Coulomb plus nuclear scattering in momentum space for coupled angular momentum states

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The Vincent-Phatak procedure for solving the momentum-space Schrödinger equation with combined Coulomb—plus—short-range potentials is extended to angular momentum states coupled by an optical potential—as occurs in spin $1/2\times1/2$ scattering. A generalization of the Blatt–Biedenharn phase shift parametrization is derived and applied to 500 MeV polarized-proton scattering from ^{3}He. The requisite high-precision partial-wave expansions and integrations are described.

I. INTRODUCTION

The theory and equations of quantum mechanics are represented equally well in coordinate or momentum space. Bound-state problems, which by definition deal with normalizable wave functions, can actually be solved equally well in either space, while scattering problems, which in the time-independent Schrödinger theory deal with non-normalizable states, are more challenging in momentum space. This challenge arises, in part, because boundary conditions are more naturally imposed in coordinate space, and, in part, because non-normalizable states contain singularities in momentum space and, accordingly, have no Fourier transforms [1]. In spite of the difficulties, momentum-space calculations are important because momentum space is where one derives the nonlocal potentials of many-body and field theories, and because there are fewer approximations needed in momentum space to handle them.

The Coulomb problem in momentum space has actually been "solved" a number of times—possibly starting with Fock's study of the hydrogen atom [2]—yet no one numerical approach appears to provides the requisite precision for all applications. The real "problem" is that the Coulomb potential between a point projectile (P) and a target (T),

$$V_c(\mathbf{k}',\mathbf{k}) = \frac{Z_P Z_T e^2}{2\pi^2 q^2}\rho(q), \tag{1}$$

has a $1/q^2$ singularity arising from the infinite range of the Coulomb potential, and this singularity must somehow be regularized before a numerical solution is implemented. In (1), $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the difference between the final and initial momenta $\mathbf{k}'$ and $\mathbf{k}$, and $\rho(q)$ is a form factor which accounts for the finite size of the target's charge distribution and makes the potential well behaved at large q (but not at $q = 0$).

Kwon and Tabakin [3] solved the bound-state problem with the potential (1) by using Landé's technique [4] of subtracting a term from (1) which makes its integral finite, and then adding in a correction. Alternatively, Cieplý *et al.* [5] solved the bound-state problem by using a modification of the Vincent-Phatak (VP) procedure [6]. This procedure gives the Coulomb potential a finite range by cutting it off beyond some radius $R_{\rm cut}$, as shown in Fig. 1, and then corrects the asymptotic behavior of the resulting wave functions. If the procedure is successful, the calculated scattering will be independent of $R_{\rm cut}$.

The VP cutoff procedure was originally formulated for intermediate-energy pion scattering from light nuclei [6] where it provided sufficient accuracy for the small number of partial waves involved [7]. However, the accuracy has become a concern for intermediate-energy proton scattering where the proton's much larger mass leads to correspondingly larger momentum transfers and correspondingly greater numbers of partial waves. Crespo and Tostevin [8] and Picklesimer *et al.* [9] have documented difficulties with the VP procedure, difficulties which appear as a sensitivity of the computed phase shifts to the cutoff radius or as a several-percent error in the phase shift when compared to coordinate-space calcula-

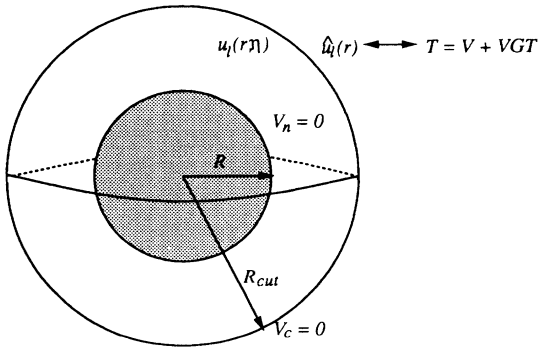

FIG. 1. The VP procedure's partition of coordinate space into a region $r > R$ in which the nuclear potential vanishes, and a region $r > R_{\rm cut}$ in which the Coulomb potential is set equal to zero. The wave function in the outer region is denoted by $\hat{u}_l(r)$ and that in the intermediate region by $u_l(\eta, r)$.

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tions. Both Refs. [8] and [9] suggest algorithms to reduce the errors. Alternatively, Elster *et al.* [10] applied the two-potential formula to the Coulomb and nuclear potentials and outlined an approach requiring multiple, numeric Fourier transforms between coordinate and momentum spaces. In contrast, the study of intermediate-energy proton scattering from spinless nuclei of Arellano *et al.* [11] simply made the VP procedure sufficiently precise by using the high-precision partial-wave expansions developed by Eisenstein and Tabakin [12] (as a check, they transformed the potentials to coordinate space and solved the equivalent integro-differential equation).

In this paper we generalize the VP procedure so that it can be applied to intermediate-energy proton scattering from spin 1/2 nuclei in which states of differing orbital or spin angular momentum are coupled. In the process, we also generalize the Blatt-Biedenharn phase shift parametrization of the scattering of two spin 1/2 particles so that it can describe channel coupling with a nonsymmetric or nonunitary S matrix, as occurs when scattering from an optical potential or when the phase shifts are complex. Although our generalizations of the VP procedure and computations emphasize working directly with S or T matrix elements, the connection to phase shifts is indicated.

In Sec. II we derive and reformulate the VP procedure for uncoupled channels. Since the basic physics can get obscured in the multiple steps of the VP procedure for coupled channels, it is important to understand the physics and notation of Sec. II before proceeding to the coupled-channels case. In Sec. III we present our formulation for coupled channels, and in Sec. IV we give some sample calculations of 500 MeV proton scattering from ^{3}He.

II. UNCOUPLED STATES (0×0, $0 \times 1/2$)

Consider scattering from a short-ranged, but nonlocal, nuclear potential $V_n(\mathbf{r}', \mathbf{r})$ and the infinite-ranged Coulomb potential $V_c(r)$. Because the nuclear potential V_n is nonlocal, the preferred method to obtain the scattering amplitude is to solve the Lippmann-Schwinger equation

$$T_j(k', k) = V_j(k', k) + \frac{2}{\pi} \int_0^\infty p^2 \, dp \frac{V_j(k', p) T_j(p, k)}{E(k_0) + i\epsilon - E(p)}, \tag{2}$$

with $V_{j=l\pm}(k', k)$ the partial-wave matrix element of the momentum-space potential $V_n(\mathbf{k}', \mathbf{k})$. Here l is the orbital angular momentum and $j = l \pm 1/2 \stackrel{\text{def}}{=} l\pm$ is the total angular momentum. For spin 0 scattering from a spin 0 or spin 1/2 target, there is no coupling of different channels in (2).

Equation (2) is valid as long as the coordinate-space potential is of finite range—which in practice means that at some radius the potential is small enough to be ignored without significantly changing the predicted scattering observables. We indicate by the shaded area in Fig. 1 the region in which the nuclear potential V_n acts and the range for the nuclear potential by R. The coordinate-space Coulomb potential does not vanish rapidly enough to be considered as having a finite range, and although its strength may be weaker than the nuclear potential, it cannot be included with the nuclear potential in (2).

The Vincent-Phatak procedure sets the coordinate-space Coulomb potential to zero (cuts it off) for all radii r greater than some fixed value $R_{\rm cut}$:

$$V_c^{\rm cut}(r) = V_c(r)\, \theta(R_{\rm cut} - r). \tag{3}$$

The coordinate-space regions are illustrated in Fig. 1 where we assume that $R_{\rm cut}$ is larger than the range R of the nuclear potential. Since the momentum-space transform

$$V_c^{\rm cut}(\mathbf{k}', \mathbf{k}) = \frac{Z_P Z_T e^2}{2\pi^2 q^2} \left[\rho(q) - \cos(qR_{\rm cut})\right] \tag{4}$$

of the truncated Coulomb potential (3) has the $q \to 0$ limit of $Z_P Z_T e^2 R_{\rm cut}^2/(6\pi^2)$, we see that the $q = 0$ singularity of (1) has indeed been removed. Because the cutoff Coulomb potential is of finite range (in coordinate space) and without singularities in momentum space, its partial-wave decomposition can be added to that of the nuclear potential,

$$V_{l\pm}(k', k) = V_{n,l\pm}(k', k) + V_{c,l}^{\rm cut}(k', k), \tag{5}$$

and when inserted into the Lippmann-Schwinger equation (2), this combined potential produces a well-defined solution.

The solutions $T_{l\pm}(k', k)$ of the Lippmann-Schwinger equation in momentum space (2) can readily be transformed into coordinate-space wave functions for all values of r [13]. Alternatively, just the on-shell element $T_{l\pm}(k_0, k_0)$ can be used to obtain the wave function anyplace outside of the shaded region in Fig. 1. In the "outer" region $r > R_{\rm cut}$, both the nuclear and cutoff Coulomb potentials vanish, and so the (unnormalized) wave function there is expressed as a linear combination of the regular plus either irregular or outgoing solutions [$F_l(k_0 r)$ plus either $G_l(k_0 r)$ or $H_l^{(+)}(k_0 r)$] of the potential-free Schrödinger equation [14]:

$$\hat{u}_{j=l\pm1/2}(r) = \begin{cases} e^{i\delta_{l\pm}(k_0)} \left[\sin\delta_{l\pm}(k_0)\, G_l(k_0 r) + \cos\delta_{l\pm}(k_0)\, F_l(k_0 r)\right] & \text{for } r > R_{\rm cut}\ , \\ F_l(k_0 r) + \hat{T}_{l\pm}(k_0)\, H_l^{(+)}(k_0 r) & \text{for } r > R_{\rm cut}\ , \\ \sin[k_0 r - l\pi/2 + \delta_{l\pm}(k_0)] & \text{for } r (> R_{\rm cut}) \to \infty\ . \end{cases} \tag{6}$$

In (6) we have used two equivalent forms for the free partial wave functions as well as the asymptotic limit. The reduced T matrix element $\hat{T}_{l\pm}(k_0)$ in (6) is related to the "preliminary" phase shift $\delta_{l\pm}$ by

$$\hat{T}_{l\pm}(k_0) = e^{i\delta_{l\pm}(k_0)} \sin\delta_{l\pm}(k_0) \tag{7}$$

and to the solution $T_{l\pm}(k',k)$ of the Lippmann-Schwinger equation (2) by

$$\hat{T}_{l\pm}(k_0) = -\rho_E T_{l\pm}(k_0,k_0), \quad \rho_E = 2k_0 \frac{E_P(k_0)E_T(k_0)}{E_P(k_0)+E_T(k_0)}. \tag{8}$$

Note, again, that we solve the Lippmann-Schwinger equation with a potential which is the sum of nuclear and cutoff Coulomb potentials. Accordingly, the preliminary or unmatched phase shift $\delta_{j=l\pm}$ [which describes the wave function (6) in the outer region of Fig. 1] incorporates the effects of the naturally finite-range nuclear potential and of the artificially truncated Coulomb potential. In particular, effects arising from the finite extent of the target's charge distribution are included in the charge form factor $\rho(q)$ in (1), and consequently are included in the preliminary phase shift $\delta_{j=l\pm}$.

To describe physical scattering observables we need a wave function which incorporates the full extent of the Coulomb force (or at least one with a cutoff of atomic dimensions, which is essentially at infinity in Fig. 1). This, in turn, requires that the preliminary phase shifts δ_j be corrected for the artificial cutoff. The heart of the VP procedure is the observation that while there is no nuclear potential acting in the intermediate region between R and $R_{\rm cut}$ of Fig. 1, there is the Coulomb potential there, and that means that the wave function in the intermediate region must be a linear combination of regular and irregular Coulomb waves:

$$u_{l\pm}(\eta,r) = \begin{cases} F_l(\eta,k_0r) + \hat{T}^c_{l\pm}(k_0)\, H^{(+)}_l(\eta,k_0r) & \text{for } R \le r \le R_{\rm cut}\ , \\ \sin\left[k_0r - l\pi/2 + \delta^c_{l\pm} + \sigma_l - \eta\ln(2k_0r)\right] & \text{for } r(\le R_{\rm cut}) \to \infty\ . \end{cases} \tag{9}$$

Here $\eta = Z_P Z_T e^2/v$ is the Sommerfeld parameter, $F_l(\eta,kr)$ is the regular Coulomb function, and $H^{(+)}_l(\eta,kr)$ is the outgoing Coulomb function.

The Coulomb-modified T matrix,

$$\hat{T}^c_{l\pm}(k_0) \stackrel{\rm def}{=} e^{i\delta^c_{l\pm}} \sin\delta^c_{l\pm} \tag{10}$$

is unknown, and the purpose of the VP procedure is to determine it or, equivalently, the phase shift $\delta^c_{l\pm}$. This is done by the requiring that at $r = R_{\rm cut}$ the intermediate region's wave function $u_{l\pm}(\eta,r)$ (a linear combination of Coulomb waves) have a logarithmic derivative which matches that of the exterior wave function $\hat{u}_{l\pm}(r)$ (a linear combination of free waves):

$$\left.\frac{u'_{l\pm}(\eta,r)}{u_{l\pm}(\eta,r)}\right|_{r\to R_{\rm cut}-} = \left.\frac{\hat{u}'_{l\pm}(r)}{\hat{u}_{l\pm}(r)}\right|_{r\to R_{\rm cut}+}. \tag{11}$$

While r is not large enough to match the phases of the asymptotic wave functions in (6) and (9), we can match the linear combination of free and Coulomb waves. This yields

$$\hat{T}^c_{l\pm}(k) = \frac{\hat{T}_{l\pm}(k)[F_l(\eta,kr),H^{(+)}_l(kr)] + [F_l(\eta,kr),F_l(kr)]}{[F_l(kr),H^{(+)}_l(\eta,kr)] + \hat{T}_{l\pm}(k)[H^{(+)}_l(kr),H^{(+)}_l(\eta,kr)]}\ , \tag{12}$$

where the brackets indicate Wronskians evaluated at $r = R_{\rm cut}$ [15].

As we expand the intermediate region by taking $R_{\rm cut} \to \infty$, the intermediate region's wave function $u_{l\pm}(k_0r,\eta)$ becomes the final physical wave function from which we can extract the experimental scattering observables. Consequently, we can now use the standard expression for the scattering amplitude describing scattering from a short-range potential in the presence of the Coulomb potential. It is informative to note that if instead of matching we had set the phase of the asymptotic limit of the intermediate-region wave function $u_{l\pm}(k_0r,\eta)$ [Eq. (9)] equal to that of the asymptotic limit of the exterior wave function $\hat{u}_{j=l\pm1/2}(r)$ [Eq. (6)], we would have obtained

$$\delta_{l\pm} \sim \delta^c_{l\pm} + \sigma_l - \eta\ln(2kR_{\rm cut}). \tag{13}$$

The $\ln(2kR_{\rm cut})$ term, which arises from the specific distortion of wave functions caused by the point Coulomb force, is problematic in the $R_{\rm cut} \to \infty$ limit. The detailed analysis [14,16] shows that for all but the most forward of scatterings, the standard expansion of the scattering amplitude can be used with

$$\delta_{l\pm} \simeq \delta^c_{l\pm} + \sigma_l. \tag{14}$$

When we determine the Coulomb-modified phase shift via matching the wave functions' logarithmic derivatives (12), we explicitly subtract the $\ln(2kR_{\rm cut})$ term.

Substitution of (14) into the usual partial-wave expansion of the scattering amplitude, and some rearrangement, leads to the final expression for the (spin-nonflip) amplitude for scattering:

$$f(\theta) = f^c_{\rm pt}(\theta) + f^{nc}(\theta), \tag{15}$$

$$f^c_{\rm pt}(\theta) = -\frac{\eta}{2k_0 \sin^2(\theta/2)} \exp\{2i\,[\sigma_0 - \eta \ln \sin(\theta/2)]\}, \tag{16}$$

$$f^{nc}(\theta) = \frac{1}{2ik_0} \sum_{l=0}^{\infty} (2l+1) e^{2i\sigma_l} \left(e^{2i\delta^c_l} - 1\right) P_l(\cos\theta) \tag{17}$$

$$= \frac{1}{k_0} \sum_{l=0}^{\infty} (2l+1) e^{2i\sigma_l} \hat{T}^c_{l\pm} P_l(\cos\theta), \tag{18}$$

where $f^c_{\rm pt}$ is the scattering amplitude for a point Coulomb potential, and f^{nc} is the amplitude for nuclear scattering in the presence of the Coulomb potential [17]. Note that since the Coulomb-modified phase shift $\delta^c_{l\pm}$ is defined in (9) relative to Coulomb waves which are already shifted by the point Coulomb phase σ_l, the amplitude f^{nc} also includes the effect of Coulomb scattering from the finite extent of the charge distribution.

III. COUPLED STATES ($\frac{1}{2} \times \frac{1}{2}$)

A. Basic analysis

If the strong interaction couples orbital or spin angular momentum states, we must generalize the VP method—even though we assume that the Coulomb interaction remains central and does not couple states. We assume rotation invariance, parity conservation, and time reversal invariance, in which case the spin-space structure of the nucleon-nucleus T matrix is [18,19]

$$2\,T(\mathbf{k}',\mathbf{k}) = a + b + (a-b)\sigma^P_n \sigma^T_n + (c+d)\sigma^P_m \sigma^T_m + (c-d)\sigma^P_l \sigma^T_l + e(\sigma^P_n + \sigma^T_n) + f(\sigma^P_n - \sigma^T_n). \tag{19}$$

Although not indicated in (19), a–f are functions of the initial and final momenta $\mathbf{k}$ and $\mathbf{k}'$. The superscripts P and T in Eq. (19) indicate the projectile and target, respectively, while the subscripts n, l, and m indicate a dot product of P's or T's σ with one of the three independent unit vectors

$$\hat{\mathbf{n}} = \frac{\mathbf{k}\times\mathbf{k}'}{|\mathbf{k}\times\mathbf{k}'|}, \quad \hat{\mathbf{m}} = \frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}, \quad \hat{\mathbf{l}} = \frac{\mathbf{k}+\mathbf{k}'}{|\mathbf{k}+\mathbf{k}'|}. \tag{20}$$

Once the a–f amplitudes are known, it is straightforward to calculate the experimental scattering observables [18–20]. For example, the differential cross section, beam analyzing power, target analyzing power, and depolarization parameters are

$$\sigma = \frac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |f|^2), \tag{21}$$

$$A_{00n0} = \frac{1}{\sigma}{\rm Re}(a^* e + b^* f), \tag{22}$$

$$A_{000n} = \frac{1}{\sigma}{\rm Re}(a^* e - b^* f), \tag{23}$$

$$D_{n0n0} = \frac{1}{2\sigma}(|a|^2 + |b|^2 - |c|^2 - |d|^2 + |e|^2 + |f|^2). \tag{24}$$

Here we use the tensor notation $X_{p't'pt}$ with the subscripts p and t denoting the direction of the initial-state projectile and target polarizations, the primes denoting the corresponding final-state quantities, and a subscript 0 denoting zero or undetected polarization. Accordingly, only P is polarized in the n direction in (22) while only T is polarized in the n direction in (23).

The origin of the partial-wave analysis [21] is the expansion of the T and V matrices in spin-angle functions:

$$\begin{pmatrix} T(\mathbf{k}',\mathbf{k}) \\ V(\mathbf{k}',\mathbf{k}) \end{pmatrix} = \frac{2}{\pi} \sum_{jm_j ll' ss'} i^{(l'-l)} \begin{pmatrix} T^{j(s's)}_{l'l}(k',k) \\ V^{j(s's)}_{l'l}(k',k) \end{pmatrix} \times \mathcal{Y}^{jm_j}_{l's'}(\hat{k}')\, \mathcal{Y}^{\dagger jm_j}_{ls}(\hat{k}). \tag{25}$$

In (25), l, s, and j are the orbital, spin, and total angular momenta of the target plus projectile, and $\mathcal{Y}^{jm_j}_{ls}$ is the spin-angle function. When we substitute the expansions (25) into the three-dimensional Lippmann-Schwinger equation, we obtain the integral equations coupling states with spin 0 and 1 (the singlet s and triplet t states), as well as those coupling triplet states with differing orbital angular momenta:

$$\begin{bmatrix} T^{j(ss)}_{jj} \\ T^{j(ts)}_{jj} \end{bmatrix} = \begin{bmatrix} V^{j(ss)}_{jj} \\ V^{j(ts)}_{jj} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V^{j(ss)}_{jj}(k'p)\; V^{j(st)}_{jj}(k',p) \\ V^{j(ts)}_{jj}(k',p)\; V^{j(tt)}_{jj}(k',p) \end{bmatrix} \begin{bmatrix} T^{j(ss)}_{jj}(p,k) \\ T^{j(ts)}_{jj}(p,k) \end{bmatrix}, \tag{26}$$

$$\begin{bmatrix} T^{j(tt)}_{jj} \\ T^{j(st)}_{jj} \end{bmatrix} = \begin{bmatrix} V^{j(tt)}_{jj} \\ V^{j(st)}_{jj} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V^{j(tt)}_{jj}(k',p)\; V^{j(ts)}_{jj}(k',p) \\ V^{j(st)}_{jj}(k',p)\; V^{j(ss)}_{jj}(k',p) \end{bmatrix} \begin{bmatrix} T^{j(tt)}_{jj}(p,k) \\ T^{j(st)}_{jj}(p,k) \end{bmatrix}, \tag{27}$$

$$\begin{bmatrix} T^{j(tt)}_{j-1j-1} \\ T^{j(tt)}_{j+1j-1} \end{bmatrix} = \begin{bmatrix} V^{j(tt)}_{j-1j-1} \\ V^{j(tt)}_{j+1j-1} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V^{j(tt)}_{j-1j-1}(k',p)\; V^{j(tt)}_{j-1j+1}(k',p) \\ V^{j(tt)}_{j+1j-1}(k',p)\; V^{j(tt)}_{j+1j+1}(k',p) \end{bmatrix} \begin{bmatrix} T^{j(tt)}_{j-1j-1}(p,k) \\ T^{j(tt)}_{j+1j-1}(p,k) \end{bmatrix}, \tag{28}$$

$$\begin{bmatrix} T^{j(tt)}_{j+1j+1} \\ T^{j(tt)}_{j-1j+1} \end{bmatrix} = \begin{bmatrix} V^{j(tt)}_{j+1j+1} \\ V^{j(tt)}_{j-1j+1} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V^{j(tt)}_{j+1j+1}(k',p)\; V^{j(tt)}_{j+1j-1}(k',p) \\ V^{j(tt)}_{j-1j+1}(k',p)\; V^{j(tt)}_{j-1j-1}(k',p) \end{bmatrix} \begin{bmatrix} T^{j(tt)}_{j+1j+1}(p,k) \\ T^{j(tt)}_{j-1j+1}(p,k) \end{bmatrix}. \tag{29}$$

For the sake of compactness, we leave off the (k',k) dependences of the leftmost T and V in (26)–(29), and use E^+ as a shorthand for $E(k_0)+i\epsilon$.

Once these partial-wave Lippmann-Schwinger equations are solved, the on-energy-shell matrix elements $T^{j(s's)}_{l'l}(k_0,k_0)$ can be converted to phase shifts or summed to form scattering amplitudes. In Sec. III B we show how to extend the spin $1/2\times 1/2$ phase shift analysis used for two nucleons [22]– [25] to nucleon-nucleus scattering with complex potentials. The summation of the partial-wave T matrices to form the spin-basis matrix element $\langle s'm_{s'}|T|sm_s\rangle$ is derived in [21] for the pure nuclear case. As discussed in Sec. III C, when the Coulomb force is present, there is a point Coulomb scattering amplitude added to the nonflip spin matrix elements, and a Coulomb phase factor $\Phi_l=\exp(2i\sigma_l)$ multiplying some of the partial-wave matrix elements:

$$T_{s1}(\mathbf{k}',\mathbf{k})\equiv\langle 0,0|T|1,1\rangle=\frac{-\sqrt{2}}{4\pi^2}\sum_{l=1}P^1_l(x=\cos\theta_{k'k})\,\frac{2l+1}{\sqrt{l(l+1)}}\,T^{l(st)}_{ll}(k',k)\ , \tag{30}$$

$$T_{ss}(\mathbf{k}',\mathbf{k})\equiv\langle 0,0|T|0,0\rangle=\frac{1}{2\pi^2}\sum_{l=0}P_l(x)(2l+1)\,\Phi_l T^{l(ss)}_{ll}(k',k)\ , \tag{31}$$

$$\begin{aligned}T_{11}(\mathbf{k}',\mathbf{k})=&\frac{1}{4\pi^2}\sum_{l=0}P_l(x)\Big\{(l+2)\Phi_l T^{l+1(tt)}_{ll}(k',k)-\sqrt{(l+1)(l+2)}\,T^{l+1(tt)}_{ll+2}(k',k)\\&+(2l+1)\Phi_l T^{l(tt)}_{ll}(k',k)+(l-1)\Phi_l T^{l-1(tt)}_{ll}(k',k)-\sqrt{(l-1)l}\,T^{l-1(tt)}_{ll-2}(k',k)\Big\}\ ,\end{aligned} \tag{32}$$

$$\begin{aligned}T_{00}(\mathbf{k}',\mathbf{k})\equiv\langle 0,0|T|0,0\rangle=&\frac{1}{2\pi^2}\sum_{l=0}P_l(x)\Big\{(l+1)\Phi_l T^{l+1(tt)}_{ll}(k',k)+l\Phi_l T^{l-1(tt)}_{ll}(k',k).\\&+\sqrt{(l+1)(l+2)}\,T^{l+1(tt)}_{ll+2}(k',k)+\sqrt{(l-1)l}\,T^{l-1(tt)}_{ll-2}(k',k)\Big\}\ ,\end{aligned} \tag{33}$$

$$T_{10}(\mathbf{k}',\mathbf{k})=\frac{\sqrt{2}}{4\pi^2}\sum_{l=1}P^1_l(x)\Big\{-T^{l-1(tt)}_{ll}(k',k)+T^{l+1(tt)}_{ll}(k',k)+\sqrt{\frac{l+2}{l+1}}T^{l+1(tt)}_{ll+2}(k',k)-\sqrt{\frac{l-1}{l}}T^{l-1(tt)}_{ll-2}(k',k)\Big\}\ , \tag{34}$$

$$\begin{aligned}T_{01}(\mathbf{k}',\mathbf{k})=&\frac{\sqrt{2}}{4\pi^2}\sum_{l=1}P^1_l(x)\Big\{-\frac{l+2}{l+1}T^{l+1(tt)}_{ll}(k',k)+\frac{2l+1}{l(l+1)}T^{l(tt)}_{ll}(k',k)\\&+\frac{l-1}{l}T^{l-1(tt)}_{ll}(k',k)+\sqrt{\frac{l+2}{l+1}}T^{l+1(tt)}_{ll+2}(k',k)-\sqrt{\frac{l-1}{l}}T^{l-1(tt)}_{ll-2}(k',k)\Big\}\end{aligned} \tag{35}$$

$$\begin{aligned}T_{1-1}(\mathbf{k}',\mathbf{k})=&\frac{1}{4\pi^2}\sum_{l=2}P^2_l(x)\Big\{\frac{1}{l+1}T^{l+1(tt)}_{ll}(k',k)-\frac{1}{\sqrt{(l+1)(l+2)}}T^{l+1(tt)}_{ll+2}(k',k)\\&-\frac{2l+1}{l(l+1)}T^{l(tt)}_{ll}(k',k)+\frac{1}{l}T^{l-1(tt)}_{ll}(k',k)-\frac{1}{\sqrt{l(l-1)}}T^{l-1(tt)}_{ll-2}(k',k)\Big\}\ .\end{aligned} \tag{36}$$

The a–f amplitudes needed to calculate the spin observables (21)–(24) are then constructed from the T matrices in the spin basis:

$$a(\mathbf{k}',\mathbf{k})=\frac{1}{2}\left(T_{11}+T_{00}-T_{1-1}\right), \tag{37}$$

$$b(\mathbf{k}',\mathbf{k})=\frac{1}{2}\left(T_{11}+T_{ss}+T_{1-1}\right), \tag{38}$$

$$c(\mathbf{k}',\mathbf{k})=\frac{1}{2}\left(T_{11}-T_{ss}+T_{1-1}\right), \tag{39}$$

$$d(\mathbf{k}',\mathbf{k})=\frac{1}{2}\left(T_{00}+T_{1-1}-T_{11}\right)/(2\cos\theta_{kk}), \tag{40}$$

$$e(\mathbf{k}',\mathbf{k})=\frac{i}{\sqrt{2}}\left(T_{10}-T_{01}\right), \tag{41}$$

$$f(\mathbf{k}',\mathbf{k})=i\sqrt{2}T_{s1}. \tag{42}$$

The partial-wave potential matrix elements used as input to (26)–(29) are obtained by first evaluating the potentials in the spin basis $\langle s'm_{s'}|V|sm_s\rangle$, and then expanding these spin-basis matrix elements in partial waves. The expansions are the same as those of the T matrix, (30)–(36), but with no Coulomb phase factors. These expansions are then inverted to obtain the partial-wave potentials $V^{j(s's)}_{l'l}(k',k)$ by numerically projecting out the different $P^m_l(x)$ dependences and then solving the resulting linear equations [21].

B. Extensions for optical potentials

Blatt and Biedenharn were the first to give the extension to the phase shift analysis needed to describe the

scattering of two spin 1/2 particles in the presence of a tensor force which mixes the orbital angular momentum states [24,25]. They assumed that the $j = l \pm 1$ states within the nucleon-nucleon spin triplet have the asymptotic forms

$$\lim_{r\to\infty} u_{j=l+1}(r) = A_+ e^{-i[kr-(j-1)\frac{\pi}{2}]} - B_+ e^{i[kr-(j-1)\frac{\pi}{2}]}, \tag{43}$$

$$\lim_{r\to\infty} u_{j=l'-1}(r) = A_- e^{-i[kr-(j+1)\frac{\pi}{2}]} - B_- e^{i[kr-(j+1)\frac{\pi}{2}]}. \tag{44}$$

The S matrix for the coupled system is then defined by the relation among the A's and B's:

$$\begin{bmatrix} B_+ \\ B_- \end{bmatrix} = \begin{bmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix}, \tag{45}$$

where we use the shorthand notation

$$\pm \stackrel{\text{def}}{=} (j = l \pm 1). \tag{46}$$

For NN scattering below the pion production threshold, the S matrix must be unitary since flux is conserved, and symmetric since all terms in the Schrödinger equation are real. For that case, the most general form for S, a unitary and symmetric 2×2 matrix, is given by a similarity transformation with a "mixing" parameter ϵ:

$$[S] = [U]^{-1}[e^{2i\Delta}][U], \tag{47}$$

$$[U] = \begin{bmatrix} \cos\epsilon_j & \sin\epsilon_j \\ -\sin\epsilon_j & \cos\epsilon_j \end{bmatrix}, \tag{48}$$

$$[e^{2i\Delta}] = \begin{bmatrix} e^{2i\delta_{++}} & 0 \\ 0 & e^{2i\delta_{--}} \end{bmatrix}. \tag{49}$$

When dealing with nonidentical particle scattering through an optical potential, the S matrix is no longer unitary (which means the phases shifts become complex), and as well, the S matrix is no longer symmetric (which means there are now two mixing parameters). To describe this more general case, we assume (47) to be valid but with a more general transformation matrix

$$[U] = \begin{bmatrix} \cos\epsilon_{+-} & \sin\epsilon_{-+} \\ -\sin\epsilon_{+-} & \cos\epsilon_{-+} \end{bmatrix}, \tag{50}$$

$$[U]^{-1} = \frac{1}{\det U} \begin{bmatrix} \cos\epsilon_{-+} & -\sin\epsilon_{-+} \\ \sin\epsilon_{+-} & \cos\epsilon_{+-} \end{bmatrix}, \tag{51}$$

$$\det U = \cos\epsilon_{+-}\cos\epsilon_{-+} + \sin\epsilon_{+-}\sin\epsilon_{-+}. \tag{52}$$

This leads to the S matrix elements $S_{\pm\pm} \equiv S_{j=l'\pm1,j=l\pm1}(k_0)$ having the form

$$S_{++} = \frac{1}{\det U}(\cos\epsilon_{+-}\cos\epsilon_{-+}e^{2i\delta_{++}} + \sin\epsilon_{+-}\sin\epsilon_{-+}e^{2i\delta_{--}}), \tag{53}$$

$$S_{+-} = \frac{1}{\det U}(\sin\epsilon_{-+}\cos\epsilon_{-+}e^{2i\delta_{++}} - \cos\epsilon_{-+}\sin\epsilon_{-+}e^{2i\delta_{--}}), \tag{54}$$

$$S_{-+} = \frac{1}{\det U}(\sin\epsilon_{+-}\cos\epsilon_{+-}e^{2i\delta_{++}} - \cos\epsilon_{+-}\sin\epsilon_{+-}e^{2i\delta_{--}}), \tag{55}$$

$$S_{--} = \frac{1}{\det U}(\cos\epsilon_{+-}\cos\epsilon_{-+}e^{2i\delta_{--}} + \sin\epsilon_{+-}\sin\epsilon_{-+}e^{2i\delta_{++}}). \tag{56}$$

The T matrix elements used in the VP procedure and computations are simply related to the S matrix elements via (7) and (8):

$$2i\rho_E T_{ll'}(k_0,k_0) = \delta_{ll'} - S_{l'=j\mp1,l=j\mp1}(k_0). \tag{57}$$

We note that (53)–(56) reduce to the standard, coupled case [24,25] if $\epsilon_{+-} = \epsilon_{-+}$, and to the standard uncoupled case if $\epsilon_{+-} = \epsilon_{-+} = 0$. For the symmetric S matrix case, Stapp *et al.* [22] also gave a parametrization of the S matrix in terms of the "bar" phase shifts, which in some cases is more convenient for the phenomenological parametrization of data. Note, however, that the bar phases are not the ones introduced here, and even for the NN case, the bar phases do not provide a diagonal representation of the S matrix as do the Blatt-Biedenharn phases.

C. VP procedure for coupled channels

The general approach we take for applying the VP procedure to channels coupled by an optical potential has three steps. First, we transform the states to a new basis in which there is no channel coupling. Second, we match the exterior wave function in this basis $\hat{u}(kr)$ to an intermediate region wavefunction $u(r,\eta)$ (a linear combination of Coulomb waves). Finally, we return to the original, nondiagonal basis to calculate the scattering observables.

A possible implementation of these steps would be to take our S matrix elements computed via (26)–(29) and (57), assume they have the forms (53)–(56) in terms of phase shifts and coupling parameters, and then search for the $(\delta_{--},\delta_{++},\epsilon_{+-},\epsilon_{-+})$ which satisfy these transcendental equations. The δ's would then be the phase shifts in the basis in which S is diagonal and we could use them for matching. Instead, we have adopted a more direct—but equivalent—approach in which we explicitly diagonalize the S matrix, do the VP matching of the wave functions in the diagonal basis to obtain the Coulomb-modified T matrix elements, and then transform the matrix elements back to the original basis where we calculate the observables.

Considering the complexity of the procedure, we enumerate the steps followed in a realistic calculation.

(1) Start with a microscopic, first-order, momentum-space optical potential [20,26,27]

$$V_n(\mathbf{k}',\mathbf{k}) = N\{(t^{Pn}_{a+b} + t^{Pn}_e \sigma^P_n)\rho^n_{mt}(q) + [t^{Pn}_{a-b}\sigma^P_n\sigma^T_n + t^{Pn}_e\sigma^T_n + t^{Pn}_{c+d}\sigma^P_m\sigma^T_m + t^{Pn}_{c-d}\sigma^P_l\sigma^T_l + t^{Pn}_{c+d}(\sigma^P_m\sigma^T_l + \sigma^P_l\sigma^T_m)]\rho^n_{sp}(q)\} + Z\{(t^{Pp}_{a+b} + t^{Pp}_e\sigma^P_n)\rho^p_{mt}(q) + [t^{Pp}_{a-b}\sigma^P_n\sigma^T_n + t^{Pp}_e\sigma^t_n + t^{Pp}_{c+d}\sigma^P_m\sigma^T_m + t^{Pp}_{c-d}\sigma^P_l\sigma^P_l + t^{Pp}_{c+d}(\sigma^P_m\sigma^T_l + \sigma^P_l\sigma^T_m)]\rho^p_{sp}(q)\} \ . \quad (58)$$

Here the subscripts a–e indicate that these terms originate from nucleon-nucleon (NN) t's with the same spin-space structure as (19). The potential (58) manifestly contains the spin $1/2 \times 1/2$ dependence of NN scattering weighted by form factors describing the distributions of spin (sp) and matter (mt) for protons and neutrons within the nucleus. The off-energy-shell NN t's in (58) are transformed to the projectile-target center of momentum frame with a Lorentz covariant prescription which also optimizes the impulse and factorization approximations, and the off-shell variation is described with a separable potential [26,27].

(2) Add the regularized Coulomb potential $V_c^{\rm cut}(\mathbf{k}',\mathbf{k})$ [Eq. (58)] to the optical potential $V_n(\mathbf{k}',\mathbf{k})$ [Eq. (58)]. Since the Coulomb potential is central, this effectively modifies the central potential term arising from t^{Pp}_{a+b} and t^{Pn}_{a+b}. The Coulomb potential is accordingly added to the diagonal spin-basis potentials V_{ss}, V_{00}, and V_{11}:

$$\begin{pmatrix} V_{ss}(\mathbf{k}',\mathbf{k}) \\ V_{00}(\mathbf{k}',\mathbf{k}) \\ V_{11}(\mathbf{k}',\mathbf{k}) \end{pmatrix} \to \begin{pmatrix} V_{ss}(\mathbf{k}',\mathbf{k}) \\ V_{00}(\mathbf{k}',\mathbf{k}) \\ V_{11}(\mathbf{k}',\mathbf{k}) \end{pmatrix} + V_c^{\rm cut}(\mathbf{k}',\mathbf{k}). \quad (59)$$

(3) Project out the partial-wave potentials from the spin-basis potentials using the V versions of (30)–(36). The spin-independent Coulomb potential is thereby included in $V^{j(ss)}_{ll}(k',k)$ and $V^{j(tt)}_{ll}(k',k)$.

(4) Solve the coupled Lippmann-Schwinger equations (26)–(29) for $T^{j(s's)}_{l'l}(k',k)$. This is equivalent to solving for the wave function in the outer region.

(5) Convert the T matrix elements into S matrix elements via (57), and construct the nondiagonal S matrix

$$[S] = \begin{bmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{bmatrix}. \quad (60)$$

(6) Explicitly diagonalize the S matrix elements with the similarity transformation

$$[S'] = [U][S][U]^{-1} = \begin{pmatrix} e^{2i\delta'_{++}} & 0 \\ 0 & e^{2i\delta'_{--}} \end{pmatrix}, \quad (61)$$

$$[U] = \begin{pmatrix} 1 & \frac{S_{+-}}{\lambda_- - S_{--}} \\ \frac{S_{-+}}{\lambda_+ - S_{++}} & 1 \end{pmatrix}, \quad (62)$$

$$[U]^{-1} = \frac{1}{\det U}\begin{pmatrix} 1 & \frac{-S_{+-}}{\lambda_- - S_{--}} \\ \frac{-S_{-+}}{\lambda_+ - S_{++}} & 1 \end{pmatrix}, \quad (63)$$

$$2\lambda_\pm = S_{++} + S_{--} \pm \sqrt{(S_{++} - S_{--})^2 + 4S_{+-}S_{-+}}. \quad (64)$$

We now know the diagonal elements $\exp(2i\delta'_{\pm\pm})$.

(7) Extract the preliminary phase shift $\delta'_{\pm\pm}$ (for the outer region's wave function) or equivalently, the normalized preliminary T' matrix [Eq. (7)]:

$$\hat{T}'_{\pm\pm} = e^{i\delta'_{\pm\pm}} \sin\delta'_{\pm\pm}. \quad (65)$$

Since we are in a basis in which there is no coupling, we can match the outer wave functions to the linear combination of Coulomb waves of the intermediate region.

(8) Do the VP matching as in (12) to obtain the Coulomb-modified amplitude $\hat{T}'^c_{\pm\pm}$ (or, equivalently, the phase shift $\delta'^c_{\pm\pm}$) from the $\hat{T}'_{\pm\pm}$:

$$\hat{T}'^c_{\pm\pm}(k) = \frac{\hat{T}'_{\pm\pm}(k)[F_l(\eta,kr), H^{(+)}_l(kr)] + [F_l(\eta,kr), F_l(kr)]}{[F_l(kr), H^{(+)}_l(\eta,kr)] + \hat{T}'_{\pm\pm}(k)[H^{(+)}_l(kr), H^{(+)}_l(\eta,kr)]} \ . \quad (66)$$

(9) Now that we know the wave function in the intermediate region, we transform back to the original basis to extract the scattering amplitude. Form a Coulomb-modified S' matrix in the diagonal basis,

$$[S'^c] = \begin{pmatrix} e^{2i\delta'^c_{++}} & 0 \\ 0 & e^{2i\delta'^c_{--}} \end{pmatrix}, \quad (67)$$

and use the U matrix of (61) to transform S'^c back to the basis in which we calculate the observables:

$$[S^{nc}] = [U]^{-1}[S'^c][U]. \quad (68)$$

(10) Even though the method is guaranteed to diagonalize the S matrix, as an independent test, check that the unitarity constraint is preserved throughout:

$$|S^{nc}_{ll'}| \le 1, \quad \left|S'_{ll'}\right| \le 1. \quad (69)$$

(11) We have split the total phase shift into the sum of the point Coulomb phase σ_l plus the Coulomb-modified phase shift $\delta^c_{\pm\pm}$. As done for the spinless case, we separate out the point Coulomb scattering amplitude from the nuclear amplitude, leaving behind the Coulomb-phase factor $\Phi_l = \exp(2i\sigma_l)$ in the diagonal ($m'_s = m_s$) spin-basis amplitudes $T_{m'_s m_s}(\mathbf{k}',\mathbf{k})$. Form the partial-wave expansions of these amplitudes via (30)–(36).

(12) Add the point Coulomb amplitude to the diagonal spin-basis amplitudes:

$$\begin{pmatrix} T_{ss}(\mathbf{k},\mathbf{k}') \\ T_{00}(\mathbf{k},\mathbf{k}') \\ T_{11}(\mathbf{k},\mathbf{k}') \end{pmatrix} \rightarrow \begin{pmatrix} T_{ss}(\mathbf{k},\mathbf{k}') \\ T_{00}(\mathbf{k},\mathbf{k}') \\ T_{11}(\mathbf{k},\mathbf{k}') \end{pmatrix} + f_{\rm pt}^{c}(\theta). \quad (70)$$

(13) Calculate the scattering amplitudes a–f via (37)–(42), and from these the experimental observables, for example, via (21)–(24).

IV. SAMPLE CALCULATIONS

As a first test of our precision with the VP method, we solved the Lippmann-Schwinger equation with only a point Coulomb potential. We confirmed that our computations reproduced the point Coulomb phase shifts σ_l [after removal of the $\eta \ln(2kR_{\rm cut})$ term in (9)]. We concluded from this severe test that factoring out the overall phase factor of $e^{2i\sigma_0}$ is helpful, and that 48–64 grid points are required in order for our solution of the Lippmann-Schwinger equation to yield four- to five-place precision in σ_l. That precision in σ_l is needed to obtain a point Coulomb scattering amplitude $f_c^{\rm pt}$ which is indistinguishable from the analytic expression on a five-decade semilogarithmic plot.

As the next test, we computed pure Coulomb scattering of 500 MeV protons from the finite charge distribution of ^{3}He. While in this case we have no exact answer to compare with, we do have the first Born approximation amplitude,

$$f_c^{\rm finite}(\theta) \simeq f_c^{\rm pt}(\theta)\rho(q) + O[(Z\alpha)^2]. \quad (71)$$

We obtained an essentially perfect reproduction of (71). This means we can include finite Coulomb effects, in addition to the long-range Coulomb force, to at least this level of precision [$(Z\alpha)^2 \simeq 0.02\%$]. To obtain this agreement we used 48 grid points in the solution of the Lippmann-Schwinger equations (26)–(29), and increased the number of Gaussian integration points used in our partial-wave projection,

$$V_l(k',k) = \pi^2 \int_{-1}^{1} V(\mathbf{k}',\mathbf{k})\, P_l(\cos\theta'_{kk})\, d(\cos\theta'_{kk}), \quad (72)$$

until the partial-wave summation

$$V(\mathbf{k}',\mathbf{k}) \simeq \frac{1}{2\pi^2} \sum_{l}^{l_{\rm max}} (2l+1) V_l(k',k) P_l(\cos\theta_{k'k}) \quad (73)$$

reproduced all oscillations present in $V_c^{\rm cut}(\mathbf{k}',\mathbf{k})$. We show a reproduction of this type in Fig. 2 where the many oscillations arising from the $\cos(qR_{\rm cut})$ term in the cut-off Coulomb potential (4) are evident. We obtained six-place reproduction of $V_c^{\rm cut}(\mathbf{k}',\mathbf{k})$ using $l_{\rm max} = 48$ partial waves and 96 integration points in the partial-wave projection (72). Ten-place reproduction demanded $l_{\rm max} = 96$. We expect these number to scale as kR, and so larger nuclei or higher energies will require more partial waves and grid points. For these calculations we used analytic nuclear form factors, although we also were successful for ^{13}C using numerical Fourier transforms of Wood-Saxon densities [21]. However, we found that noise and instability appear for form factors which falloff slowly in q. In those cases, a cure is to impose a rapid falloff for q values beyond the limit of experimental measurements.

An important requirement on the VP method is that the matching radius, which we take equal to $R_{\rm cut}$, be

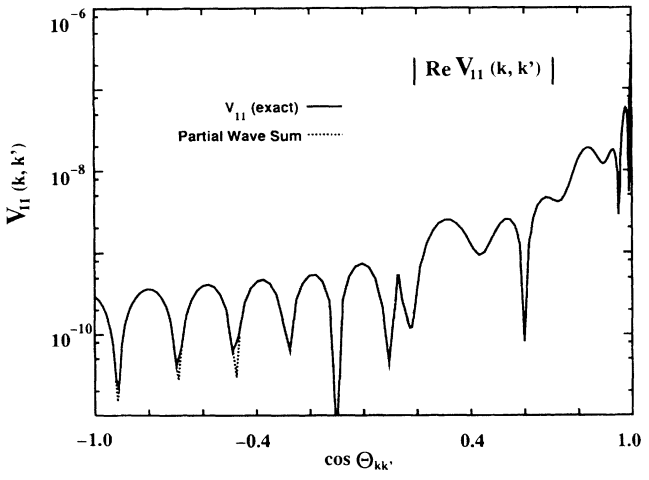

FIG. 2. The nuclear-plus-Coulomb potentials in momentum space for the spin triplet state with $m_s = m_{s'} = 1$ as a function of the cosine of the angle between $\mathbf{k}$ and $\mathbf{k}'$. The summation (73) of partial-wave potentials essentially overlaps the input potential.

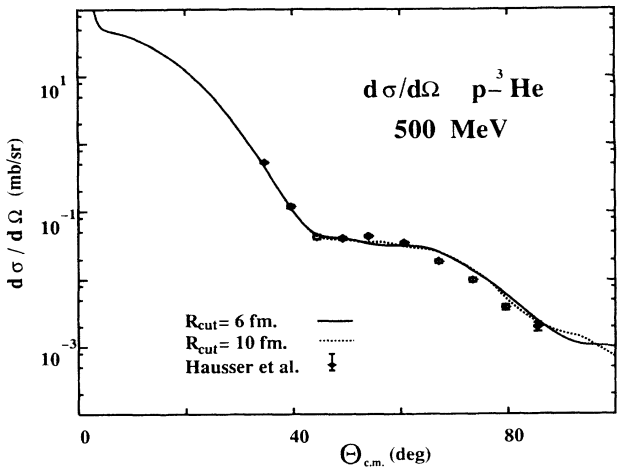

FIG. 3. The differential cross section for 500 MeV proton scattering from ^{3}He. Calculations performed using a cutoff radius in the range 6 fm $\leq R_{\rm cut} \leq$ 10 fm fall within the two curves. The experimental data are from Häusser *et al.* [28].

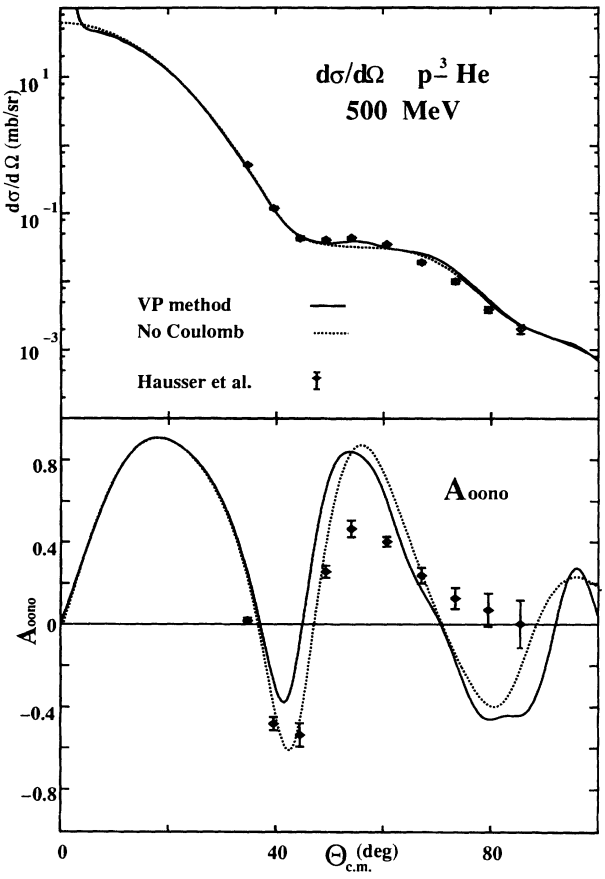

FIG. 4. The differential cross section and analyzing power (unpolarized target, projectile polarized in normal direction) for 500 MeV proton scattering from ^{3}He. The solid curves give the exact results using the VP method and the dashed curves give the results if no Coulomb force is included. The experimental data are from Häusser *et al.* [28].

larger than the range of the nuclear force (in order to be able to express the intermediate region's wave function as a linear combination of pure Coulomb waves). However, increasing $R_{\rm cut}$ makes the cutoff Coulomb potential more oscillatory and more difficult to handle in momentum space. In fact, it was the sensitivity to changes in $R_{\rm cut}$ which led the authors of Ref. [8] to search for an alternative to the matching method. For $p\,^3$He at 500 MeV, we find that using $R_{\rm cut} \leq 5$ fm produces unstable results (presumably cutting off some of the nuclear potential), but as seen in Fig. 3, we obtain stable results for $6\,{\rm fm} \leq R_{\rm cut} \leq 10\,{\rm fm}$.

In Fig. 4 we compare the nuclear-plus-Coulomb cross section and polarization (solid curves) to those calculated without Coulomb (dashed curves). The exact handling of the Coulomb potential is seen to have a significant, although small, effect in the semilogarithmic plot of $d\sigma/d\Omega$, and a more pronounced effect for A_{00n0}.

V. CONCLUSION

We have extended the Vincent-Phatak procedure for the exact inclusion of the Coulomb potential in momentum space to calculations of proton scattering from spin 1/2 nuclei in which spin-dependent forces couple orbital and spin angular momenta channels. As part of that extension we also generalized the Blatt-Biedenharn phase shift analysis for the scattering of two spin 1/2 particles to cases where the S matrix is no longer symmetric (optical potentials or complex phase shifts). Although our formulation and calculation is for a more complicated spin dependence than examined by Arrellano *et al.* [11], we confirm their finding that the VP procedure can be made sufficiently accurate for applications to intermediate-energy proton scattering by using high-precision partial-wave expansions and large numbers of partial waves.

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[1] E. Hernàndez and A. Mondragon, Phys. Rev. C **29**, 722 (1984); G. Garcia-Calderon and R. Peierls, Nucl. Phys. **A265**, 443 (1976).
[2] V. Fock, Z. Phys. **98**, 145 (1935).
[3] Y.R. Kwon and F. Tabakin, Phys. Rev. C **18**, 932 (1978); D.P. Heddle, Y.R. Kwon, and F. Tabakin, Comput. Phys. Commun. **38**, 71 (1985).
[4] A. Landé (private communication).
[5] A. Cieplý, M. Gmitro, R. Mach, and S.S. Kamalov, Phys. Rev. C **44**, 713 (1991); M. Gmitro, S.S. Kamalov, and R. Mach, *ibid.* **36**, 1105 (1987).
[6] C.M. Vincent and S.C. Phatak, Phys. Rev. **10**, 391 (1974).
[7] R.H. Landau, S.C. Phatak, and F. Tabakin, Ann. Phys. (N.Y.) **78**, 299 (1973); R.H. Landau, Comput. Phys. Commun. **28**, 109 (1982).
[8] R. Crespo and J.A. Tostevin, Phys. Rev. C **41**, 2615 (1990).
[9] A. Picklesimer, P.C. Tandy, R.M. Thaler, and D.H. Wolfe, Phys. Rev. C **30**, 2225 (1984).
[10] Ch. Elster, L.C. Liu, and R.M. Thaler, Los Alamos National Laboratory Report No. LA-OR-90-2126, 1990; C.R. Chinn, Ch. Elster, and R.M. Thaler, Phys. Rev. C. **44**, 1569 (1991).
[11] H.F. Arellano, F.A. Brieva, and W.G. Love, Phys. Rev. C **41**, 2188 (1990).

[12] R.A. Eisenstein and F. Tabakin, Phys. Rev. C **26**, 1 (1982).

[13] The Coulomb waves used here are consistent with a Lippmann-Schwinger equation (2) in which the nonrelativistic expression $E(p) = p^2/2\mu$ is used for the energy. In our applications we use the relativistic expression $E(p) = \sqrt{p^2 + m_P^2} + \sqrt{p^2 + m_T^2}$ which is not fully consistent with these Coulomb waves (although the forms for the Coulomb waves appropriate to the Klein-Gordon and Dirac equations are known, we are not aware of any discussion of the forms appropriate to the relativisic Schrödinger equation we use). While this introduces some inaccuracy into our calculation, it does not affect the thrust of this paper which is to outline the extensions to the VP method for two spin 1/2 particles and to demonstrate that the requisite precision is attainable. We suspect it would be a better approximation to use relativistic Coulomb waves.

[14] We use the conventions and normalizations found in R.H. Landau, *Quantum Mechanics II* (Wiley, New York, 1990).

[15] Since this Wronskian is a functional of the solutions from two different Schrödinger equations, it is *not* independent of r.

[16] L.S. Rodberg and R.M. Thaler, *Introduction to the Quantum Theory of Scattering* (Academic Press, New York, 1967).

[17] M.J. Páez, M.E. Sagen, and R.H. Landau, Comput. Phys. Commun. **52**, 141 (1988). Note that the factorization of $\exp(2i\sigma_l)$ was not made properly for all parts of the scattering amplitude in this code and this may lead to numerical problems.

[18] J. Bystricky, F. Lehar, and P. Winternitz, J. Phys. (Paris) **39**, 1 (1978).

[19] P. La France and P. Winternitz, J. Phys. (Paris) **41**, 1391 (1980); correction in L. Ray *et al.*, Phys. Rev C **37**, 1169 (1988).

[20] R.H. Landau, M. Sagen, and G. He, Phys. Rev. C **41**, 50 (1990).

[21] T. Mefford, R.H. Landau, L. Berge, and K. Amos, Phys. Rev. C **50**, 1648 (1994).

[22] H.P. Stapp, T.J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[23] A. Gersten, Phys. Rev. C **18**, 2252 (1978); **24**, 2174 (1981).

[24] J.M. Blatt and L.C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

[25] J.M Eisenberg and W. Greiner, *Nuclear Theory* (North Holland, Amsterdam, 1972), Vol. 3, p. 39.

[26] M.J. Paez and R.H. Landau, Phys. Rev. C **29**, 2267 (1984); **30**, 1757 (1984); Phys. Lett. **142B**, 235 (1984).

[27] R.H. Landau and M. Sagen, Phys. Rev. C **33**, 447 (1986).

[28] O. Häusser, J. Phys. (Paris) Colloq. **22**, C6-99 (1990).

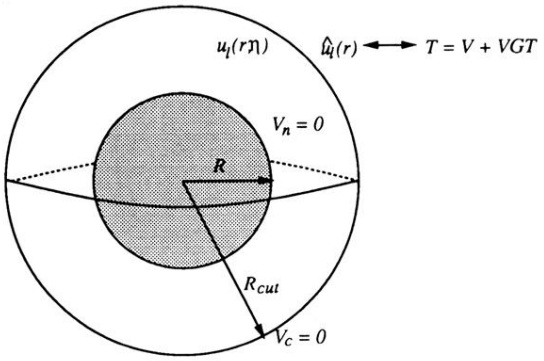

FIG. 1. The VP procedure's partition of coordinate space into a region $r > R$ in which the nuclear potential vanishes, and a region $r > R_{\rm cut}$ in which the Coulomb potential is set equal to zero. The wave function in the outer region is denoted by $\hat{u}_l(r)$ and that in the intermediate region by $u_l(\eta, r)$.