 + 5.31} \ . \tag{14}$$

In Figs. 2(a)–2(d) we compare values of $\hbar\omega$ against A ($A = N + Z$) for various isotopes of representative nuclei, i.e, O, Ca, Sn and Pb in the region $8 \le Z \le 82$. These values are obtained with relation (7) (denoted Pomorski in the figures) and relation (14) (denoted Sym. Fermi). For the sake of comparison we also include the asymptotic expressions $\hbar\omega = 41A^{-1/3}$ [1] (denoted Moszkowski) and $\hbar\omega = 39.0A^{-1/3} - 36.8A^{-1}$ from [10] (denoted Gambhir).

It is seen that as the atomic number increases from Z=8 (O isotopes) to Z=82 (Pb isotopes) the two curves which correspond to the isospin dependent $\hbar\omega$ (Pomorski and Sym. Fermi, respectively) for small Z are close to the asymptotic expression based on the Gambhir-Patil distribution and differ a lot from the Moszkowski formula. For large Z they are closer to the Moszkowski formula, although the difference of the two asymptotic expressions is less than 0.5 MeV in the case of Pb nuclei.

The effect on $\hbar\omega$ of the diffuseness of the nuclear surface is also seen in the figures. It is observed that the curve corresponding to the SF distribution is always somewhat lower than the curve corresponding to the uniform distribution. This difference ranges from about 1 MeV for Z=8 to about 0.3 MeV for Z=82. This is due to the fact that a diffuse distribution leads to higher values for the MS radii and consequently to smaller values of $\hbar\omega$. As is expected, this difference is larger for light nuclei where the surface effects are more important.

IV. ASYMPTOTIC FORMULAS FOR $\hbar\omega$

The dependence of $\hbar\omega$ on N, Z and A can be shown explicitly by finding the following asymptotic expression of (7):

$$\hbar\omega = 38.6A^{-1/3} - 127.0A^{-4/3} + 14.75A^{-4/3}(N - Z) \ . \tag{15}$$

Another asymptotic expression can be found employing (14), i.e., for the SF distribution:

$$\hbar\omega = 40.0A^{-1/3} - 56.0A^{-1} - 208.8A^{-5/3} + 68.8A^{-5/3}(N - Z). \tag{16}$$

In Table I we compare the exact and asymptotic expressions obtained from the uniform nucleon density distribution [relations (7) and (15)] and the SF distribution [relations (14) and (16)]. The comparison is made for

TABLE I. The values of $\hbar\omega$ calculated with the exact and asymptotic expressions using a uniform nucleon density distribution (columns 3,4) and a symmetrized Fermi density distribution (columns 5,6). It is seen that the asymptotic expressions are quite accurate and thus they can be used in practice.

A	N-Z	Exact (7)	Asympt. (15)	Exact (14)	Asympt. (16)
^{18}O	2	12.86	12.66	12.06	11.58
^{44}Ca	4	10.51	10.50	10.22	10.18
^{68}Ni	12	9.64	9.64	9.44	9.52
^{106}Zr	26	8.69	8.67	8.51	8.59
^{132}Sn	32	8.12	8.09	7.96	8.01
^{208}Pb	44	6.96	6.94	6.85	6.87

some nuclei increasing the neutron excess. It is observed that the asymptotic expressions are very accurate and can be used in practice.

V. DETERMINATION OF $\hbar\omega$ TAKING INTO ACCOUNT CORRECTIONS AND THE VALENCE NUCLEONS

The average harmonic oscillator shell model square radius for nucleons may be written [7]

$$\overline{\langle r^2\rangle}_{(K+n)} = \frac{\hbar}{m\omega}\frac{4\sum_{p=1}^{K}(p+1/2)N(p)+(K+3/2)n}{4\sum_{p=1}^{K}N(p)+n}\,, \tag{17}$$

where n is the number of valence nucleons and K the number of the highest filled shell. It is found that K satisfies the equation

$$\frac{2}{3}K(K+1)(K+2)+n=A\,. \tag{18}$$

Using (17) and taking into account the corrections due to the center of mass and to the proton and neutron finite size effects, we obtain

$$\hbar\omega = \frac{3}{4}\frac{\hbar^2}{mA}\frac{[(K+1)(A+\frac{1}{3}n)+\frac{2}{3}n-2]}{[\langle r^2\rangle-(\langle r^2\rangle_p+\langle r^2\rangle_n)]}\,, \tag{19}$$

where $(\langle r^2\rangle_p+\langle r^2\rangle_n)\simeq 0.659\ \mathrm{fm}^2$.

However, in the present paper we take into account an additional isospin dependence in the numerator of (17), i.e., the sum over nucleons is replaced by a sum over protons and neutrons separately. Expression (17) is modified as follows:

$$\overline{\langle r^2\rangle} = \frac{\hbar}{m\omega}\frac{2\sum_{p=1}^{K_n}(p+1/2)N(p)+2\sum_{p=1}^{K_p}(p+1/2)N(p)+(K_n+3/2)n_n+(K_p+3/2)n_p}{A}\,, \tag{20}$$

where K_n (K_p) is the number of the highest filled shell of neutrons (protons) and n_n (n_p) is the number of valence neutrons (protons).

Performing the summations, we obtain

$$\overline{\langle r^2\rangle} = \frac{\hbar}{m\omega}\frac{(K_n+1)(3N+n_n)+2n_n+(K_p+1)(3Z+n_p)+2n_p}{4A}\,. \tag{21}$$

Note that instead of (18) the following relations hold:

$$\begin{aligned}\tfrac{1}{3}K_n(K_n+1)(K_n+2)+n_n&=N\,,\\ \tfrac{1}{3}K_p(K_p+1)(K_p+2)+n_p&=Z\,.\end{aligned} \tag{22}$$

Using (20), and taking into account the corrections referred to above, we find

$$\hbar\omega = \frac{3}{4}\frac{\hbar^2}{mA}\frac{(K_n+1)(N+\frac{1}{3}n_n)+\frac{2}{3}n_n+(K_p+1)(Z+\frac{1}{3}n_p)+\frac{2}{3}n_p-2}{[\langle r^2\rangle-(\langle r^2\rangle_p+\frac{N}{Z}\langle r^2\rangle_n)]}\,. \tag{23}$$

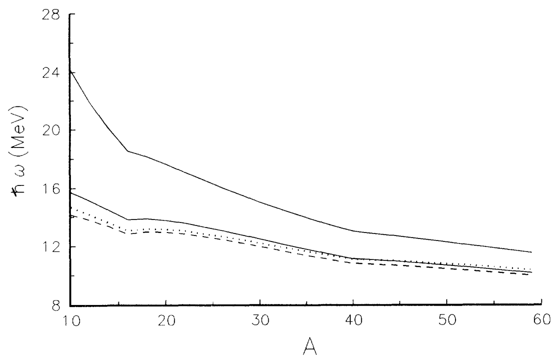

FIG. 3. Oscillator spacing $\hbar\omega$ as a function of A taking into account corrections and the valence nucleons, for four cases: (1) with the simple liquid drop formula (upper solid curve), (2) with expression (6) (lower solid curve), (3) using the SF distribution (short dashed curve) and (4) using the Fermi distribution (dotted curve).

Next we calculate numerically $\hbar\omega$ as a function of N and Z using as input in (23) the MS radii corresponding to the following four cases: (i) the simple formula $R_{00} = 1.2A^{1/3}$, (ii) expression (6), (iii) the SF distribution with parameters determined in Sec. III, and (iv) the Fermi distribution with parameters from [7].

In Fig. 3 we plot for the special case N=Z the corresponding curves of $\hbar\omega$ as a function of A for the four cases mentioned above. It is seen in Fig. 3 that the old formula [case (i)] gives again a curve which is higher than the other cases (ii)–(iv).

The isospin dependence of $\hbar\omega$ can be seen in Fig. 4(a), where we plot $\hbar\omega = f(N)$ for various values of Z $(8 \leq Z \leq 20)$, i.e., for various isotopes, calculated numerically from (23) using as input the mean square radius corresponding to the formula (6). In Fig. 4(b) we plot the corresponding values obtained with the SF distribution [relations (12) and (13)]. Shell effects (i.e., a discontinuity in the slope of the curve) are observed at the closed shells with N=8 and N=20. In Fig. 4(c) we compare the two cases [i.e., $\hbar\omega$ calculated using relations (6) and (12), respectively] for three nuclei increasing the neutron excess. It is seen that, in accord with Figs. 2(a)–2(d) and the comments made above, the curve corresponding to SF distribution lies lower than the curve obtained with the uniform distribution. It is also seen that an increase of the atomic number Z results to a decrease of the difference of the two curves.

VI. CONCLUSIONS

In the present paper we exploit the extra input provided by very recent and accurate experimental data for the isotopic shifts in order to obtain expressions for the MS radius of nuclei as functions of N and Z. These expressions allow us to propose formulas for $\hbar\omega$ using a uniform distribution from [14] and the symmetrized Fermi density distribution. Thus we are able to study the effect on $\hbar\omega$ of neutron excess and the diffuseness of the nuclear surface, as well as the variation of $\hbar\omega$ with N.

Our study has shown the following.

(i) As seen from Figs. 2(a)–2(d) and the comments made in Sec. III, the isospin dependence of $\hbar\omega$ is important for relatively light and medium heavy nuclei. For very heavy nuclei, all the formulas examined in the present paper give practically similar results.

(ii) Shell effects, i.e., discontinuities in the slope of the curve of $\hbar\omega$ as a function of N [Figs. 4(a)–4(c)] are observed at the closed shells (N=8 and N=20).

(iii) We derive very accurate approximate asymptotic formulas for $\hbar\omega$ as functions of N and Z, which can be used in practice.

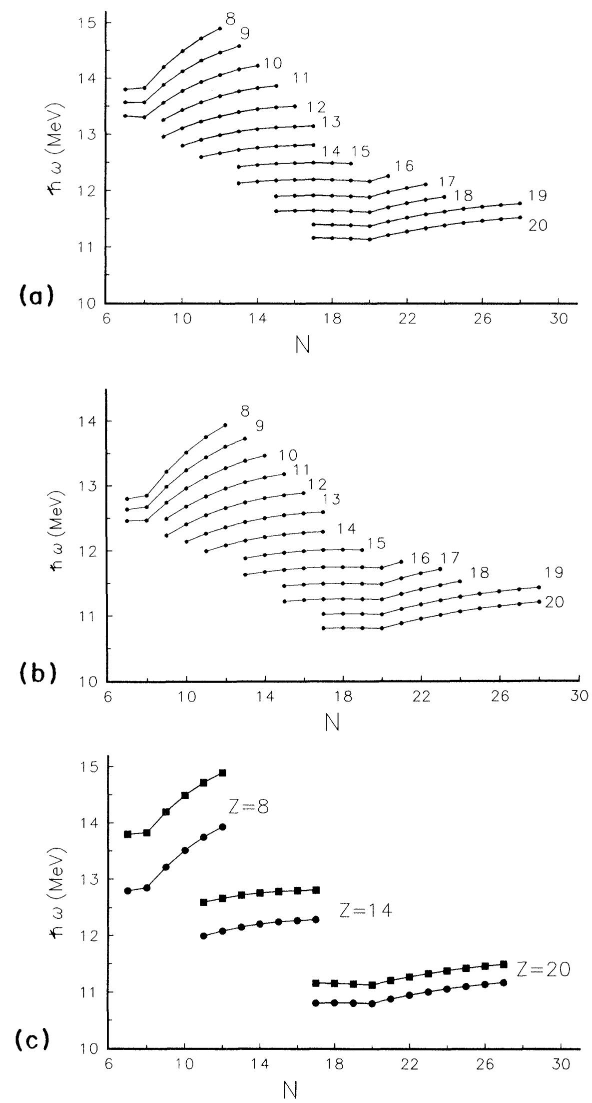

FIG. 4. (a) The variation of the oscillator spacing $\hbar\omega$ as a function of N for various isotopic chains. The values next to each curve denote the atomic number Z. For the calculation formula (23) is used with the MS radius of (6). All the corrections are included. (b) The same as in Fig. 4(a) but with the MS radius from the SF distribution [relations (12) and (13)]. (c) Comparison of $\hbar\omega$ as a function of N for two cases: The solid boxes correspond to the uniform distribution from Pomorski [Fig. 4(a)] and the solid circles to the SF distribution [Fig. 4(b)]. The difference of the curves decreases as Z increases.

(iv) The effect of the nuclear surface on $\hbar\omega$, which is studied by comparing the results using the uniform distribution of Pomorski with those coming from the SF density distribution, is that the distribution with a surface has a larger radius compared with the radius of a uniform distribution and consequently the values of $\hbar\omega$ for the SF distribution are lower than the corresponding values obtained with a uniform distribution. The difference ranges from 1 MeV for light nuclei to 0.3 MeV for heavy nuclei.

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