# Collective isospin excitations in nuclear matter droplets

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The isovector giant resonances in heavy nuclei are considered. These excitations correspond to a gapless Goldstone mode which comes along due to the broken symmetry in the isospin space, and can exhibit themselves in a nucleus where the numbers of neutrons and protons are not equal. These modes can be interpreted as weakly damped fluctuations of the transverse isospin components, which propagate through the nuclear matter. The dispersion law of such isospin waves is calculated on the basis of the Fermi-liquid theory as well as by means of the semiclassical variational method. The finite size of the nuclei leads to quantization of the spectra and results in a series of excited states. The number of long-living excited states is strongly affected by the Landau damping. The orthogonality relations and energy-weighted sum rule are formulated. The collective isospin modes of $^{208}$Pb and other heavy nuclei are computed.

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## I. INTRODUCTION

A system of interacting fermions can exhibit a great variety of collective Bose excitations. These collective modes result from the interaction between fermions and do not exist at all in a perfect noninteracting gas. The interaction plays an extremely important role when calculating spectra or formulating the existence criteria for weakly damped excitations. Even at zero temperature, $T = 0$, Landau damping can make the existence of a long-living collective mode impossible. For this reason one cannot *a priori* make a certain prediction that some particular collective mode undoubtedly exists in *any* Fermi system. Such a statement would strongly depend on the specific features of the interaction potential that pertains to the system under consideration. This is valid for both the excitations in configuration space (like zero sound) and spin modes.

The situation is quite different in the case of states with broken symmetry, e.g., in spin-polarized Fermi systems. In this case a gapless Goldstone mode must come along as a result of the broken symmetry. (If the polarization is produced by means of a static magnetic field, there appears a gap in the spectrum, which corresponds to the Larmor precession frequency.) In other words a collective magnetic excitation certainly exists in *any* interacting Fermi system independently of what is the interaction between particles. The appropriate calculations for an infinite system were carried out primarily on the basis of the Fermi-liquid theory [1–5]. A series of experiments which confirmed the theoretical predictions was performed with liquid $^3$He, quantum $^3$He-$^4$He mixtures, and alkaline metals in an external magnetic field [6–8]. Collective spin waves of a quantum-mechanical origin can propagate even in a rarefied polarized gas at high temperatures [9–13], i.e., when all particles obey the classical Maxwell-Boltzmann statistics. Thus the phenomenon in question can manifest itself in various many-body systems and under different conditions.

In this paper we focus on the transverse spin excitations in polarized nuclear matter. We will start with the description of collective spin waves in an infinite nuclear system (for instance, neutron stars). However, our primary goal is to consider spin modes in finite nuclear matter like heavy nuclei. There might be many branches of collective Bose excitations due to both the spin and isospin degrees of freedom. To be specific we will concentrate on the transverse isospin modes only. In view of this a nucleus where the number of neutrons $N$ is not equal to the number of protons $Z$ provides us with a natural "polarized" nuclear system with a broken symmetry in the isospin space. The effective degree of polarization $\alpha$ can simply be defined as

$$\alpha = \frac{Z - N}{A}, \quad A = N + Z. \tag{1}$$

The dispersion law of collective excitations in an infinite polarized nuclear matter can be found on the basis of the standard Boltzmann transport equation for the density matrix. The finite-size effects, i.e., quantization of the excitation spectrum, may be obtained by applying the appropriate boundary conditions. We will restrict ourselves to considering large isotropic spherical nuclei (the droplet model). To calculate the spectrum of collective isospin modes we will apply a Fermi-liquid-like approach [1,5] as well as a Lagrangian-based formalism [14] which allows us to conveniently take into account the finite-size effects in nuclear systems.

 

## II. FERMI-LIQUID APPROACH

The dynamics and collective properties of an infinite Fermi fluid at the zero temperature can be described in terms of the collisionless quasiclassical transport equation [2] for the nucleon density matrix (distribution function) $\hat{n}$:

$$\partial_t \hat{n} + \tfrac{1}{2}[\partial_{\mathbf{p}}\hat{\epsilon} \cdot \nabla \hat{n} + \nabla \hat{n} \cdot \partial_{\mathbf{p}} \hat{\epsilon}] - \tfrac{1}{2}[\nabla \hat{\epsilon} \cdot \partial_{\mathbf{p}} \hat{n} + \partial_{\mathbf{p}} \hat{n} \cdot \nabla \hat{\epsilon}] + \frac{i}{\hbar}[\hat{\epsilon}, \hat{n}] = 0. \quad (2)$$

Here $\hat{\epsilon}$ is the self-consistent single-particle excitation energy. For small perturbations of the density matrix, $\delta\hat{n}$, the self-consistent energy can be expressed as a linear functional of $\delta\hat{n}$:

$$\delta\hat{\epsilon}_\tau(\mathbf{p}) = \mathrm{Tr}_{\tau'} \sum_{p'} \hat{f}_{\tau\tau'}(\mathbf{p}, \mathbf{p}')\delta\hat{n}_{\tau'}(\mathbf{p}'), \quad (3)$$

where $\tau$ labels different isospin states. As was mentioned earlier, no spin effects will be considered in this paper. That is why all spin indices are omitted here and throughout, and averaging over the spin states reduces to the extra factor 2 in the density of states when integrating over momentum $\mathbf{p}$. The function $\hat{f}$ describes the nucleon-nucleon interaction and is the main quantitative feature of the Fermi-liquid theory. In an isotropic "polarized" Fermi fluid the interaction function takes the form

$$\hat{f}_{\tau\tau'}(\mathbf{p}, \mathbf{p}') = \psi(\mathbf{p}, \mathbf{p}')\hat{I}\hat{I}' + \zeta(\mathbf{p}, \mathbf{p}')\hat{\tau}_i\hat{\tau}_i' + \phi(\mathbf{p}, \mathbf{p}')(\hat{\tau}_3\hat{I}' + \hat{\tau}_3'\hat{I}) + \chi(\mathbf{p}, \mathbf{p}')(\hat{\tau}_3\hat{\tau}_3'). \quad (4)$$

Here, the summation over the index $i = 1, 2, 3$ is used. As will be demonstrated later, only the term with the function $\zeta$ in Eq. (4) contributes to the spectrum of transverse isospin modes.

Both $\hat{n}$ and $\hat{\epsilon}$ are, indeed, linear functions of the isospin operator, i.e., of the Pauli matrices, $\hat{\tau}_i$, $i = 1, 2, 3$. In equilibrium the density matrix $\hat{n}$ is diagonal in the isospin space:

$$\hat{n}_\tau^{(0)}(\mathbf{p}) = \frac{n_p + n_n}{2}\hat{I} + \frac{n_p - n_n}{2}\hat{\tau}_3, \quad (5)$$

where $\hat{I}$ is the unity matrix, and $n_p$ and $n_n$ are the Fermi distribution functions for protons and neutrons, respectively:

$$n_p = \theta(\mu_p - \epsilon_p), \quad n_n = \theta(\mu_n - \epsilon_n). \quad (6)$$

Here, $\mu_p$ and $\mu_n$ are the Fermi energies for the proton and neutron components. The single-particle energy spectra, $\epsilon_p$ and $\epsilon_n$, for both species determine the self-consistent energy $\hat{\epsilon}^{(0)}$ in equilibrium:

$$\hat{\epsilon}^{(0)} = \frac{\epsilon_p + \epsilon_n}{2}\hat{I} + \frac{\epsilon_p - \epsilon_n}{2}\hat{\tau}_3 = a(\mathbf{p})\hat{I} + b(\mathbf{p})\hat{\tau}_3. \quad (7)$$

The number of exceeding neutrons (or protons), i.e., the effective degree of "polarization," $\alpha$, is normally small $|\alpha| \ll 1$. (Moreover, the standard Fermi-liquid theory does not hold to describe the transverse spin modes in a strongly polarized, $|\alpha| \leq 1$, system [15]. Formally carrying out such a calculation would require integrating over the Fermi sea, which is in an obvious contradiction with the *cornerstone* of the Fermi-liquid approach.) Keeping the term linear in $\alpha$ one can easily obtain

$$n_p + n_n \approx n_0 = \theta(\mu_0 - \epsilon_0), \qquad n_p - n_n \approx -\frac{4}{3}\frac{\partial n_0}{\partial \epsilon_0}\epsilon_F\alpha, \quad \epsilon_F = \frac{p_F^2}{2m^*}, \quad (8)$$

where the Fermi momentum, $p_F = (3\pi^2 n/2)^{1/3}\hbar$, has been introduced. Here $n$ is the density of nucleons, $m^*$ is the nucleon effective mass, and the index "0" refers to the characteristics of the "unpolarized" system, i.e., where $N = Z$. On the Fermi surface (with radius $p_F$) the interaction function $\hat{f}_{\tau\tau'}(\mathbf{p}, \mathbf{p}')$ depends only on the angle $\gamma$ between vectors $\mathbf{p}$ and $\mathbf{p}'$ as $|\mathbf{p}| = |\mathbf{p}'| = p_F$. It is convenient to define the dimensionless interaction function $g_0\hat{f}$, where $g_0$ is the density of states on the Fermi surface, and extend it in a series of the Legendre polynomials $P_n(\cos\gamma)$ as usual, namely:

$$\frac{2p_F m^*}{\pi^2\hbar^3}\zeta(\gamma) = \sum_{n=0}^{\infty} Z_n P_n(\cos\gamma), \qquad \frac{2p_F m^*}{\pi^2\hbar^3}\psi(\gamma) = \sum_{n=0}^{\infty} F_n P_n(\cos\gamma). \quad (9)$$

Combining Eqs. (3)–(9) one can calculate the isospin-dependent term in the self-consistent energy of a quasiparticle in equilibrium [see Eq. (7)]:

$$b(\mathbf{p}) = \frac{\hbar\Omega_{\rm int}}{2}, \quad \Omega_{\rm int} = \frac{4}{3}\alpha\frac{\epsilon_F}{\hbar}Z_0. \quad (10)$$

A small deviation of $\hat{n}$ from its equilibrium value (5) is a linear function of $\hat{\tau}$ as well, and will be sought in the form

$$\delta\hat{n}_\tau(\mathbf{p}) = \hat{n}_\tau(\mathbf{p}) - \hat{n}_\tau^{(0)}(\mathbf{p}) = \nu(\mathbf{p})\hat{I} + \lambda_k(\mathbf{p}) \cdot \hat{\tau}_k, \quad (11)$$

where $k = 1, 2, 3$, and the summation over dummy indices is used here and throughout. The extra terms in the self-consistent energy due to the fluctuations of the density matrix can be calculated from Eqs. (3), (4), and (11) and read

$$\hat{\epsilon} - \hat{\epsilon}^{(0)} = \delta\hat{\epsilon}_\tau(\mathbf{p}) = 2\hat{I}\sum_{p'}[\psi(\mathbf{p}, \mathbf{p}')\nu(\mathbf{p}') + \phi(\mathbf{p}, \mathbf{p}')\lambda_3(\mathbf{p}')] + 2\hat{\tau}_3\sum_{p'}[\phi(\mathbf{p}, \mathbf{p}')\nu(\mathbf{p}') + \chi(\mathbf{p}, \mathbf{p}')\lambda_3(\mathbf{p}')] + 2\hat{\tau}_i\sum_{p'}\zeta(\mathbf{p}, \mathbf{p}')\lambda_i(\mathbf{p}'), \quad i = 1, 2, 3. \quad (12)$$

Now we have a complete set of equations in order to calculate the spectra of collective modes in nuclear matter with finite isospin.

When substituting Eqs. (5)–(12) in the linearized transport equation (2) one can easily convince oneself that the whole system of equations falls into two separate parts. The first part consists of two coupled equations for $\nu(\mathbf{p})$ and $\lambda_3(\mathbf{p})$, which describe the oscillations of the total nucleon density in the collisionless regime (zero sound) and the longitudinal isospin waves (spatially inhomogeneous oscillations of the relative concentration of neutrons and protons). We will not consider these types of excitations in this section.

The two remaining equations for $\lambda_1$ and $\lambda_2$ describe the dynamics of the off-diagonal components of the density matrix in isospin space, i.e., the transverse isospin modes which we are interested in. Instead of $\lambda_1$ and $\lambda_2$ it is more convenient to use circular variables: $\lambda_\pm = \lambda_1 \pm \iota\lambda_2$. In the case of a slightly "polarized" system, $|\alpha| \ll 1$ , it is natural to seek a solution in the form

$$\lambda_\pm = \frac{\partial n_0}{\partial \epsilon_0}\eta_\pm(\theta,\varphi), \tag{13}$$

where $\theta$ and $\varphi$ are the polar angle and azimuth in momentum space, i.e., on the Fermi sphere. We will be seeking the eigenvalues of the transport equation for the Fourier components: $\eta_\pm \propto \exp(\iota\mathbf{k}\cdot\mathbf{r} - \iota\omega t)$. After some manipulations of Eqs. (2)–(13) we get

$$(\omega + \Omega_{\rm int} - \mathbf{k}\cdot\mathbf{v})\eta_+(\theta,\varphi) - \left(\mathbf{k}\cdot\mathbf{v} + \frac{\Omega_{\rm int}}{Z_0}\right)\int Z(\theta,\varphi,\theta',\varphi')\eta_+(\theta',\varphi')\frac{do'}{4\pi} = 0. \tag{14}$$

Here $\mathbf{v} = p_F\mathbf{n}(\theta,\varphi)/m^*$, and $\mathbf{n}$ is a unit vector on the Fermi sphere. The equation (and solution) for $\eta_-$ can be obtained from Eq. (14) simply by replacing $\omega \to -\omega$ and $\mathbf{k} \to -\mathbf{k}$. However, such a solution does not provide an extra branch of collective excitations. One can easily see that both solutions correspond to the same physical process.

Expanding the quantities $\omega$ and $\nu$ in a power series of the small wave vector $kv/|\Omega_{\rm int}| \ll 1$ (similar to the procedure used in Refs. [2] and [5]) one obtains the following dispersion law:

$$\hbar\omega = \frac{(\hbar k)^2}{2M}, \quad M = m^*\alpha\frac{Z_0 - \frac{1}{3}Z_1}{(1+Z_0)(1+\frac{1}{3}Z_1)}. \tag{15}$$

Here, the quantity $M$ can be considered as the effective mass of the new elementary excitation. These collective modes correspond to fluctuations of the transverse components of the macroscopic isospin. Spectrum (15) looks, indeed, similar to that of spin waves in a magnetized Fermi liquid [15] due to the formal analogy in the quantitative description. The physical meaning of the solution obtained is, however, quite different. The excitation in question, "isospinon," can be interpreted as a delocalized neutron (neutronic state) which can travel through the system as a plane wave with the dispersion law (15). It is fundamentally a collective mode due to the interaction between nucleons rather than a single-particle excitation. Such an excitation exists neither in a perfect (noninteracting gas) nor in an "unpolarized," $\alpha = 0$, nuclear matter. That is why the spectrum (15) has singularities at $\alpha = 0$ and $Z_k = 0$, $k = 0,1$. A formal crossover to the case of $\alpha = 0$ or $Z_k = 0$ nevertheless exists. Although the spectrum (15) diverges if $\alpha \to 0$ or $Z_k \to 0$, the wavelength range, $kv \le |\Omega_{\rm int}|$, in which the transverse isospin waves exist, shrinks simultaneously. The number of the "isospinons" is not fixed: they may appear and annihilate. The isospin excitations obey Bose-Einstein statistics.

Let us emphasize that the dispersion law (15) is the exact solution in the long-wavelength limit, $kv \ll |\Omega_{\rm int}|$. The spectrum of the isospin modes depends only on the first two harmonics $Z_0$ and $Z_1$ whichever interaction between particles is considered. (Indeed, the effective mass $m^*$ of a single-particle excitation also depends on $F_1$ in accordance with the well-known relation of the Fermi-liquid theory

$$\frac{m^*}{m_0} = 1 + \frac{F_1}{3}, \tag{16}$$

where $m_0$ is the bare nucleon mass.)

In the simplest (but quite reasonable, in the case in question) model the nucleon-nucleon interaction can be considered as a pointlike one:

$$U(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{4\pi\hbar^2}{m_0}(c\hat{I}_1\hat{I}_2 + d\hat{\tau}_{1i}\hat{\tau}_{2i})\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{17}$$

Then all Fermi-liquid harmonics $Z_n$ and $F_n$ from Eq. (9) with $n \ge 1$ are equal to zero, and the "isospinon" effective mass $M$ in the spectrum of the transverse modes becomes simplified:

$$M = m_0\alpha\frac{Z_0}{1+Z_0}, \quad Z_0 = -\frac{16}{\pi}\frac{p_F d}{\hbar}. \tag{18}$$

The quantity $d$ which determines the strength of the nucleon-nucleon interaction can be both negative and positive. If the effective mass $M$ turns out to be negative, one should choose the conjugate solution for $\eta_-$, which can be obtained by replacing $\omega \to -\omega$ and $\mathbf{k} \to -\mathbf{k}$, and corresponds to a positive effective mass.

The Coulomb interaction has been ignored in the calculations (although it is the Coulomb interaction that leads to a finite "degree of polarization," $\alpha \ne 0$, in equilibrium). The results obtained for the spectrum of transverse isospin fluctuations can be applied not only to stable nuclei in equilibrium but to long-living excited nuclei with $\alpha \ne 0$ as well. The Coulomb interaction could appreciably affect the spectrum (15) only for extremely large wavelengths irrelevant for a finite-size droplet of any reasonable diameter. The role played by the Coulomb interaction in the transverse isospin dynamics is similar to that of the spin-spin dipole interaction (or magnetostatic contribution) when considering the transverse spin waves in spin-polarized systems.

Stringari and Lipparini proposed an interesting hydrodynamic model to describe isovector collective modes [16], which includes to an extent the effects considered above. Despite the fact that this model could provide a quite correct qualitative description of the phenomena in question, the quantitative results might be rather crude. The hydrodynamic equations (a low-gradient expansion) are not valid in Fermi systems at $T = 0$, as the mean free path for single-particle excitations becomes infinitely large. Under these conditions one should solve the collisionless transport equation directly rather than derive the hydrodynamic equations and then seek their solutions. The hydrodynamic approach does not involve the Landau damping, which plays an extremely important role in a finite-size system and, as will be shown below, can destroy most of the collective transverse modes. The spectrum of collective isospin excitations obtained within the model [16] is linear in the wave number $k$ and does not have any singularity as a function of $\alpha$. The spectrum of the Goldstone mode, which results from the broken symmetry, must be quadratic in $k$ and cannot be formally defined in an unpolarized system, i.e., it should possess a singularity at $\alpha = 0$. Thus the model [16] seems to work satisfactorily when describing longitudinal isospin modes but requires some modifications in order to describe the transverse isospin dynamics.

## III. QUANTIZATION OF THE EXCITATION SPECTRUM

The dispersion law derived in the previous section is valid only in the case of an infinite nuclear system and cannot directly be applied to finite nuclei. In the latter case the continuous momentum $\mathbf{k}$ is no longer a good quantum number to characterize excited states of nuclei. To find the correct classification of the energy levels one needs to quantize the excitation spectrum using the boundary conditions. To simplify the problem we will consider the nucleus as a spherical droplet. The boundary conditions for the macroscopic isospin $T_i$, $i = 1, 2, 3$,

$$T_i = \mathrm{Tr}_\tau \sum_p \hat{\tau}_i \hat{n}_\tau(\mathbf{p}), \tag{19}$$

reduce to the criterion that there should be no isospin current through the surface of the droplet. The equation of motion for the circular component, $T_+ = T_1 + \iota T_2$,

$$-i\partial_t T_+ = \frac{\hbar}{2M}\nabla^2 T_+, \tag{20}$$

coincides, in fact, with the Schrödinger equation for a particle with mass $M$. The solutions describing standing isospin waves in a spherical droplet correspond to the eigenfunctions of the stationary states, which have the usual form [17]:

$$T_+(r,\theta,\phi) = A\left(\frac{k}{r}\right)^{1/2} J_{l+1/2}(kr) Y_{lm}(\theta) e^{\iota m\phi},$$
$$k^2 = \frac{2ME}{\hbar^2}. \tag{21}$$

Here, the spherical coordinates $r, \theta, \phi$ are used, $A$ is the normalization factor, $l$ and $m$ are the polar and azimuthal quantum numbers, $Y_{lm}(\theta)$ are the standard spherical functions, $J_{l+1/2}(z)$ is the Bessel function, and $E$ denotes the energy of the stationary states. The eigenvalues $E$ are determined by the boundary conditions, $J_{l+1/2}(kR) = 0$, where $R$ is the radius of the droplet. In other words, the energy spectrum of the isospin excitation is quantized and may be described in terms of two quantum numbers $l$ and $n_l$:

$$E(n_l, l) = E_{nl} = \frac{\hbar^2}{2MR^2} z_{ln}^2, \tag{22}$$

where $z_{ln}$ are the zeros of the Bessel function, $J_{l+1/2}(z_{ln}) = 0$, and the index $n_l$ labels the zeros of $J_{l+1/2}$ at a given $l$. The sequence of the different stationary states is thus determined by the sequence of the zeros of $J_{l+1/2}$ for all possible $l$. When using the traditional classification scheme the sequence of discrete levels on increasing the energy can be displayed as follows:

$$1s, 1p, 1d, 2s, 1f, 2p, 1g, 2d, 1h, \ldots . \tag{23}$$

Indeed, it coincides with the sequence of levels in an infinite spherically symmetric potential well. The energy spectra of the isospin excitations in the $s$ states have a particularly simple numerical form:

$$E(n_0, 0) = E(ns) = \frac{\pi^2 n^2 \hbar^2}{2MR^2}, \quad n = 1, 2, 3, \ldots . \tag{24}$$

In contrast to an infinite potential well the number of collective excited states (transverse isospin modes) in a heavy nucleus may be finite (or even zero). The Landau damping provides a mechanism that destroys collective excitations with high enough wave numbers $k$. Microscopically a collective mode of the Fermi-liquid type can be interpreted as a bound state of a particle (nucleon) and a hole in the Fermi sea. If the collective excitation moves too fast, it turns out to be unstable with respect to the decay in a particle-hole pair. The threshold above which the Landau damping comes into effect and kills the isospin waves can easily be seen from Eq. (14). The corresponding criterion thus reads

$$\omega + \Omega_{\rm int} - \mathbf{k}\cdot\mathbf{v} > 0. \tag{25}$$

In the long-wavelength limit where the frequency $\omega$ is given by Eq. (15), this criterion reduces approximately to $kv \le |\Omega_{\rm int}|$, which is equivalent to the following condition:

$$\frac{z_{ln} v}{R} \le \frac{4}{3}|\alpha Z_0|\frac{\epsilon_F}{\hbar}. \tag{26}$$

One can easily conclude that the transverse isospin mode $(n_l, l)$ exists if the following inequality is fulfilled:

$$\frac{2}{3 z_{ln}} \frac{p_F R}{\hbar}|\alpha Z_0| > 1. \tag{27}$$

It should be pointed out that criterion (27) applies to the transverse isospin excitations in the long-wavelength limit only. At high wave numbers, $kv \gg |\Omega_{\rm int}|$, the influence of the "isospin polarization" on the spectrum of collective excitations becomes negligible. Under certain conditions (in the one-harmonic approximation, $Z_n = 0$, $n = 1, 2, 3, \ldots$, the conditions reduce to the criterion $Z_0 > 0$) the isospin waves of the zero-sound type with the linear dispersion law $\omega \propto ck$ can still propagate in the system. In this case the dispersion curve $\omega \propto k^2$ for the isospin modes in the long-wavelength limit, $kv \ll |\Omega_{\rm int}|$, converts smoothly into the linear dispersion law $\omega \propto k$ at high $k$. However, in this paper we will be limiting ourselves to considering the low-lying excitations only.

In the opposite case ($Z_0 < 0$ in the one-harmonic approximation) high-frequency isospin waves with a linear spectrum do not exist at all. Nevertheless, the transverse modes with quadratic spectrum certainly propagate in a "polarized" system provided criterion (27) is satisfied. When increasing the quantum numbers $n_l$ and $l$ (i.e., effectively the wave number $k$) Landau damping comes into effect and destroys the collective modes at $kv \sim |\Omega_{\rm int}|$.

The radius of the nuclear droplet, $R$, is, of course, determined by the mass number $A$ of the nucleus, and can be estimated as [17]

$$R = r_0 A^{1/3}, \quad r_0 \approx 1.1 \times 10^{-13} \text{ cm}. \tag{28}$$

Then the Fermi momentum $p_F$ can be easily calculated as follows:

$$p_F = \frac{(9\pi)^{1/3}}{2}\frac{\hbar}{r_0} = 1.4 \times 10^{-14} \text{ g cm s}^{-1}. \tag{29}$$

Substituting Eqs. (28)–(29) into Eq. (27) we finally arrive at the existence criterion for transverse isospin excitations in heavy nuclei:

$$\frac{1}{z_{ln}}\left(\frac{\pi}{3}A\right)^{1/3}|\alpha Z_0| > 1. \tag{30}$$

A simple model of contact interaction (17) provides us with the range of the mass and charge numbers, $A$ and $Z$, in which one can expect the existence of transverse isospin excitations. When combining Eqs. (18) and (30) we obtain the following relation:

$$Z < \frac{1}{2}\left[A - \frac{1}{8}\left(\frac{\pi}{3}\right)^{1/3} A^{2/3}\frac{r_0}{|d|}z_{ln}\right]. \tag{31}$$

For $|d| = 0.35$ fm (see Appendix) one can easily convince oneself that there is a large variety of heavy, stable or long-living, nuclei (for instance, $^{238}_{92}$U, $^{244}_{94}$Pu, $^{248}_{96}$Cm, $^{250}_{96}$Cm, $^{257}_{98}$Cf, etc.) for which criterion (31) is fulfilled, and which are good candidates to exhibit the transverse collective isospin resonances. For such nuclei the spectrum of "isospinons" can fairly well be described in terms of the quadratic dispersion law and Eq. (22). Inasmuch as the Fermi-liquid harmonic $Z_0$ is positive, violating criterion (31) does not mean that the collective isobaric excitations disappear due to the Landau damping (see above). It means only that these fluctuations possess a linear dispersion law rather than the quadratic spectrum (15). $^{208}_{82}$Pb is probably the most interesting objective for experimental study, which lies, however, right at the edge of criterion (31) and therefore requires a more detailed and accurate calculation. Such calculations will be carried out in the next two sections.

## IV. VARIATIONAL APPROACH TO COLLECTIVE EXCITATIONS

In its conventional formulation, the Fermi-liquid approach has been designed to describe the properties of extended systems. Finite systems present peculiar features which deserve special attention. One may ask how the presence of a boundary will affect the spectrum of collective excited states and what is the correct formulation of the boundary conditions which are expected to apply not only to the mass flux but also to the momentum flux. A rather successful variational approach has recently been developed to deal with this kind of situation and to answer these questions [14]. The method starts from a Lagrangian formulation of the collisionless transport equation and uses an approximation based on a variational assumption which has the effect of replacing the Landau damped modes by a discrete set of modes. The boundary couples the modes, so that, in the presence of the boundary, undamped modes may have a small admixture of the modes which would be damped in the absence of the boundary. In any case, the Landau damping is not regarded as the sign of an instability but as the result of an interference effect [18].

We now adapt the method developed in Ref. [14] to the description of the propagation of isospin fluctuations in finite systems such as droplets of nuclear matter. To this end we consider the phenomenological Hamiltonian in agreement with the pointlike nucleon-nucleon interaction (17),

$$H = \sum_i \frac{p_i^2}{2m_0} + \tfrac{1}{2}\sum_{i,j}[c'\hat{I}(i)\hat{I}(j) + d'\boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)]\delta(\mathbf{r}_i - \mathbf{r}_j), \tag{32}$$

where $c' = -(4\pi\hbar^2/m_0)c$, $d' = -(4\pi\hbar^2/m_0)d$, and $\boldsymbol{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)$.

A semiclassical approach is adopted, so that the system of nucleons is described by a distribution function $\hat{n}$ which depends on isospin. The equilibrium distribution function reads

$$\hat{n} = \begin{pmatrix} n_p & 0 \\ 0 & n_n \end{pmatrix} = \frac{\hat{I} + \hat{\tau}_3}{2}n_p + \frac{\hat{I} - \hat{\tau}_3}{2}n_n, \tag{33}$$

with $n_p = \theta(\epsilon_F - h_p)$, $n_n = \theta(\epsilon_F - h_n)$. We introduce the self-consistent densities $n_0 = \frac{1}{2}(n_p + n_n)$, $n_3 = \frac{1}{2}(n_p - n_n)$ and the self-consistent energies $\epsilon_0 = \frac{1}{2}(\epsilon_p + \epsilon_n)$, $\epsilon_3 = \frac{1}{2}(\epsilon_p - \epsilon_n)$. Then $\hat{n} = n_0\hat{I} + n_3\hat{\tau}_3$. The equilibrium self-consistent (effective) Hamiltonian $\hat{\epsilon} = \epsilon_0\hat{I} + \epsilon_3\hat{\tau}_3$ reads

$$\hat{\epsilon} = \frac{p^2}{2m_0} + \mathrm{tr}_2 \int d\Gamma_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) \times [c'\hat{I}(1)\hat{I}(2) + d'\boldsymbol{\tau}(1)\cdot\boldsymbol{\tau}(2)]\hat{n}(2) = \left(\frac{p^2}{2m_0} + c'\rho_0\right)\hat{I} + d'\rho_3\hat{\tau}_3 = \epsilon_0\hat{I} + \epsilon_3\hat{\tau}_3, \quad (34)$$

with $\rho_0 = 2g\int[d^3p/(2\pi)^3]n_0$, $\rho_3 = 2g\int[d^3p/(2\pi)^3]n_3$, $\epsilon_0 = p^2/2m_0 + c'\rho_0$, and $\epsilon_3 = d'\rho_3$.

By $[\ ,\ ]_{\mathrm{SC}}$ we denote the semiclassical limit of the commutator. Let $\Delta, \Lambda$ be operators in the isospin space and let $f, g$ be either operators in the configuration space or their Wigner transforms, depending on the context. We write

$$[f\Delta, g\Lambda]_{\mathrm{SC}} = \frac{i}{2}(\Delta\Lambda + \Lambda\Delta)\{f,g\} + fg[\Delta,\Lambda]. \quad (35)$$

In the right-hand side, $f, g$ are functions of $\mathbf{r}, \mathbf{p}$. In the left-hand side they are operators.

The self-consistency condition may be written in the form

$$[\hat{\epsilon}, \hat{n}]_{\mathrm{SC}} = 0. \quad (36)$$

Indeed, we have

$$[\hat{\epsilon}, \hat{n}]_{\mathrm{SC}} = \{\epsilon_p, n_p\}\frac{\hat{I} + \hat{\tau}_3}{2} + \{\epsilon_n, n_n\}\frac{\hat{I} - \hat{\tau}_3}{2} = 0. \quad (37)$$

By $\hat{S}$ we denote the generator of isospin fluctuations of the distribution function,

$$\hat{S} = S_1\hat{\tau}_1 + S_2\hat{\tau}_2, \quad (38)$$

where $S_1$ and $S_2$ are $c$-number functions of $\mathbf{r}$, $\mathbf{p}$, and $t$. The dynamics of small-amplitude fluctuations is controlled by the second-order Lagrangian

$$L^{(2)} = \frac{i}{2}\mathrm{tr}\int d\Gamma\, \hat{n}[\partial_t\hat{S}, \hat{S}]_{\mathrm{SC}} + \tfrac{1}{2}\mathrm{tr}\int d\Gamma[\hat{S}, \hat{\epsilon}]_{\mathrm{SC}}[\hat{S}, \hat{n}]_{\mathrm{SC}} - \frac{d'}{2}\mathrm{tr}_1\mathrm{tr}_2\int d\Gamma_1 \int d\Gamma_2 [\hat{S}, \hat{n}]_{\mathrm{SC}}(1)[\hat{S}, \hat{n}]_{\mathrm{SC}}(2)\boldsymbol{\tau}(1)\cdot\boldsymbol{\tau}(2)\delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (39)$$

Simple algebra leads to

$$[\hat{S}, \hat{n}]_{\mathrm{SC}} = \iota(\{S_1, n_0\} + 2S_2 n_3)\hat{\tau}_1 + \iota(\{S_2, n_0\} - 2S_1 n_3)\hat{\tau}_2, \quad (40)$$

$$[\hat{S}, \hat{\epsilon}]_{\mathrm{SC}} = \iota(\{S_1, \epsilon_0\} + 2S_2\epsilon_3)\hat{\tau}_1 + \iota(\{S_2, \epsilon_0\} - 2S_1\epsilon_3)\hat{\tau}_2, \quad (41)$$

$$[\hat{S}, \partial_t\hat{S}]_{\mathrm{SC}} = \iota(\{S_1, \dot{S}_1\} + \{S_2, \dot{S}_2\})\hat{I} + 2\iota(S_1\dot{S}_2 - S_2\dot{S}_1)\hat{\tau}_3. \quad (42)$$

The fluctuation of the distribution function is $\delta\hat{n} = \iota[\hat{S}, \hat{n}]_{\mathrm{SC}}$. Its time evolution is determined by the Lagrangian (39).

## V. LAGRANGIAN AND EULER-LAGRANGE EQUATIONS

By $p_{\mathrm{FP}}$ and $p_{\mathrm{FN}}$ we denote the Fermi momenta for protons and neutrons, respectively. It is convenient to introduce some notation: $b_0 = \frac{1}{3}(p_{\mathrm{FP}}^3 + p_{\mathrm{FN}}^3)$, $b_3 = \frac{1}{3}(p_{\mathrm{FP}}^3 - p_{\mathrm{FN}}^3)$, $t_0 = \frac{1}{15}(p_{\mathrm{FP}}^5 + p_{\mathrm{FN}}^5)$, $t_3 = \frac{1}{15}(p_{\mathrm{FP}}^5 - p_{\mathrm{FN}}^5)$, $\epsilon_3 = [2g/(2\pi)^2]d'b_3$.

Also the following approximation is introduced: $S_k(\mathbf{p}, \mathbf{r}, t) = \phi_k(\mathbf{r}, t) + \mathbf{p}\cdot\boldsymbol{\psi}_k(\mathbf{r}, t)$, $k \in \{1, 2\}$. This approximation preserves the Goldstone mode and has the effect of replacing the Landau damped modes by a discrete set of modes. We follow now a variational approach which is similar to the one developed in Ref. [14]. The Lagrangian which determines the time evolution of the fields $\phi_j, \boldsymbol{\psi}_j$ is derived from Eq. (39) and reduces to

$$\begin{aligned} L^{(2)} = \frac{g}{(2\pi)^2}\int d^3r\{ & b_0(\partial_\alpha\phi_1\dot{\psi}_{1\alpha} + \partial_\alpha\phi_2\dot{\psi}_{2\alpha} - \partial_\alpha\dot{\phi}_1\psi_{1\alpha} - \partial_\alpha\dot{\phi}_2\psi_{2\alpha}) + 2b_3(\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1) \\ & + 2t_3(\boldsymbol{\psi}_1\cdot\dot{\boldsymbol{\psi}}_2 - \boldsymbol{\psi}_2\cdot\dot{\boldsymbol{\psi}}_1) + \frac{b_0}{m_0}[(\nabla\phi_1)^2 + (\nabla\phi_2)^2] \\ & + \frac{t_0}{m_0}(\partial_\alpha\psi_{1\alpha}\partial_\beta\psi_{1\beta} + \partial_\alpha\psi_{1\beta}\partial_\alpha\psi_{1\beta} + \partial_\alpha\psi_{1\beta}\partial_\beta\psi_{1\alpha} + \partial_\alpha\psi_{2\alpha}\partial_\beta\psi_{2\beta} + \partial_\alpha\psi_{2\beta}\partial_\alpha\psi_{2\beta} + \partial_\alpha\psi_{2\beta}\partial_\beta\psi_{2\alpha}) \\ & - 2\epsilon_3 b_0\nabla\cdot(\phi_2\boldsymbol{\psi}_1 - \phi_1\boldsymbol{\psi}_2) + \frac{2t_3}{m_0}(\boldsymbol{\psi}_1\cdot\nabla\phi_2 + \phi_1\nabla\cdot\boldsymbol{\psi}_2 - \boldsymbol{\psi}_2\cdot\nabla\phi_1 - \phi_2\nabla\cdot\boldsymbol{\psi}_1) \\ & - 4\epsilon_3 t_3(\boldsymbol{\psi}_1\cdot\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2\cdot\boldsymbol{\psi}_2) + \frac{\epsilon_3 b_0^2}{b_3}[(\nabla\cdot\boldsymbol{\psi}_1)^2 + (\nabla\cdot\boldsymbol{\psi}_2)^2]\}. \end{aligned} \quad (43)$$

The Euler-Lagrange equations read

$$2\dot{\phi}_2 - \frac{b_0}{b_3}\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\psi}}_1 = \frac{b_0}{m_0 b_3}\nabla^2\phi_1 - 2\frac{t_3}{m_0 b_3}\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_2, \quad (44)$$

$$2\dot{\phi}_1 + \frac{b_0}{b_3}\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\psi}}_2 = -\frac{b_0}{m_0 b_3}\nabla^2\phi_2 - 2\frac{t_3}{m_0 b_3}\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_1, \quad (45)$$

$$2t_3\dot{\boldsymbol{\psi}}_2 - b_0\boldsymbol{\nabla}\dot{\phi}_1 = -2\frac{t_3}{m_0}\boldsymbol{\nabla}\phi_2 + 4t_3\epsilon_3\boldsymbol{\psi}_1 + \frac{t_0}{m_0}\nabla^2\boldsymbol{\psi}_1 + \left(\frac{2t_0}{m_0} + \frac{\epsilon_3 b_0^2}{b_3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_1), \quad (46)$$

$$2t_3\dot{\boldsymbol{\psi}}_1 + b_0\boldsymbol{\nabla}\dot{\phi}_2 = -2\frac{t_3}{m_0}\boldsymbol{\nabla}\phi_1 - 4t_3\epsilon_3\boldsymbol{\psi}_2 - \frac{t_0}{m_0}\nabla^2\boldsymbol{\psi}_2 - \left(\frac{2t_0}{m_0} + \frac{\epsilon_3 b_0^2}{b_3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_2). \quad (47)$$

The boundary conditions at $r = a$ (the nuclear radius) are also derived from the Lagrangian and read

$$\mathbf{r}\cdot\left[\dot{\boldsymbol{\psi}}_1 + \frac{1}{m_0}\boldsymbol{\nabla}\phi_1 + \left(\epsilon_3 - \frac{t_3}{m_0 b_0}\right)\boldsymbol{\psi}_2\right] = 0, \quad (48)$$

$$\mathbf{r}\cdot\left[\dot{\boldsymbol{\psi}}_2 + \frac{1}{m_0}\boldsymbol{\nabla}\phi_2 - \left(\epsilon_3 - \frac{t_3}{m_0 b_0}\right)\boldsymbol{\psi}_1\right] = 0, \quad (49)$$

$$x_\alpha\left[\left(\epsilon_3 b_0 + \frac{t_3}{m_0}\right)\phi_1 + \left(\frac{t_0}{m_0} + \frac{\epsilon_3 b_0^2}{b_3}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_2\right] + \frac{t_0}{m_0}x_\beta(\partial_\alpha\psi_{2\beta} + \partial_\beta\psi_{2\alpha}) = 0, \quad (50)$$

$$x_\alpha\left[-\left(\epsilon_3 b_0 + \frac{t_3}{m_0}\right)\phi_2 + \left(\frac{t_0}{m_0} + \frac{\epsilon_3 b_0^2}{b_3}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_1\right] + \frac{t_0}{m_0}x_\beta(\partial_\alpha\psi_{1\beta} + \partial_\beta\psi_{1\alpha}) = 0. \quad (51)$$

In the first step we disregard the boundary conditions and concentrate on obtaining a complete set of local solutions of the Euler-Lagrange equations. In the second step we construct, with the help of the boundary conditions, the correct linear combinations of the previously obtained local solutions. The Euler-Lagrange equations admit two types of solutions: longitudinal and transverse modes. We observe that these words are here used with a different meaning than in Sec. II, where they describe directions in isospace. Here, they specify directions in configuration space. Thus "longitudinal" means along the direction of propagation and "transverse" means perpendicularly to that direction. The transverse modes (such that $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_1 = \phi_1 = \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_2 = \phi_2 = 0$) may be expressed in terms of the quantities $\boldsymbol{\Omega}_1 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_1$ and $\boldsymbol{\Omega}_2 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_2$. Then, $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ obey the equations

$$\dot{\boldsymbol{\Omega}}_2 = 2\epsilon_3\boldsymbol{\Omega}_1 + \frac{t_0}{2m_0 t_3}\nabla^2\boldsymbol{\Omega}_1, \quad (52)$$

$$\dot{\boldsymbol{\Omega}}_1 = -2\epsilon_3\boldsymbol{\Omega}_2 - \frac{t_0}{2m_0 t_3}\nabla^2\boldsymbol{\Omega}_2. \quad (53)$$

Making use of $\nabla^2\boldsymbol{\Omega}_i = -k^2\boldsymbol{\Omega}_i$ we get

$$\omega^2 = \left(2\epsilon_3 - \frac{t_0 k^2}{2m_0 t_3}\right)^2. \quad (54)$$

In order to obtain the equations for the longitudinal modes (such that $\boldsymbol{\nabla}\times\boldsymbol{\psi}_1 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_2 = 0$) it is convenient to define the scalar functions $F_i = \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_i$, $i \in \{1,2\}$, and to make the local ansatz $\phi_1 = \phi_{1n}\cos\omega_n t$, $\boldsymbol{\psi}_2 = \boldsymbol{\psi}_{2n}\cos\omega_n t$, $\phi_2 = \phi_{2n}\sin\omega_n t$, $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_{1n}\sin\omega_n t$. In terms of the quantities $C = t_3/m_0 b_3$, $B = 3t_0/m_0 b_3 + h_3 b_0^2/b_3^2$, and $\mathcal{K}^2 = k^2 b_3/t_3$, the local equations of motion get the form

$$\omega_n\left(2\phi_{2n} - \frac{b_0}{b_3}F_{1n}\right) = -C\left(\frac{b_0}{b_3}\mathcal{K}^2\phi_{1n} + 2F_{2n}\right), \quad (55)$$

$$\omega_n\left(2\phi_{1n} + \frac{b_0}{b_3}F_{2n}\right) = -C\left(\frac{b_0}{b_3}\mathcal{K}^2\phi_{2n} - 2F_{1n}\right), \quad (56)$$

$$\omega_n\left(2F_{1n} - \frac{b_0}{b_3}\mathcal{K}^2\phi_{2n}\right) = 2\mathcal{K}^2 C\phi_{1n} - (4\epsilon_3 - \mathcal{K}^2 B)F_{2n}, \quad (57)$$

$$\omega_n\left(2F_{2n} + \frac{b_0}{b_3}\mathcal{K}^2\phi_{1n}\right) = -2\mathcal{K}^2 C\phi_{2n} - (4\epsilon_3 - \mathcal{K}^2 B)F_{1n}. \quad (58)$$

Assuming $\phi_{1n} = \phi_{2n}$ and $F_{1n} = -F_{2n}$, one solution of these equations is obtained. The dispersion relation is

$$\mathcal{K}^2\left(\frac{b_0}{b_3}\omega_n + 2C\right)^2 = \left(2\omega_n + \frac{b_0}{b_3}C\mathcal{K}^2\right)(2\omega_n - 4\epsilon_3 + \mathcal{K}^2 B). \quad (59)$$

For small $\mathcal{K}$, one of the solutions behaves like $\omega = \alpha\mathcal{K}^2$, $\alpha = -C^2/2\epsilon_3 - b_0 C/2b_3$ (recall that $\epsilon_3$ and $b_3$ are both negative). This result is in agreement with the "isospin" effective mass (15). This is the Goldstone mode.

Assuming $\phi_{1n} = -\phi_{2n}$ and $F_{1n} = F_{1n}$, another solution to these equations is obtained. The dispersion relation, in this case, is equivalent to changing $\omega_n$ into $-\omega_n$ in the previous expression,

$$\mathcal{K}^2\left(\frac{b_0}{b_3}\omega_n - 2C\right)^2 = \left(2\omega_n - \frac{b_0}{b_3}C\mathcal{K}^2\right)(2\omega_n + 4\epsilon_3 - \mathcal{K}^2 B). \quad (60)$$

It may be appropriate to discuss briefly the meaning of the different solutions described here. Beyond the expected Goldstone mode, the other solutions obtained arise as a result of the polynomial structure imposed on the generators. The replacement of the Landau damped modes by a discrete set of modes was a consequence of this approximation. The boundary condition mixes the modes, so that, in the presence of the boundary, undamped modes may have an admixture of modes which are damped in the absence of the boundary. We remark that Landau damping should not be regarded as the sign of an instability but as the result of an interference effect which is quenched by the boundary [18].

The fields $\phi_- = \frac{1}{2}(\phi_1 - \phi_2)$ are coupled to the fields $\boldsymbol{\psi}_+ = \frac{1}{2}(\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2)$. The fields $\phi_+ = \frac{1}{2}(\phi_1 + \phi_2)$ are coupled to the fields $\boldsymbol{\psi}_- = \frac{1}{2}(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2)$. The global normal modes are linear combinations of local modes determined by the boundary equations. In Table I, the energy levels of $^{208}$Pb associated with normal modes described by the fields $\phi_-$ and $\boldsymbol{\psi}_+$ are shown. These correspond to modes excited by a generator which increases the $Z$ number $(\tau_+)$. The particle-hole force (17) with $c' = 380 \times 0.0685$ MeV fm$^3$ and $d' = 380 \times 0.3315$ MeV fm$^3$, taken from Ref. [19], was used. We wish to point out that the low-lying levels for $L = 1, 2, 3, 4$ (shown in parentheses in Table I) are very unstable with respect to small changes of the parameter $d'$. They fluctuate strongly and may even disappear. The remaining levels are quite stable. Only the stable modes should be interpreted as true collective modes.

The level at 15.75 MeV for $l = 0$ may be identified with the 19 MeV isospin analog resonance reported in Ref. [20]. Further experimental results on isovector resonances may be found in Ref. [21]. The normal modes described by the fields $\phi_+$ and $\boldsymbol{\psi}_-$ are excited by a generator which decreases the $Z$ number $(\tau_-)$. All these modes have an excitation energy higher than 50 MeV except for the $l = 0$ mode which has the energy $\omega_n = -2\epsilon_3 = \Omega_{\text{int}}$.

When numerically calculating the energy spectrum of "isospinons" in $^{208}$Pb the "bulk" coupling constants, $c'$ and $d'$, in the Migdal interaction potential are used, and no "surface" contribution is taken into account. It may not be true in the case of light nuclei (small droplets). As was shown earlier the developed theoretical approach should work pretty well for heavy enough nuclei (the larger the droplet, the better the accuracy) where the surface-to-volume ratio is very small. For that reason the use of the "bulk" interaction strengths seems to be a quite reasonable approximation.

TABLE I. Energy levels of $^{208}$Pb associated with the normal modes.

| | | | | |
|---|---|---|---|---|
| $L = 0$ | $\omega_{n,0} = 15.75$ | 35.08 | 36.36 | MeV |
| $L = 1$ | $\omega_{n,1} = (3.49)$ | 11.78 | 32.19 | MeV |
| $L = 2$ | $\omega_{n,2} = (1.02)$ | 22.13 | 34.42 | MeV |
| $L = 3$ | $\omega_{n,3} = (6.94)$ | 34.43 | 47.83 | MeV |
| $L = 4$ | $\omega_{n,4} = (11.80)$ | 44.50 | 63.64 | MeV |

In this section we restrict ourselves to considering the two-body nucleon-nucleon interaction only. A three-body interaction, which is responsible for the saturation effects, can indeed be included in the developed calculation scheme. However the traditional Skyrme potential, which is widely used in nuclear physics, does not contain any isospin-dependent terms responsible for the isospin dynamics. One can easily convince oneself that adding this kind of interaction to the initial Hamiltonian does not affect the spectrum of collective isospin waves at all. A three-body effective potential, which contains the isospin variables, could influence the spectrum of transverse isospin excitations. Such an interaction has, however, not been considered so far and lies far beyond the scope of this paper.

## VI. ORTHOGONALITY RELATIONS AND SUM RULE

When writing for the normal modes $\phi_{1n}(\mathbf{r},t) = \phi_{1n}(\mathbf{r})\cos\omega_n t$, $\boldsymbol{\psi}_{2n}(\mathbf{r},t) = \boldsymbol{\psi}_{2n}(\mathbf{r})\cos\omega_n t$, $\phi_{2n}(\mathbf{r},t) = \phi_{2n}(\mathbf{r})\sin\omega_n t$, and $\boldsymbol{\psi}_{1n}(\mathbf{r},t) = \boldsymbol{\psi}_{1n}(\mathbf{r})\sin\omega_n t$, the modes may be normalized so that the following orthonormality relations hold:

$$\frac{g}{(2\pi)^2}\int d^3r[b_0(\partial_\alpha\phi_{1n}\psi_{1m\alpha} - \partial_\alpha\phi_{2m}\psi_{2n\alpha}) + 2b_3(\phi_{1n}\phi_{2m}) + 2t_3(-\boldsymbol{\psi}_{2n}\cdot\boldsymbol{\psi}_{1m})]$$

$$= \frac{g}{(2\pi)^2}\int d^3r[b_0(\partial_\alpha\phi_{1m}\psi_{1n\alpha} - \partial_\alpha\phi_{2n}\psi_{2m\alpha}) + 2b_3(\phi_{1m}\phi_{2n}) + 2t_3(-\boldsymbol{\psi}_{2m}\cdot\boldsymbol{\psi}_{1n})] = \delta_{mn}. \quad (61)$$

Arbitrary fields (initial conditions) $\phi_2(\mathbf{r})$ and $\boldsymbol{\psi}_1(\mathbf{r})$ may be expanded in normal modes

$$\phi_2(\mathbf{r}) = \sum_n C_n\phi_{2n}(\mathbf{r}), \quad \boldsymbol{\psi}_1(\mathbf{r}) = \sum_n C_n\boldsymbol{\psi}_{1n}(\mathbf{r}), \quad (62)$$

where

$$C_n = \frac{g}{(2\pi)^2}\int d^3r[b_0(\partial_\alpha\phi_{1n}\psi_{1\alpha} - \partial_\alpha\phi_2\psi_{2n\alpha}) + 2b_3(\phi_{1n}\phi_2) + 2t_3(-\boldsymbol{\psi}_{2n}\cdot\boldsymbol{\psi}_1)]. \quad (63)$$

Similarly,

$$\phi_1(\mathbf{r}) = \sum_n D_n \phi_{1n}(\mathbf{r}), \quad \boldsymbol{\psi}_2(\mathbf{r}) = \sum_n D_n \boldsymbol{\psi}_{2n}(\mathbf{r}), \tag{64}$$

where

$$D_n = \frac{g}{(2\pi)^2} \int d^3r [b_0(\partial_\alpha \phi_{2n} \psi_{2\alpha} - \partial_\alpha \phi_1 \psi_{1n\alpha}) + 2b_3(\phi_{2n}\phi_1) + 2t_3(-\boldsymbol{\psi}_{1n} \cdot \boldsymbol{\psi}_2)]. \tag{65}$$

The following sum rule holds:

$$\sum_n \omega_n C_n^2 = -\frac{g}{(2\pi)^2} \int d^3r \left( \frac{b_0}{m_0}[(\nabla\phi_2)^2] + \frac{t_0}{m_0}(\partial_\alpha \psi_{1\alpha} \partial_\beta \psi_{1\beta} + \partial_\alpha \psi_{1\beta} \partial_\alpha \psi_{1\beta} + \partial_\alpha \psi_{1\beta} \partial_\beta \psi_{1\alpha}) - 2\epsilon_3 b_0 \boldsymbol{\nabla} \cdot (\phi_2 \boldsymbol{\psi}_1) + \frac{2t_3}{m_0}(\boldsymbol{\psi}_1 \cdot \boldsymbol{\nabla}\phi_2 - \phi_2 \boldsymbol{\nabla} \cdot \boldsymbol{\psi}_1) - 4\epsilon_3 t_3 (\boldsymbol{\psi}_1 \cdot \boldsymbol{\psi}_1) + \frac{\epsilon_3 b_0^2}{b_3}[(\boldsymbol{\nabla} \cdot \boldsymbol{\psi}_1)^2] \right). \tag{66}$$

The quantities $C_n^2$ and $\omega_n C_n^2$ should be regarded as the transition probability and the transition strength respectively.

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## APPENDIX

The parameter $d$ in Eq. (17) or the parameter $d'$ in Eq. (32) are obtained from the potential contribution for the total symmetry energy of the system [22],

$$E_{\text{sym}} = \frac{V}{8} \frac{(N-Z)^2}{A}, \quad V = 100 \text{ MeV}.$$

In the model presented in Sec. IV this contribution is given by

$$E_{\text{sym}} = \frac{d'}{2} \mathrm{tr}_1 \mathrm{tr}_2 \int d\Gamma_1 d\Gamma_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) \boldsymbol{\tau}(1) \cdot \boldsymbol{\tau}(2) \hat{n}_1 \hat{n}_2 = \frac{d'}{8} \frac{3}{\pi r_0^3} \frac{(N-Z)^2}{A}.$$

For $r_0 = 1.2$ fm we get

$$d' = 181 \text{ MeV} \quad \text{and } d = -\frac{m_0 d'}{4\pi\hbar^2} = -0.35 \text{ fm}.$$

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