Multiparticle scattering theory and inclusive cross sections in nuclear collisions

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The dynamics of inclusive breakup processes is studied in the framework of an exact multichannel multiparticle scattering theory. Due to the proper physical description cross sections are uniquely defined in terms of multiparticle transition operators without the problem of post-prior asymmetry. Unlike previous treatments, the present work includes Coulomb and exchange effects. In order to derive tractable expressions certain mathematical approximations have to be invoked. These, and the corresponding physical approximations, are discussed in detail.

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I. INTRODUCTION

Inclusive reactions until recently played only a minor role in low and intermediate energy nuclear reaction studies. Udagawa and Tamura [1] found, however, that massive transfer reactions involving heavy ions can be successfully described in terms of a two-step breakup fusion mechanism. In these processes the first step proceeds very rapidly by a direct mechanism, so that the ejectile can be reasonably assumed to have been part of the projectile. The second step of the reaction proceeds much more slowly, and the various decay modes of the highly excited residual system are irrelevant from the point of view of the primary ejected particle. Thus, in order to account for the angular and energy distribution of the observed ejectile, a special type of process, inclusive breakup, has to be considered.

Following the initiative by Udagawa and Tamura [1] a variety of theories [2–5] were proposed for the reaction

$$a + A \to b + \text{ anything}, \tag{1.1}$$

where the ejectile b is a definite fragment of projectile a. The various formulations yielded different results, however, producing considerable confusion [6,7] over the dynamical treatment of the basic process. Even though Ichimura [8] later tried systematically to describe relations between the proposed formulations, no proper breakup fusion cross section has yet been derived in an unambiguous way.

The aim of the present work is to use the methods of exact multiparticle scattering theory to clarify the basic theoretical situation and obtain from first principles an expression for the desired inclusive cross section. Our formulation includes Coulomb interactions and the antisymmetrization required by the Pauli principle, which were not treated by previous authors. It is shown that the very notion of an inclusive breakup mechanism is in itself an approximation that precludes particle exchange effects between the ejectile and the residual system. In addition it is shown that even the definition of a differential inclusive cross section, which is central to previous treatments, depends on neglecting certain polarization effects.

In particular, it turns out that the multitude of channels emerging from the decay of the residual system can be treated in a unified way in terms of an absorptive effective potential only if that system is described in terms of two basic clusters. This realization leads naturally to a three-body model of the collision process, as was pointed out by Austern and his co-workers [5]. The reaction mechanisms used to describe the final stage of the reaction are logically independent, however, from those used in the treatment of the initial (direct) stage. This fact was not exploited by Austern *et al.* but is essential if exchange effects are to be included in a fundamental way.

In Sec. II the necessary notations and facts of nonrelativistic multichannel scattering theory are summarized in a way that accommodates Coulomb and exchange effects. In Sec. III an exact general expression is derived for the total inclusive cross section of the reaction of Eq. (1.1). It is pointed out that a similar exact formula cannot be derived for the differential inclusive cross section if the ejectile b in Eq. (1.1) has nonzero electric charge. In Sec. IV it is shown how a sequence of approximations leads to the formulas of previous authors [1–8]. Concluding remarks are presented in Sec. V.

II. NOTATION AND DEFINITIONS

In this section the standard notation and a number of basic facts about the nonrelativistic scattering theory of systems of identical charged particles are collected for the reader's convenience. For details one is referred to a previous paper [9], the presentation of which we closely follow.

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A. Scattering operators for distinguishable particles

Consider a system of N distinguishable charged particles. Let $\mathcal{H}_N$ denote the Hilbert space of vectors that represent states of the system, and let H_N denote the Hamiltonian operator that determines the temporal evolution of those vectors. Let $P_{\alpha m}$ denote the orthogonal projection operators that project $\mathcal{H}_N$ onto the subspaces $\mathcal{H}_{\alpha m} \equiv P_{\alpha m}\mathcal{H}_N$ of vectors that represent the various asymptotic states of the system, and let H_α denote the Hamiltonian operator that determines the (asymptotic) temporal evolution of vectors in $\mathcal{H}_{\alpha m}$. Here α denotes the partition of the index set $\{1, 2, \ldots, N\}$ that represents the division of the particles into different bound clusters and m denotes the collection of quantum numbers that specify the quantum states of those clusters. The scattering processes of the system are governed by the wave operators $\Omega^{(\pm)}_{\alpha m} : \mathcal{H}_{\alpha m} \to \mathcal{H}_N$,

$$\Omega^{(\pm)}_{\alpha m} \equiv s - \lim_{t \to \pm\infty} e^{iH_N t} D_\alpha(t) e^{-iH_\alpha t} P_{\alpha m} , \tag{2.1}$$

where $D_\alpha(t)$ is an operator necessitated by the long range of the Coulomb interaction [10]. The scattering operators $S_{\beta n:\alpha m} : \mathcal{H}_{\alpha m} \to \mathcal{H}_{\beta n}$ for the system are defined by

$$S_{\beta n:\alpha m} \equiv \Omega^{(+)*}_{\beta n} \Omega^{(-)}_{\alpha m} , \tag{2.2}$$

where $\Omega^{(+)*}_{\beta n}$ is the adjoint of $\Omega^{(+)}_{\beta n}$.

A property of N-body systems that is particularly prominent in the present considerations is that of asymptotic completeness. If I_N denotes the identity operator on $\mathcal{H}_N$, and if P_{bound} denotes the projection of $\mathcal{H}_H$ onto the space spanned by the bound states of the Hamiltonian H_N, then the equation that expresses asymptotic completeness is [11]

$$I_N = P_{\text{bound}} + \sum_{\alpha m} \Omega^{(\pm)}_{\alpha m} \Omega^{(\pm)*}_{\alpha m} . \tag{2.3}$$

The sum extends over all partitions α with at least two clusters. Equation (2.3) holds separately for the + and the − wave operators.

B. Permutations

When some of the particles are indistinguishable, the group $\mathcal{S}$ of permutations of those particles is a symmetry group of the system. The permutations p of $\mathcal{S}$ are represented on $\mathcal{H}_N$ by linear unitary operators that will be denoted by the same symbol, p, with the context specifying which meaning is intended. In order to treat bosons and fermions in a unified manner, it is convenient to define operators $\widehat{p} : \mathcal{H}_N \to \mathcal{H}_N$ by

$$\widehat{p} \equiv f_p \, p , \tag{2.4}$$

where f_p is -1 if p involves an odd number of fermion permutations and is $+1$ otherwise. All the operators $\widehat{p}$ commute with the Hamiltonian H_N of the system.

The operators that represent the permutations of $\mathcal{S}$ form an algebra. In particular, the Young operator $Y^{\mathcal{S}} : \mathcal{H}_N \to \mathcal{H}_N$ defined by the idempotent element of the group algebra

$$Y^{\mathcal{S}} \equiv |\mathcal{S}|^{-1} \sum_{p \in \mathcal{S}} \widehat{p} , \tag{2.5}$$

where $|\mathcal{S}|$ denotes the number of elements of $\mathcal{S}$, is an orthogonal projection of $\mathcal{H}_N$ onto the subspace of states with the correct symmetries under interchange of indistinguishable particles. The Young operator has also the important "absorption" property that $\widehat{p} Y^{\mathcal{S}} = Y^{\mathcal{S}}$ for every $p \in \mathcal{S}$.

While a permutation $p \in \mathcal{S}$ will in general change a partition α into a different partition, permutations that interchange indistinguishable particles inside a given cluster of α leave α unchanged. For any given partition α the set of permutations

$$\mathcal{S}_\alpha \equiv \{p : p\alpha = \alpha, \; p \in \mathcal{S}\} \tag{2.6}$$

is a subgroup of $\mathcal{S}$. All the operators $\widehat{p}$, $p \in \mathcal{S}_\alpha$, commute with both H_α and $D_\alpha(t)$.

The operators that represent the permutations of $\mathcal{S}_\alpha$ form a subalgebra of the algebra associated with $\mathcal{S}$. The operator $Y^{\mathcal{S}_\alpha} : \mathcal{H}_N \to \mathcal{H}_N$ defined by

$$Y^{\mathcal{S}_\alpha} \equiv |\mathcal{S}_\alpha|^{-1} \sum_{p \in \mathcal{S}_\alpha} \widehat{p} \tag{2.7}$$

is trivially the Young operator of that subalgebra (though not of the full algebra). Consequently, it is an orthogonal projection that projects the space $\mathcal{H}_{\alpha m}$ onto the subspace of asymptotic states in which the bound states of the clusters have correct symmetries with respect to permutations of indistinguishable particles. Of course, by construction it also has the "absorption" property $\widehat{p} Y^{\mathcal{S}_\alpha} = Y^{\mathcal{S}_\alpha}$ for every $p \in \mathcal{S}_\alpha$.

Because a permutation of identical particles does not change any physical property of the system, any two partitions α and α' are physically equivalent if $\alpha' = p\alpha$, $p \in \mathcal{S}$. This property leads in a natural way to an equivalence relation which divides the set of all partitions into equivalence classes

$$[\alpha] \equiv \{\alpha' : \alpha' = p\alpha, \; p \in \mathcal{S}\} . \tag{2.8}$$

The number of distinct elements of $[\alpha]$ is given by

$$N_{[\alpha]} = |\mathcal{S}|/|\mathcal{S}_\alpha| . \tag{2.9}$$

The group properties of $\mathcal{S}$ imply that $[\alpha'] = [\alpha]$ for all $\alpha' \in [\alpha]$.

For any given partition α there are two extremely useful factorization formulas:

$$Y^{\mathcal{S}} = N^{-1}_{[\alpha]} \sum_{\alpha' \in [\alpha]} \widehat{p}_{\alpha'\alpha} Y^{\mathcal{S}_\alpha} = N^{-1}_{[\alpha]} \sum_{\alpha' \in [\alpha]} Y^{\mathcal{S}_\alpha} \widehat{p}_{\alpha\alpha'} . \tag{2.10}$$

Here $p_{\alpha'\alpha}$ denotes the permutation that maps α into α',

$p_{\alpha\alpha'}$ denotes the permutation that maps α' into α, and the sums are over the $N_{[\alpha]}$ distinct partitions in $[\alpha]$.

C. Scattering operators for indistinguishable particles

If some of the particles are indistinguishable the definitions of the scattering operators must be changed to reflect that fact [10]. The correctly symmetrized wave operator $\Omega^{\mathcal{S}(\pm)}_{[\alpha^0]m} : Y^{\mathcal{S}_{\alpha^0}}\mathcal{H}_{\alpha^0 m} \to Y^{\mathcal{S}}\mathcal{H}_N$ for the equivalence class $[\alpha^0]$ is defined by

$$\Omega^{\mathcal{S}(\pm)}_{[\alpha^0]m} \equiv N^{1/2}_{[\alpha^0]}\, Y^{\mathcal{S}}\, \Omega^{(\pm)}_{\alpha^0 m}\,, \tag{2.11}$$

where $Y^{\mathcal{S}}$ and $\Omega^{(\pm)}_{\alpha^0 m}$ are defined by Eqs. (2.5) and (2.1). The correctly symmetrized scattering operator $S^{\mathcal{S}}_{[\beta^0]n:[\alpha^0]m} : Y^{\mathcal{S}_{\alpha^0}}\mathcal{H}_{\alpha^0 m} \to Y^{\mathcal{S}_{\beta^0}}\mathcal{H}_{\beta^0 n}$ is defined by

$$S^{\mathcal{S}}_{[\beta^0]n:[\alpha^0]m} \equiv \Omega^{\mathcal{S}(+)*}_{[\beta^0]n}\, \Omega^{\mathcal{S}(-)}_{[\alpha^0]m}\,. \tag{2.12}$$

If $E^{\mathcal{S}(\pm)} : Y^{\mathcal{S}}\mathcal{H}_N \to Y^{\mathcal{S}}\mathcal{H}_N$ is defined by

$$E^{\mathcal{S}} \equiv \sum_{\alpha^0 m} \Omega^{\mathcal{S}(\pm)}_{[\alpha^0]m}\, \Omega^{\mathcal{S}(\pm)*}_{[\alpha^0]m}\,, \tag{2.13}$$

it follows directly from Eq. (2.11) that

$$E^{\mathcal{S}} = Y^{\mathcal{S}} \left(\sum_{\alpha^0 m} N_{[\alpha^0]}\, \Omega^{(\pm)}_{\alpha^0 m}\, \Omega^{(\pm)*}_{\alpha^0 m} \right) Y^{\mathcal{S}}\,. \tag{2.14}$$

On the other hand, by using the factorization formulas for $Y^{\mathcal{S}}$, the property that $\widehat{p}\,Y^{\mathcal{S}} = Y^{\mathcal{S}}$ for all $p \in \mathcal{S}$, and the label transforming property $p\,\Omega^{(\pm)}_{\alpha m} = \Omega^{(\pm)}_{p\alpha\, m}\,p$, one obtains

$$E^{\mathcal{S}} = \sum_{\alpha^0 m} \sum_{\alpha \in [\alpha^0]} \Omega^{(\pm)}_{\alpha m}\, \Omega^{(\pm)*}_{\alpha m}\, \widehat{p}_{\alpha\alpha^0} Y^{\mathcal{S}_{\alpha^0}} Y^{\mathcal{S}} \tag{2.15}$$

$$= \sum_{\alpha^0 m} \sum_{\alpha \in [\alpha^0]} \Omega^{(\pm)}_{\alpha m}\, \Omega^{(\pm)*}_{\alpha m}\, Y^{\mathcal{S}}\,. \tag{2.16}$$

Similarly,

$$E^{\mathcal{S}} = Y^{\mathcal{S}} \sum_{\alpha^0 m} \sum_{\alpha \in [\alpha^0]} \Omega^{(\pm)}_{\alpha m}\, \Omega^{(\pm)*}_{\alpha m}\,. \tag{2.17}$$

It now follows from Eq. (2.3) that the equation expressing asymptotic completeness for systems of identical particles is

$$Y^{\mathcal{S}} = Y^{\mathcal{S}} P_{\text{bound}} Y^{\mathcal{S}} + E^{\mathcal{S}}\,. \tag{2.18}$$

III. INCLUSIVE CROSS SECTION

In this section a fundamental formula for the total inclusive cross section is derived. It is exact within the framework of a multiparticle scattering theory that includes both Coulomb and exchange effects.

The notation for the representative initial states is as follows. Let α^0 denote a two-cluster partition representative of the initial state of the reaction, and let m denote the quantum numbers of the two bound states. Let the product of kets of the two bound states and of a plane wave for their free relative motion (with relative momentum $\mathbf{k}$) be denoted by $|\alpha^0 m\mathbf{k}\rangle$. Although only properly normalized linear superpositions of these plane-wave states can belong the Hilbert space $\mathcal{H}_\alpha$, for the purposes of this paper mathematical rigor can be relaxed and plane-wave states used to represent the asymptotic states of the systems.

Similarly, let γ^0 denote a partition representative of a final state of the reaction, and let n denote the quantum numbers of the bound states of the clusters of γ^0. The plane-wave states that will represent the asymptotic states with this clustering are denoted by $|\gamma^0 n\mathbf{q}_{\gamma^0}\rangle$, where $\mathbf{q}_{\gamma^0}$ denotes the Jacobi momenta of the free relative motion of the bound clusters.

It is now useful to introduce certain auxiliary wave operators. Let $\beta^0 = (b, B)$ denote a two-cluster partition in which b is a representative cluster for the particle b of Eq. (1.1). Imagine a fictitious particle with the total mass and charge of the cluster b located at the center of mass of that cluster. Define the potential $\bar{V}_{\beta^0}$ to be an effective interaction between that fictitious particle and the individual particles of cluster B. The (distinguishable particle) wave operators $\omega^{(\pm)}_{\beta^0:\gamma n} : \mathcal{H}_{\gamma n} \to \mathcal{H}_N$ that represent the distorted waves are then defined by

$$\omega^{(\pm)}_{\beta^0:\gamma n} \equiv s - \lim_{t\to\pm\infty} e^{\,i[H_{\beta^0}+\bar{V}_{\beta^0}]t}\, D_\gamma(t)\, e^{\,-iH_\gamma t}\, P_{\gamma n}\,, \qquad \gamma \subseteq \beta^0\,. \tag{3.1}$$

The notation $\gamma \subseteq \beta^0$ signifies that the partition γ is either equal to β^0 or can be obtained from β^0 by breaking apart the cluster B into subclusters.

There is no flexibility in the way that Coulomb interactions are included in $\bar{V}_{\beta^0}$. They must be included as the sum of Coulomb interactions between the fictitious particle and each of the individual particles of the cluster B. This follows from the requirement the structure of $\bar{V}_{\beta^0}$ be consistent with the structure of the operators $D_\gamma(t)$ for all $\gamma \subseteq \beta^0$. Any other way of including the Coulomb potentials would, in particular, be inconsistent with $D_\gamma(t)$ for the partition in which each particle of B is assigned to a separate cluster. This fact has an interesting consequence which is discussed at the end of this section.

There is, however, some freedom in choosing the effective nuclear interactions included in $\bar{V}_{\beta^0}$. It is only important that they cannot overcome Coulomb repulsion to create a bound state of the particles in B and the fictitious particle. In other words, it is necessary that $\bar{V}_{\beta^0}$ be chosen so that the Hamiltonian $H_{\beta^0} + \bar{V}_{\beta^0}$ has no discrete spectrum (proper eigenvalues) but only continuous spectrum.

Finally, the interchange of indistinguishable particles in the cluster B must leave $\bar{V}_{\beta^0}$ invariant:

$$p\,\bar{V}_{\beta^0} = \bar{V}_{\beta^0}\,p \quad \text{for all } p \in \mathcal{S}_{\beta^0}\,. \tag{3.2}$$

The corresponding modified transition operators $T_{[\gamma^0]n:[\alpha^0]m} : \mathcal{H}_{\alpha^0 m} \to \mathcal{H}_{\gamma^0 n}$ are given by

$$T_{[\gamma^0]n:[\alpha^0]m} \equiv N^{1/2}_{[\gamma_0]}\, \omega^{(+)*}_{\beta^0:\gamma^0 m} \left(H_N - H_{\beta^0} - \bar{V}_{\beta^0}\right) \Omega^{\mathcal{S}(-)}_{[\alpha_0]m}\,. \tag{3.3}$$

The γ^0 are representatives of the equivalence classes (generated by permutations in $\mathcal{S}_{\beta^0}$) of partitions $\gamma \subseteq \beta^0$.

The total inclusive cross section $\sigma^{\rm inc}$ is now given by the following formula:

$$\sigma^{\rm inc} = \frac{(2\pi)^4 \mu_{\alpha^0}}{|\mathbf{k}|} \sum_{\gamma^0 n''} \int d\mathbf{q}_{\gamma^0}\, \delta(E_{\alpha^0 m} - E_{\gamma^0 n}) \left| \langle \gamma^0 n\, \mathbf{q}_{\gamma^0} | T_{[\gamma^0]n:[\alpha^0]m} | \alpha^0 m \mathbf{k} \rangle \right|^2 , \tag{3.4}$$

where μ_{α^0} denotes the reduced mass of the two initial bound clusters and $n = (n_b, n'')$ with n_b denoting the quantum numbers of the detected particle b. Equation (3.4) can be rewritten with the aid of standard manipulations [8]:

$$\sigma^{\rm inc} = \lim_{\epsilon \to 0^+} {\rm Im}\, \left\{ \left[-(2\pi)^4 \mu_{\alpha^0} / \pi |\mathbf{k}| \right] I(\epsilon) \right\} , \tag{3.5}$$

with

$$I(\epsilon) = \langle \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} | \, \widetilde{G}_{\beta^0}(E_{\alpha^0} + i\epsilon)\, F_{\beta^0} \, | \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} \rangle , \tag{3.6}$$

where

$$F_{\beta^0} \equiv \sum_{\gamma^0 n''} \int d\mathbf{q}_{\gamma^0}\, N_{[\gamma^0]}\, \omega^{(+)}_{\beta^0:\gamma^0 n} \, |\gamma^0 n\, \mathbf{q}_{\gamma^0}\rangle\langle \gamma^0 n\, \mathbf{q}_{\gamma^0}| \omega^{(+)*}_{\beta^0:\gamma^0 n} \tag{3.7}$$

$$= \sum_{\gamma^0 n''} N_{[\gamma^0]}\, \omega^{(+)}_{\beta^0:\gamma^0 n}\, \omega^{(+)*}_{\beta^0:\gamma^0 n} . \tag{3.8}$$

In Eq. (3.6) the definitions

$$|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle \equiv (H_N - H_{\beta^0} - \bar{V}_{\beta^0})\, \Omega^{\mathcal{S}(-)}_{[\alpha^0]m} \, |\alpha^0 m\mathbf{k}\rangle \tag{3.9}$$

and $\widetilde{G}_{\beta^0}(z) \equiv (z - H_{\beta^0} - \bar{V}_{\beta^0})^{-1}$ have been introduced.

Because H_{β^0} is a sum of the internal Hamiltonian for the cluster b and a Hamiltonian that refers only to variables external to the cluster, and because $\bar{V}_{\beta_0}$ is independent of the internal coordinates of the cluster b, the wave operators $\omega^{(+)}_{\beta^0:\gamma^0 n}$ have the form

$$\omega^{(+)}_{\beta^0:\gamma^0 n} = \Pi_b \otimes \widetilde{\omega}^{(+)}_{\beta^0:\gamma^0 n''} . \tag{3.10}$$

The operator Π_b acts on the Hilbert space $\mathcal{H}_b$ of functions of variables internal to the cluster b, and on that space it is a projection operator onto the bound states of b with the quantum numbers n_b. The operators $\widetilde{\omega}^{(+)}_{\beta^0:\gamma^0 n''}$ act on the Hilbert space $\mathcal{H}_b^{\perp}$ of functions of variables external to the cluster b, and on that space they are the wave operators for a system consisting of the particles in the cluster B and the fictitious particle with the total mass and charge of cluster b located at the center of mass of that cluster. Since it is assumed that $\bar{V}_{\beta^0}$ is chosen so that $H_{\beta^0} + \bar{V}_{\beta^0}$ has no discrete spectrum, the equation expressing asymptotic completeness is

$$Y^{\mathcal{S}_{\beta^0}} (\Pi_b \otimes I_b^{\perp})\, Y^{\mathcal{S}_{\beta^0}} = Y^{\mathcal{S}_{\beta^0}}\, F_{\beta^0}\, Y^{\mathcal{S}_{\beta^0}} , \tag{3.11}$$

where $I_b^{\perp}$ is the identity operator on $\mathcal{H}_b^{\perp}$.

Because the wave operator $\Omega^{\mathcal{S}(-)}_{[\alpha^0]m}$ has the proper symmetry under the interchange of identical particles, and $Y^{\mathcal{S}_{\beta^0}}$ commutes with both H_N and $H_{\beta^0} + \bar{V}_{\beta^0}$, then

$$Y^{\mathcal{S}_{\beta^0}} \, |\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle = |\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle . \tag{3.12}$$

Consequently, with the aid of Eqs. (3.10) and (3.11), Eq. (3.6) can be rewritten in the form

$$\begin{aligned} I(\epsilon) &= \langle \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} | \, \widetilde{G}_{\beta^0}(E_{\alpha^0} + i\epsilon)\, Y^{\mathcal{S}_{\beta^0}}\, F_{\beta^0}\, Y^{\mathcal{S}_{\beta^0}} \, | \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} \rangle \\ &= \langle \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} | \, \widetilde{G}_{\beta^0}(E_{\alpha^0} + i\epsilon)^{-1}\, Y^{\mathcal{S}_{\beta^0}} \, \left(\Pi_b \otimes I_b^{\perp}\right) Y^{\mathcal{S}_{\beta^0}} \, | \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} \rangle \\ &= \langle \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} | \, \widetilde{G}_{\beta^0}(E_{\alpha^0} + i\epsilon)\, (\Pi_b \otimes I_b^{\perp}) \, | \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} \rangle . \end{aligned} \tag{3.13}$$

A formal, but instructive, simplification of the states $|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle$ in Eq. (3.13) is obtained with the aid of the factorization formulas:

$$|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle = N^{-1/2}_{[\alpha^0]} \, \left(H_N - H_{\beta^0} - \bar{V}_{\beta^0}\right) \sum_{\alpha \in [\alpha^0]} \Omega^{(-)}_{\alpha m} |\alpha m\mathbf{k}\rangle , \tag{3.14}$$

where

$$\Omega^{(-)}_{\alpha m} |\alpha m\mathbf{k}\rangle \equiv \widehat{p}_{\alpha\alpha^0}\, \Omega^{(-)}_{\alpha^0 m} |\alpha^0 m\mathbf{k}\rangle . \tag{3.15}$$

Here it has been assumed that $Y^{\mathcal{S}_{\alpha^0}}\Omega^{(-)}_{\alpha^0 m}|\alpha^0 m\mathbf{k}\rangle = \Omega^{(-)}_{\alpha^0 m}|\alpha^0 m\mathbf{k}\rangle$. It is evident from Eq. (3.14) that states corresponding to all distinct physically equivalent initial partitions appear on an equal footing.

Finally, the desired exact formula for the inclusive cross section is provided by combining Eqs. (3.5) and (3.13):

$$\sigma^{\rm inc} = \lim_{\epsilon\to 0^+}\left[-(2\pi)^4\mu_{\alpha^0}/\pi|\mathbf{k}|\right]\langle\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}|\,\widetilde{G}_{\beta^0}(E_{\alpha^0}+i\epsilon)\,(\Pi_b\otimes I_b^{\perp})\,|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle \tag{3.16}$$

with the states $|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle$ being given by Eq. (3.14).

It is important to remember that this derivation yields a total, not a differential, inclusive cross section. To derive a formula for a differential inclusive cross section one would begin with a version of Eq. (3.4) in which the relative momentum of clusters b and B would be held fixed rather than being an integration variable. The present derivation then fails at the step leading from Eq. (3.7) to Eq. (3.8). To proceed further it is then necessary for the wave operators $\widetilde{\omega}^{(+)}_{\beta^0:\gamma^0 n''}$ to be a product of a wave operator in the internal variables of cluster B and a wave operator for the motion of b relative to B. If the cluster b has zero charge, such a factorization occurs if $\bar{V}_{\beta^0}$ is chosen to be independent of the internal coordinates of the cluster B, and an exact expression for the differential cross section is obtained. If cluster b has nonzero charge, the required factorization is not compatible with the form of the Coulomb part of $\bar{V}_{\beta^0}$ which, as mentioned earlier, is determined by internal consistency requirements. In this case the spectator model approximation discussed in the next section must be introduced in order to obtain a useful expression for a differential cross section.

IV. APPROXIMATIONS

In the following a series of approximations are introduced to show how to obtain formulas that can be compared with those of previous authors, all of whom confine themselves to the case in which the ejectile b can be considered a fragment of the projectile a.

A. The spectator model assumption

The first major approximation is use of the spectator model [1–8] which assumes that in Eq. (3.13) the effect of the interaction $\bar{V}_{\beta^0}$ in $\widetilde{G}_{\beta^0}$ can be accurately approximated by replacing $\bar{V}_{\beta^0}$ by an interaction U_{bB} that depends only on the relative coordinates between b and the center of mass of the rest of the particles (cluster B). In other words, cluster b moves in a mean field generated by the particles in cluster B. In particular, Coulomb polarization effects are ignored, an approximation equivalent to the "channel distortion approximation" proposed by Bencze more than 20 years ago [12]. In the model it is further assumed that the wave operators

$$\omega^{(+)}_{bB} \equiv \lim_{t\to\infty} e^{\,i(K_{bB}+U_{bB})t}\,D_{bB}(t)\,e^{\,-iK_{bB}t} \tag{4.1}$$

exist and are asymptotically complete. Here K_{bB} denotes the kinetic energy operator for the relative motion of the centers of mass of the clusters b and B and $D_{bB}(t)$ is the operator whose presence is required by the long range of the Coulomb interaction between the clusters b and B [10]. Since $\bar{V}_{\beta^0}$ is chosen so that it cannot support a bound state of the clusters b and B, it is assumed that U_{bB} also supports no bound state. Consequently, the equation expressing asymptotic completeness is $I_{bB} = \omega^{(+)}_{bB}\,\omega^{(+)*}_{bB}$, where I_{bB} is the identity operator for the coordinates of cluster b relative to cluster B. With the help of the intertwining relation $(K_{bB}+U_{bB})\omega^{(+)}_{bB} = \omega^{(+)}_{bB}\,K_{bB}$, the different parts of the spectator model assumption can be combined into the mathematical form

$$\widetilde{G}_{\beta^0}(z) \approx \omega^{(+)}_{bB}\,G_{\beta^0}(z)\,\omega^{(+)*}_{bB}\,, \tag{4.2}$$

where $G_{\beta^0}(z) \equiv (z - H_{\beta^0})^{-1}$.

It is the mathematical structure of Eq. (4.2) that facilitates the definition of differential cross sections later in Sec. IV C.

B. Three-body reaction mechanism

The second major approximation follows from an assumption that the reaction proceeds via a three-body reaction mechanism that allows Eq. (1.1) to be rewritten as

$$a + A = (b,x) + A \to \begin{cases} b + (x,A), \\ b + x + A. \end{cases} \tag{4.3}$$

Like the spectator model assumption, this assumption is common to all previous treatments [1–8] of inclusive cross sections for the reaction of Eq. (1.1).

In order that a mathematical form for this second assumption can be developed, it is supposed, without loss of generality, that the representative partitions $\alpha^0 = (a,A)$ and $\beta^0 = (b,B)$ are chosen so that $a = (b,x)$ and $B = (x,A)$. The three-body reaction mechanism of Eq. (4.3) is then represented by the three-cluster partition $\delta^0 = (b,x,A)$. The projection operators $P_{\delta^0} : \mathcal{H}_N \to \mathcal{H}_N$ and $Q_{\delta^0} : \mathcal{H}_N \to \mathcal{H}_N$ are defined by

$$P_{\delta^0} = P_{\delta^0\bar{n}} \text{ and } Q_{\delta^0} = I_N - P_{\delta^0}\,. \tag{4.4}$$

Here $\bar{n} = (n_b, n_x, n_A)$ with n_b denoting the fixed quantum numbers of the cluster b, as before, and n_x and n_A denoting the quantum numbers that the respective clusters x and A have in their ground states. The approximation then consists of the approximation

$$G_{\beta^0}(z) \approx P_{\delta^0}\,G_{\beta^0}(z)\,P_{\delta^0} = [z - H_{\delta^0}(z)]^{-1}\,P_{\delta^0}\,, \tag{4.5}$$

where the effective Hamiltonian $H_{\delta^0}(z)$ has the familiar Feshbach [13] form,

$$H_{\delta^0}(z) = P_{\delta^0} H_{\beta^0} P_{\delta^0} + P_{\delta^0} H_{\beta^0} Q_{\delta^0} (z - Q_{\delta^0} H_{\beta^0} Q_{\delta^0})^{-1} Q_{\delta^0} H_{\beta^0} P_{\delta^0} . \tag{4.6}$$

Equation (4.5) allows Eq. (3.13) to be rewritten in the form

$$I(\epsilon) \approx \int d^3\mathbf{q} \, \langle \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) | G_{xA} \, | \, \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) \rangle \, , \tag{4.7}$$

where

$$G_{xA} \equiv \left(E_{\alpha^0 m} - \varepsilon_b - [|\mathbf{q}|^2/2\mu_{bB}] + i\epsilon - K_{xA} - U_{xA} \right)^{-1} \tag{4.8}$$

and

$$\langle \mathbf{p} | \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) \rangle \equiv \langle \delta^0 \bar{n} \, \mathbf{q}_{\delta^0} | \omega^{(+)*}_{bB} \, | \Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}} \rangle \Big|_{\mathbf{q}_{\delta^0} = (\mathbf{q},\mathbf{p})} . \tag{4.9}$$

Here $\mathbf{q}$ is the relative momentum and μ_{bB} the reduced mass of the two clusters b and B. The quantity $\mathbf{p}$ is the relative momentum of the clusters x and A, and K_{xA} is the kinetic energy operator for the relative motion of those clusters. The effective pair interaction U_{xA} between clusters x and A is energy dependent and otherwise is a function of the relative coordinates of their centers of mass. The parameter ε_b is the bound state energy of the cluster b.

It is now useful to compute the imaginary part of $I(\epsilon)$. From Eq. (4.7) it follows that

$$\text{Im } I(\epsilon) \approx J(\epsilon) \equiv (2i)^{-1} \int d^3\mathbf{q} \langle \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) | \, G_{xA} - G^*_{xA} \, | \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) \rangle \, . \tag{4.10}$$

The resolvent equation is then used to obtain

$$G_{xA} - G^*_{xA} = -G^*_{xA} \, (2i\epsilon + [U_{xA} - U^*_{xA}]) \, G_{xA} \tag{4.11}$$

and, hence,

$$J(\epsilon) = - \int d^3\mathbf{q} \, \langle \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) | \epsilon + W_{xA} | \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) \rangle \, . \tag{4.12}$$

In this last equation the definition

$$|\Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon)\rangle \equiv G_{xA} | \Upsilon^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}) \rangle \tag{4.13}$$

and the notation $W_{xA} \equiv (2i)^{-1} [U_{xA} - U^*_{xA}]$ have been introduced. The potential W_{xA} is, of course, the absorptive part of the effective two-body interaction U_{xA}.

It is also usual to assume that $H_N - H_{\beta^0} - \bar{V}_{\beta^0} \approx V_{bx}$, where V_{bx} depends only the relative coordinates of the clusters b and x and not on their internal coordinates. Combining this assumption with Eq. (3.14) yields the approximate expression:

$$\langle \mathbf{p} \, | \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) \rangle \approx N^{-1/2}_{[\alpha^0]} \sum_{\alpha \in [\alpha^0]} \langle \delta^0 \bar{n} \mathbf{q}_{\delta^0} \, | \omega^{(+)*}_{bB} \, G_{xA} \, V_{xb} \, \Omega^{(-)}_{\alpha m} | \alpha m \mathbf{k} \rangle \, , \tag{4.14}$$

where $\mathbf{p}_{\delta^0} = (\mathbf{q}, \mathbf{p})$ with $\mathbf{q}$ and $\mathbf{p}$ being the relative momenta of the clusters b and B and of x and A, respectively. It is important that the Hamiltonian $K_{xA} + U_{xA}$ acts only on the relative coordinates of the clusters x and A.

C. Approximate inclusive cross sections

The foregoing discussion implies that the exact expression in Eq. (3.16) for the inclusive cross section can be approximated as an expression involving differential cross sections,

$$\sigma^{\text{inc}} \approx \int d^3\mathbf{q} \, \left[\sigma^{\text{inc,EB}}(\mathbf{q}) + \sigma^{\text{inc,NEB}}(\mathbf{q}) \right] . \tag{4.15}$$

The differential cross section

$$\sigma^{\text{inc,EB}}(\mathbf{q}) = \lim_{\epsilon \to 0^+} \left[(2\pi)^4 \, \mu_{\alpha^0} / |\mathbf{k}| \right] \left[(\epsilon/\pi) \, \langle \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) | \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) \rangle \right] \tag{4.16}$$

is often called the elastic breakup cross section. The differential cross section

$$\sigma^{\text{inc,NEB}}(\mathbf{q}) = \lim_{\epsilon \to 0^+} \left[(2\pi)^4 \, \mu_{\alpha^0} / \pi |\mathbf{k}| \right] \langle \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) | \, W_{xA} \, | \Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q}, \epsilon) \rangle \tag{4.17}$$

is correspondingly the inelastic breakup, or breakup fusion, cross section. It is the expressions for differential cross sections given in Eqs. (4.16) and (4.17), with $\Psi^{\mathcal{S}(-)}_{\delta^0\beta^0:\alpha^0 m\mathbf{k}}(\mathbf{q},\epsilon)$ defined by Eq. (4.14), that are directly comparable to those given by previous authors.

V. CONCLUSIONS

The study of inclusive cross sections is not typical to low and intermediate energy nuclear reactions. Inclusive cross sections involve an incoherent sum of the cross sections of various processes proceeding via different reaction mechanisms. At low energies, due to the relatively small number of open asymptotic reaction channels no feasible statistical assumptions can be made about the contributing processes. A careful theoretical treatment that goes beyond the notion of an inclusive cross section as an incoherent sum is essential.

In this paper the cross sections are carefully defined in terms of on-shell multiparticle transition operators, avoiding any confusion over the basic dynamics. A new feature of the present formulation is the exact treatment of particle identity and of Coulomb interactions. The resulting exact formula for the total inclusive breakup cross section is given by Eq. (3.16) which then serves as the basis for introducing approximations.

Approximations introduced into Eq. (3.16) for the states $|\Phi^{\mathcal{S}(-)}_{\beta^0:\alpha^0 m\mathbf{k}}\rangle$ and for the operator $\widetilde{G}_{\beta^0}$ are logically independent. Indeed, if the Pauli principle is to be taken into account, the ejectile can also be produced by an exchange mechanism, something precluded in a theory in which a single reaction mechanism is assumed for the entire inclusive breakup process. It is an important feature of our treatment, therefore, that the distinction between the approximations is retained.

In order to be able to derive practical formulas certain approximations, enumerated in Sec. IV, have to be introduced for the operator $\widetilde{G}_{\beta^0}$. These approximations are necessarily associated with simplifications in the underlying dynamical model. The spectator model assumption is of particular importance because it is essential to the very definition of the differential cross sections. The assumption of a three-body reaction mechanism, which follows the work of Austern *et al.* [5], is less fundamental, as it is easy to extend the treatment to include mechanisms with a larger number of clusters.

The two basic types of approximations introduced in Sec. IV provide a general framework within which to build dynamical models of the inclusive process. Our final results, given in Eqs (4.15)–(4.17), are the starting point for introducing further approximations which must be considered as less important, and essentially of technical nature. Therefore care must be taken not to let these further approximations confuse the fundamental dynamical picture.

Since the present formulation is based on exact multiparticle scattering theory, we do not investigate the details of previous three-body approaches. We only point out that aside from the DWBA-type picture of Udagawa and Tamura [1], the only previous systematic treatment of the reaction dynamics, if particle identity is ignored, is the that of Austern *et al.* [5] based on a simple three-body model. Our results provide a first step toward more elaborate models.

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