Theory of coupled π-trinucleon systems

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We derive the dynamical equations which couple the four-body (πNNN) system to the underlying three-nucleon system. Our treatment can be considered the proper generalization of the Afnan-Blankleider equations for the coupled NN-πNN system. The resulting connected-kernel equations resemble in structure the Yakubovskiĭ-Grassberger-Sandhas equations for the standard four-body problem, but involve 24 chain-labeled components (rather than the usual 18 ones) and, within the scheme where the Hilbert space is truncated to states with at most one pion, allow for a consistent evaluation of reaction amplitudes involving π absorption/production.

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Consider a picture where three nucleons interact among themselves through two-body (NN) potentials and with the additional pion through another two-body (πN) potential; then the problem involves (only) the complexities of a conventional four-body problem. It can be completely (and, unambiguously) solved by considering connected-kernel equations such as those arising from the Grassberger-Sandhas-Yakubovskiĭ formalism [1]. However, even if this solution had been pursued thoroughly (and with the related outstanding numerical efforts) our understanding of the system and related processes would be rather limited. This is because a fundamental aspect—the absorption process—is still missing in the theory. This channel couples the original four-body system with the corresponding three-nucleon system. Clearly, the argument applies as well if we start with a pure three-nucleon system because, above the pion threshold, the (inverse) pion production process couples the three-nucleon space to the one with the additional pion.

In the past decades, much attention has been devoted to the treatment of the πNN system [2]. In fact, there have been a few different diagrammatic formulations of the coupled πNN problem by Thomas, Rinat, Afnan, and Blankleider [3, 4], and by Avishai and Mizutani [5]. The situation has been recently reviewed in Ref. [6]. In spite of the specific differences among the various formulations, these approaches have in common the reduction of the original field-theoretic problem to an effective three-body problem. This is obtained through a truncation of the original space (with an infinite number of pions) to states with at most one pion only. By now it is well understood [7] that such a truncation introduces inconsistencies in the nucleon dressings. Two nucleon lines cannot be renormalized in the same way one single nucleon is, because simultaneous dressing is prohibited by the truncation. Moreover, one is compelled to a different treatment of the nucleon renormalization in the NN and πNN sectors, since one-pion truncation does not allow nucleon dressing in the presence of another pion. One practical consequence is that the effective πNN coupling constant is modified (weakened) by the presence of a second nucleon, and this makes it difficult to define unambiguously the πN input starting from the sub-system dynamics.

Besides the difficulties with the nucleon renormalization, the above-mentioned truncation of the Hilbert space to states with at most one pion seems inadequate for an accurate description of the πNN reactions also because important mechanisms (such as the Jennings mechanism [8]) require the inclusion of states with at least two pions.

Very recently, in the description of the πNN dynamics, important progress has been made. In fact, by resorting to a Green's function convolution technique, Kvinikhidze and Blankleider [9] overcame the nucleon renormalization problem, which was plaguing earlier πNN theories. The same authors, by resorting to a connected kernel formulation similar in structure to the one originally proposed by Stingl and Stelbovics [10], included in the πNN dynamics certain intermediate states with more than one pion, as these are known to give important contributions for an accurate description of the πNN processes. Alternatively, other authors [11] have suggested to circumvent the above-mentioned inconsistencies by abandoning the nonrelativistic time-ordered formulation in favor of a four-dimensional approach based on covariant perturbation theory. With reference to this point, a covariant set of three-body equations has been recently derived in Ref. [12] for Φ^3 field theory.

For the π-trinucleon system, the situation is much more primitive, since the inclusion of an extra nucleon greatly complicates the dynamical equations. Here, the level of knowledge and state of art of the theory cannot be compared with the level of sophistication achieved in the description of the πNN problem. To illustrate this point we recall that the Afnan-Blankleider equations, which in the πNN problem are presently considered superseded [9], *have never been properly generalized* to the πNNN system. Formally, equations of the Afnan-Blankleider type hold also for the coupled NNN-πNNN problem [13,14], but in this case a new difficulty arises because the disconnectedness of the associated kernel makes the equations ill conditioned, in that unphysical (spurious) solutions join up with the correct one. Here, we have

drawn our attention to this very problem, and obtained a new connected-kernel formulation of the πNNN problem, by generalizing the Yakubovskiĭ chain-of-partition labeled formalism to include the effects of the meson-two-nucleon vertex. The result is that our formulation of the πNNN problem is endowed with connectedness, much like the Afnan-Blankleider equations are connected for the πNN system.

The formulation we have developed for the πNNN system, being originated by a truncation of the underlying field theory to states with at most four particles, leaves obviously unresolved the problem of the nucleon renormalization, which has already plagued approaches of the Afnan-Blankleider type in the πNN case. In addition, from the experience gained in the πNN problem, it is clear that the inclusion of diagrams with more than one intermediate pion (i.e., at least five particles) is desirable, but also extremely difficult. Therefore, the results herein contained have not to be considered as final, but represent in our opinion an important step toward a comprehensive description of the dynamics ruling the π-trinucleon system.

To our knowledge, there has been only one previous work [13] making substantial improvements on the connectedness problem for the πNNN equations. In their approach, the authors relied substantially on the isobar picture, which allowed them to recast the original NNN-πNNN problem into an effective multichannel problem coupling isobars (also called, in other contexts, correlated pairs, or "quasiparticles") with the underlying three-nucleon space. Connectedness was achieved first in the "standard" four-body sector using the conventional Grassberger-Sandhas scheme, and then introducing later the additional complexities due to pion absorption/production, via a formal separation based on a two-potential scattering formula. By the admission of the authors themselves, their final equations are extremely involved, and their approach in fact demands for "more complete and aesthetic descriptions." We think we have fulfilled here both these requirements. In fact, we have obtained connected-kernel equations without using the isobar (quasiparticle) representation; rather, we used the genuine chain-labeled formalism of the four-body theory and generalized it to include the effects of pion absorption/production. This allowed us to treat consistently and *at the same level* the diagrams involving standard two-body potentials as well as those involving meson-two-baryon vertices. The resulting closed set of coupled equations yields the unique, exact, and complete solution of the problem [15]. We begin by defining the notation. By $T_{(1|1)}$ we represent the fully unclusterized *scattering* amplitude $4\leftarrow 4$ for the πNNN system. In the fully unclusterized *absorption* amplitude, denoted by $T_{(0|1)}$, the initial channel is similarly made by the four unbound particles, while in the final channel there are three free nucleons only (without the pion). Similarly, the quantities $T_{(1|0)}$ and $T_{(0|0)}$ are the corresponding amplitudes for the fully unclusterized π production and three-nucleon scattering, respectively.

It is most convenient to express these physical fully unclusterized amplitudes in terms of the solutions of the Afnan-Blankleider (AB) equations. To this end, one first introduces new, auxiliary amplitudes by factoring out the two-body t-matrices t_a in the channels where the pion is actually present, namely,

$$T_{(1|1)} = \sum_a t_a + \sum_{ab} t_a G_0 U_{ab} G_0 t_b,$$

$$T_{(0|1)} = \sum_a U_a^\dagger G_0 t_a,$$

$$T_{(1|0)} = \sum_a t_a G_0 U_a,$$

$$T_{(0|0)} = U.$$

The auxiliary amplitudes $U_{ab}, U_a^\dagger, U_a, U$ satisfy the AB coupled equations [13, 14]

$$U_{ab} = G_0^{-1}\bar{\delta}_{ab} + \sum_c \bar{\delta}_{ac} t_c G_0 U_{cb} + F_a g_0 U_b^\dagger,$$

$$U_a^\dagger = F_a^\dagger + \mathcal{V} g_0 U_a^\dagger + \sum_c F_c^\dagger G_0 t_c G_0 U_{ca},$$

$$U_a = F_a + \sum_c \bar{\delta}_{ac} t_c G_0 U_c + F_a g_0 U,$$

$$U = \mathcal{V} + \mathcal{V} g_0 U + \sum_c F^\dagger{}_c G_0 t_c G_0 U_c.$$

Here, as usual in few-body scattering theory, indices such as a, b, c denote the three-cluster partitions one has in the four-body (π-trinucleon) sector. As a consequence, they identify also the six pairs one can make out of four bodies. In the present situation, it is often convenient to distinguish between NN and πN pairs. Let i label each one of the three nucleons ($i = 1, 2, 3$); clearly, it can also be used to denote the specific pair (πN_i). The operators t_a represent either NN or πN t matrices in the four-body sector. With reference to this point, it is worth to recall that only the nonpolar part of the π nucleon t matrices has to be retained, to be consistent with the explicit allowance of the πNN vertex in the theory [4]. The operators G_0 and g_0 represent the free propagators in the four-body (πNNN) and three-nucleon sectors, respectively. It is a distinctive feature of formalisms retaining states with at most one pion, that nucleon dressing must be treated differently in g_0 and G_0 [4, 5, 13, 14]. In the former, equal-time dressings have to be excluded, whereas in the latter the nucleons ought to be (at least partially) undressed. To avoid different nucleon masses in the two sectors of the theory, the pion-nucleon interaction is separated into a polar and a background term, so that one-loop self-energy insertions cancel out on the mass shell [16]. In the AB equations, the dressed πNN vertex referring to the ith nucleon, $f(i)$, has been recast into a new vertex operator labeled by the three-cluster index a, namely,

$$F_a = \sum_{i=1}^{3} \bar{\delta}_{ia} f(i), \qquad F_a^\dagger = \sum_{i=1}^{3} \bar{\delta}_{ia} f^\dagger(i), \tag{1}$$

for production and absorption, respectively. Because of their importance, the structure of these operators deserves further comments. By its very definition, F_a is

given by the sum of *all* the πNN vertices when a represents a two-nucleon pair (NN), whereas the elementary vertex $\pi N_i N_i$ has to be excluded in the sums for a associated to a πN_i pair ($a \equiv i$). The explicit expression for $\mathcal{V}$ is given, for instance, by Eq. (2.25) of Ref. [14] and contains the sum over all the pair potentials in the three-nucleon sector. These pair potentials include to their lowest order the explicit one-pion exchange contribution, namely, $\sum_{j,i=1}^{3} f^\dagger(j) G_0 f(i) \bar{\delta}_{ji}$. Higher-order contributions represent diagrams with more than one pion in the intermediate states. Part of these higher-order diagrams can be approximately described by heavy-boson exchange potentials. Other diagrams give rise to possible three-body forces in the three-nucleon sector. The treatment of such higher-order effects has been considered by some authors in the literature [5, 13] while other authors preferred to consider the lowest-order potential only [4]. We observe that the choice of including or excluding these higher-order terms does not affect our discussion about the connectivity of the π-trinucleon equations.

The AB equations couple together the fully unclusterized amplitude for the $3 \leftarrow 3$ process *without pions* with all the three-cluster amplitudes U_{ab} one can have *with the pion.* In other words, the set of equations for $U_{ab}, U_a^\dagger, U_a, U$ couple together the amplitudes for *all* the three-cluster partitions (i.e., with or without the pion) one can have out of a system consisting of one pion and three nucleons. The number of such partitions is 7.

For two nucleons the AB scheme is fully connected and it is possible to evaluate all the amplitudes without ambiguities.

With three nucleons we formally rewrite the AB equation as a *super*-Lippmann-Schwinger equation for matrices defined in *all* the partitions in three clusters of the system

$$\mathsf{T}^{(3)} = \mathsf{V}^{(3)} + \mathsf{V}^{(3)} \mathsf{G}_0^{(3)} \mathsf{T}^{(3)}, \tag{2}$$

with the definitions

$$\mathsf{G}_0{}^{(3)} \equiv \begin{vmatrix} G_{0\,(a|b)}^{(3)} & G_{0\,(a|0)}^{(3)} \\ G_{0\,(0|b)}^{(3)} & G_{0\,(0|0)}^{(3)} \end{vmatrix} = \begin{vmatrix} G_0 t_a G_0 \delta_{ab} & 0 \\ 0 & g_0 \end{vmatrix}, \tag{3}$$

$$\mathsf{V}^{(3)} \equiv \begin{vmatrix} V_{(a|b)}^{(3)} & V_{(a|0)}^{(3)} \\ V_{(0|b)}^{(3)} & V_{(0|0)}^{(3)} \end{vmatrix} = \begin{vmatrix} G_0{}^{-1} \bar{\delta}_{ab} & F_a \\ F_b^\dagger & \mathcal{V} \end{vmatrix}, \tag{4}$$

$$\mathsf{T}^{(3)} \equiv \begin{vmatrix} T_{(a|b)}^{(3)} & T_{(a|0)}^{(3)} \\ T_{(0|b)}^{(3)} & T_{(0|0)}^{(3)} \end{vmatrix} = \begin{vmatrix} U_{ab} & U_a \\ U_b^\dagger & U \end{vmatrix}. \tag{5}$$

In the above equations, the superscript (3) reminds us that they refer to transitions between *three-cluster* channels.

Then, we introduce partitions of the system into *two* clusters. In the four-body sector we denote these partitions by a', while in the three-nucleon sector we use the label a_1. For a' we distinguish three classes. Type I denotes the cluster partitions $(\pi NN)N$. Type II denotes the two pairs $(\pi N)(NN)$. Type III represents the single $(NNN)\pi$ partition.

As a first step we decompose both the matrix potential $V_{(a|b)}^{(3)}$ and the potential $\mathcal{V}$ into operators labeled by two-cluster-partition indices, namely, we write [17]

$$V_{(a|b)}^{(3)} \equiv G_0{}^{-1} \bar{\delta}_{ab} = \sum_{a'} (G_0{}^{-1} \bar{\delta}_{ab} \delta_{a,b \subset a'}) \equiv \sum_{a'} (v_{a'}^{(3)})_{ab}, \tag{6}$$

$$\mathcal{V} = \sum_{a_1} \mathcal{V}_{a_1}, \tag{7}$$

for the separated four-body and three-nucleon sectors, respectively. Equation (6) follows at once from the fact that there is a unique two-cluster partition a' containing two different pairs a and b [18]. Here, we adhered to few-body theory notation, namely, by $\delta_{a,b \subset a'}$ we mean a matrix labeled by three-cluster-partition indices, whose elements are different from zero only if partitions a and b can be obtained by breaking a cluster of a' ($a, b \subset a'$). Clearly, one has as many matrices as the two-cluster partitions are. In Eq. (7), on the other hand, we stressed the fact that a_1 represents a two-cluster partition (namely, an NN pair plus a spectator nucleon) in the pure *three-nucleon sector*; in other words, Eq. (7) is nothing else but the total potential (in the three-nucleon space) written as the sum of pairwise potentials.

It is an important observation to realize that a similar two-cluster sum rule holds also for the corresponding vertices:

$$F_a = \sum_{a'} (f_{a'})_a, \quad F_a^\dagger = \sum_{a'} (f_{a'}^\dagger)_a, \tag{8}$$

with the "internal vertex" operator defined as

$$(f_{a'})_a = \sum_{i=1}^{3} \bar{\delta}_{ia} \delta_{i,a \subset a'} f(i) \tag{9}$$

and similarly for $(f_{a'}^\dagger)_a$. This is proved very easily once it has been observed again that two different pairs (a, i) unambiguously identify one single partition a' into two clusters that contains both of them.

We may also add that, if the pairs (a, i) have one particle in common (including the pion) then a' is of type I, while it is of type II otherwise. Therefore there is an internal vertex, and hence a coupling to the absorption channel, only for partitions of types I and II. This has very important consequences because only such two types of partitions unambiguously identify one single nucleon-nucleon pair a_1 in the three-nucleon sector.

For each of the two-cluster partitions we have to introduce the corresponding subamplitudes.

Type III. Here, since the vertex operators are not effective, we have standard four-body-like subamplitudes. The III-type subamplitudes are well defined through the AGS (Alt-Grassberger-Sandhas) equations, describing the multiple scattering among the three nucleons in the presence of a spectator pion

$$(u_{a'})_{ab} = G_0^{-1} \bar{\delta}_{ab} + \sum_c \bar{\delta}_{ac} t_c G_0 (u_{a'})_{cb}, \tag{10}$$

with $a, b, c \subset a'$.

Type I. Because of the coupling with three-nucleon space, here we do not have a standard AGS equation. Rather, we have an AB set of equations with corresponding subamplitudes $(u_{a'})_{ab}, (u^\dagger_{a'})_a$ and $(u_{a'})_a, (u_{a'})$. These equations describe a πNN-NN system in the presence of a spectator nucleon, and couple together all the subamplitudes for the three-cluster partitions *including* the absorption sector, when partitioned in three single nucleons. The pair a_1 is unambiguously determined, since it corresponds to the unique nucleon-nucleon pair internal to the given I-type partition a'. Thus, we get the I-type subamplitudes from the connected-kernel coupled equations

$$
\begin{aligned}
(u_{a'})_{ab} &= G_0^{-1}\bar{\delta}_{ab} + \sum_c \bar{\delta}_{ac} t_c G_0 (u_{a'})_{cb} + (f_{a'})_a g_0 (u^\dagger_{a'})_b, \\
(u^\dagger_{a'})_a &= (f^\dagger_{a'})_a + \mathcal{V}_{a_1} g_0 (u^\dagger_{a'})_a + \sum_c (f^\dagger_{a'})_c G_0 t_c G_0 (u_{a'})_{ca}, \\
(u_{a'})_a &= (f_{a'})_a + \sum_c \bar{\delta}_{ac} t_c G_0 (u_{a'})_c + (f_{a'})_a g_0 (u_{a'}), \\
(u_{a'}) &= \mathcal{V}_{a_1} + \mathcal{V}_{a_1} g_0 (u_{a'}) + \sum_c (f^\dagger{}_{a'})_c G_0 t_c G_0 (u_{a'})_c,
\end{aligned}
\tag{11}
$$

with $a, b, c, a_1 \subset a'$.

Type II. Like the I-type ones, these subamplitudes couple the four-body sector with the three-nucleon one. Differently from standard four-body theory, however, where the $3+1$ and $2+2$ subamplitudes can be treated on equal footing [1, 17], here the equations for the $2+2$ subamplitudes have to be modified in a nontrivial way with respect to the above I-type equations. This is due to a delicate double-counting problem, which can be most easily recognized by observing the sum rule Eq. (7) for the three-nucleon sector. This sum rule has been completely exhausted since it has been already used for the AB equations for the I-type subamplitudes, and hence, the equations for the II-type subamplitudes must take into account this fact:

$$
\begin{aligned}
(u_{a'})_{ab} &= G_0^{-1}\bar{\delta}_{ab} + \sum_c \bar{\delta}_{ac} t_c G_0 (u_{a'})_{cb} + (f_{a'})_a g_0 (u^\dagger_{a'})_b, \\
(u^\dagger_{a'})_a &= (f^\dagger_{a'})_a + \sum_c (f^\dagger_{a'})_c G_0 t_c G_0 (u_{a'})_{ca}, \\
(u_{a'})_a &= (f_{a'})_a + \sum_c \bar{\delta}_{ac} t_c G_0 (u_{a'})_c + (f_{a'})_a g_0 (u_{a'}), \\
(u_{a'}) &= \sum_c (f^\dagger{}_{a'})_c G_0 t_c G_0 (u_{a'})_c.
\end{aligned}
\tag{12}
$$

Obviously, $a, b, c \subset a'$.

The different structure between Eqs. (11) and (12) can be understood on physical grounds. Whereas the former set describes the "standard" dynamics of a πNN system in the presence of a spectator nucleon, the latter is associated to the less familiar situation of two noninteracting pairs, namely, an NN subsystem plus a πN pair, which can undergo emission/absorption processes. Thus, in Eqs. (11) and (12) for $(u^\dagger_{a'})_a$, absorption can take place through the "internal vertex" $(f^\dagger_{a'})_a$, or through a coupling with the full four-body sector, plus four-body propagation and pairwise rescattering followed by a final absorption process $[\sum_c (f^\dagger_{a'})_c G_0 t_c G_0 (u_{a'})_{ca}]$. In the $3+1$ case, moreover, one can have also any absorption graph, followed by pure nucleon propagation and a last rescattering between the two nucleons which belonged to the πNN system identified by a'. This possibility [described in Eqs. (11) by the term $\mathcal{V}_{a_1} g_0 (u^\dagger_{a'})_a$] is clearly missing in the $2+2$ case. Similar considerations apply to the equations for the subamplitudes $(u_{a'})$ in Eqs. (11) and (12). We finally observe that, if the coupling between the πNNN and NNN spaces is switched off [$f(i) = f^\dagger(i) = 0$], both Eqs. (11) and (12) reduce themselves to standard AGS equations of the form 10, for subamplitudes associated to two-cluster partitions. The pure nucleon-nucleon scattering in the three-nucleon sector is then described by the Lippmann-Schwinger equation

$$
(u_{a'}) = \mathcal{V}_{a_1} + \mathcal{V}_{a_1} g_0 (u_{a'}),
\tag{13}
$$

namely, by the surviving term in the last of Eqs. (11). Had we retained this term in Eqs. (12) also, nucleon-nucleon scattering in the (decoupled) three-nucleon sector would have been counted twice.

We observe that Eqs. (10)–(12) provide the dynamical input to our full four-body equation. We use AB- (or Faddeev-)like schemes to *define* these subamplitudes (and subsequent off-shell extensions) because the corresponding equations, having a connected kernel within the two-cluster partitions, yield a well-defined and unique solution for the subsystem dynamics. The main point is that by using such subamplitudes as input, we can prove that the full four-body equation is connected and provides the physical solution to the πNNN-NNN problem.

For any type (I, II, and III) of two-cluster partitions of the four-body sector, the above equations can be formally recast into a *super*-Lippmann-Schwinger matrix equation, namely,

$$
\mathsf{t}^{(3)}_{a'} = \mathsf{v}^{(3)}_{a'} + \mathsf{v}^{(3)}_{a'} \mathsf{G}^{(3)}_0 \mathsf{t}^{(3)}_{a'},
\tag{14}
$$

with $\mathsf{G}^{(3)}_0$ defined in Eq. (3).

Since the new T-matrices $\mathsf{t}^{(3)}_{a'}$ are defined in the space of all the three-cluster partitions, their definitions depend on whether a' is of type I and II, or III. In fact, we have

$$
\mathsf{t}_{a'}{}^{(3)} \equiv \begin{vmatrix} t_{a'}{}^{(3)}_{(a|b)} & t_{a'}{}^{(3)}_{(a|0)} \\ t_{a'}{}^{(3)}_{(0|b)} & t_{a'}{}^{(3)}_{(0|0)} \end{vmatrix} = \begin{vmatrix} (u_{a'})_{ab} & (u_{a'})_a \\ (u^\dagger{}_{a'})_b & u_{a'} \end{vmatrix},
$$

$$
\mathsf{t}_{a'}{}^{(3)} \equiv \left| t_{a'}{}^{(3)}_{(a|b)} \right| = \left| (u_{a'})_{ab} \right|,
$$

for partitions of type (I, II) or (III), respectively.

For III-type partitions the identification of $\mathsf{v}^{(3)}_{a'}$ corresponds to the standard four-body choice in the four-body sector,

$$
\mathsf{v}_{a'}{}^{(3)} \equiv \left| v_{a'}{}^{(3)}_{(a|b)} \right| = \left| G_0{}^{-1} \bar{\delta}_{ab} \delta_{a,b \subset a'} \right|,
$$

because there is no coupling with the three-nucleon sector. For I-type partitions, $\mathsf{v}^{(3)}_{a'}$ has 4 non-null sectors

$$\mathsf{v}_{a'}^{(3)} \equiv \begin{vmatrix} v_{a'}{}^{(3)}_{(a|b)} & v_{a'}{}^{(3)}_{(a|0)} \\ v_{a'}{}^{(3)}_{(0|b)} & v_{a'}{}^{(3)}_{(0|0)} \end{vmatrix} = \begin{vmatrix} G_0{}^{-1}\bar{\delta}_{ab}\delta_{a,b\subset a'} & (f_{a'})_a \\ (f^\dagger{}_{a'})_b & \mathcal{V}_{a_1} \end{vmatrix},$$

and for II-type partitions, $\mathsf{v}_{a'}^{(3)}$ is defined according to Eq. (12),

$$\mathsf{v}_{a'}{}^{(3)} \equiv \begin{vmatrix} v_{a'}{}^{(3)}_{(a|b)} & v_{a'}{}^{(3)}_{(a|0)} \\ v_{a'}{}^{(3)}_{(0|b)} & v_{a'}{}^{(3)}_{(0|0)} \end{vmatrix} = \begin{vmatrix} G_0{}^{-1}\bar{\delta}_{ab}\delta_{a,b\subset a'} & (f_{a'})_a \\ (f^\dagger{}_{a'})_b & 0 \end{vmatrix}.$$

Thus, all the previous sum rules [Eqs. (6)–(8)] can be formally replaced by one single *super* sum rule

$$\mathsf{V}^{(3)} = \sum_{a'} \mathsf{v}_{a'}^{(3)}. \tag{15}$$

Equations (2), (14), and (15) are all the basic ingredients we need to derive a new set of connected-kernel equations. The solution of these equations is then used to calculate the amplitudes $\mathsf{T}^{(3)}$. On the contrary, a straightforward solution of the original AB equations would lead to nonunique results since that kernel is not connected.

The fundamental ansatz

$$\mathsf{T}^{(3)} = \sum_{a'} \mathsf{t}_{a'}^{(3)} + \sum_{a'b'} \mathsf{t}_{a'}^{(3)} \mathsf{G}_0{}^{(3)} \mathsf{U}_{a'b'}^{(3)} \mathsf{G}_0{}^{(3)} \mathsf{t}_{b'}^{(3)}, \tag{16}$$

allows one to write the AB amplitudes in terms of new unknowns $\mathsf{U}_{a'b'}^{(3)}$. Clearly, Eq. (16) is an implicit definition of the new unknowns.

For each pair $(a'\ b')$, $\mathsf{U}_{a'b'}^{(3)}$ is a matrix spanning the three-cluster partitions, and thus the number of such matrices is as large as the square of 7. However, we can also view the set of unknowns $\mathsf{U}_{a'b'}^{(3)}$ as one *single* matrix defined in the chain space labeled by one partition in two clusters and one three-cluster partition *internal* to the given two-cluster partition. Contrary to what happens in standard four-body theory, however, here the absorption channel has been explicitly included. Thus, there are here 24 components rather than the 18 Yakubovskiĭ ones of the standard four-body problem. The occurrence of six extra components is explained by the fact that six different two-cluster partitions (namely, all the I- and II-type partitions) have one additional component for the absorption channel.

The strategy is to find new equations for the unknowns $\mathsf{U}_{a'b'}^{(3)}$ and then use the ansatz Eq. (16) to obtain the AB amplitudes $\mathsf{T}^{(3)}$.

To get the new equations for $\mathsf{U}_{a'b'}^{(3)}$, we substitute Eq. (16) into the original AB equation Eq. (2), and use for $\mathsf{V}^{(3)}$ the sum rule Eq. (15). Then, we take into account Eq. (14) for all the various types of subamplitudes. In so doing, we get an equation for $\mathsf{U}_{a'b'}^{(3)}$ involving sums over all the two-cluster partitions a', b'. If we make an identification *term by term* of the resulting equation we obtain

$$\mathsf{U}_{a'b'}^{(3)} = \bar{\delta}_{a'b'} \mathsf{G}_0^{(3)^{-1}} + \sum_{c'} \bar{\delta}_{a'c'} \mathsf{t}_{c'}^{(3)} \mathsf{G}_0^{(3)} \mathsf{U}_{c'b'}^{(3)}. \tag{17}$$

These equations represent our final result, and it is important to observe that they have formally the same structure as standard four-body equations, written in the Yakubovskiĭ-Grassberger-Sandhas formulation. However, the nature of these equations is different, in that they have six extra components (24 instead of 18) and couple together two sectors with different numbers of particles. We have verified that the kernel of such a set is connected. More precisely, we have verified that one sector of the kernel (the pure NNN space) is already connected after one iteration, whereas to achieve connectedness also in the remaining three sectors (namely, π scattering, production, and absorption) three iterations are required. The corresponding proof is somewhat lengthy, and requires the methods of few-body scattering theory. We briefly summarize the proof in the Appendix. For more details we refer to a more specialized paper [19], where these results have been resumed and extended to the more general framework of π-multinucleon dynamics.

The above equations can also be recast into a formal LS equation:

$$\mathsf{T}^{(2)} = \mathsf{V}^{(2)} + \mathsf{V}^{(2)} \mathsf{G}_0^{(2)} \mathsf{T}^{(2)}. \tag{18}$$

This LS equation operates at a higher hierarchic level, namely, in the standard chain-labeled space for the four-body sector, and in a new space labeled by hybrid chains for the three-nucleon sector. The two spaces are spanned by the chain indices $(a'a)$ $[a \subset a']$, and $(a'a_1)$ $[a_1 \subset a']$, respectively. The hybrid nature of the second chain is obvious, since a' is a two-cluster partition in the four-body sector, and a_1 is another two-cluster partition in the three-nucleon sector.

At this level of the hierarchy, "Green's functions," "potentials," and "T matrices" are defined as

$$\mathsf{G}_0{}^{(2)} \equiv \begin{vmatrix} G_{0\,(a'a|b'b)}^{(2)} & G_{0\,(a'a|b'b_1)}^{(2)} \\ G_{0\,(a'a_1|b'b)}^{(2)} & G_{0\,(a'a_1|b'b_1)}^{(2)} \end{vmatrix} = \begin{vmatrix} G_0 t_a G_0 (u_{a'})_{ab} G_0 t_b G_0 \delta_{a'b'} & G_0 t_a G_0 (u_{a'})_a g_0 \delta_{a'b'} \\ g_0 (u^\dagger_{b'})_b G_0 t_b G_0 \delta_{a'b'} & g_0 (u_{a'}) g_0 \delta_{a'b'} \end{vmatrix},$$

$$\mathsf{V}^{(2)} \equiv \begin{vmatrix} V_{(a'a|b'b)}^{(2)} & V_{(a'a|b'b_1)}^{(2)} \\ V_{(a'a_1|b'b)}^{(2)} & V_{(a'a_1|b'b_1)}^{(2)} \end{vmatrix} = \begin{vmatrix} (G_0 t_a G_0)^{-1} \delta_{ab} \bar{\delta}_{a'b'} & 0 \\ 0 & g_0^{-1} \bar{\delta}_{a'b'} \end{vmatrix},$$

$$\mathsf{T}^{(2)} \equiv \begin{vmatrix} T_{(a'a|b'b)}^{(2)} & T_{(a'a|b'b_1)}^{(2)} \\ T_{(a'a_1|b'b)}^{(2)} & T_{(a'a_1|b'b_1)}^{(2)} \end{vmatrix} = \begin{vmatrix} U_{a'ab'b} & U_{a'ab'b_1} \\ U^\dagger_{a'a_1b'b} & U_{a'a_1b'b_1} \end{vmatrix}.$$

It must be clearly understood that the hybrid-chain components occur only when a', b' are of types I and II. As stated before, for any given I- or II-type partition a', there is one and only one a_1 partition in the three-nucleon sector; hence, $\delta_{a'b'} \Rightarrow \delta_{a_1b_1}$, as well as $\bar{\delta}_{a_1b_1} \Rightarrow \bar{\delta}_{a'b'}$. Another useful identity is $\bar{\delta}_{a'b'} = \bar{\delta}_{a'b'}\delta_{a_1b_1} + \bar{\delta}_{a_1b_1}$.

By application of the residue method, it is possible to obtain amplitudes referring to more clusterized processes. For instance, the amplitude for the reaction $^3\mathrm{He}(\pi, NN)N$ is given by the expression

$$\sum_{a'}\langle\phi_0|\mathsf{t}^{(3)}_{a'}\mathsf{G}^{(3)}_0\mathsf{U}^{(3)}_{a'b'}|\tilde{\Phi}_{b'}\rangle = \sum_{a',a,b}\langle\phi_0|(u^\dagger_{a'})_a G_0 t_a G_0 U_{a'ab'b}|\Phi_{b';b}\rangle + \sum_{a',b}\langle\phi_0|(u_{a'})g_0 U^\dagger_{a'a_1b'b}|\Phi_{b';b}\rangle.$$

Here, the sum over a' ranges only over types I and II, while b' is of type III, and $|\tilde{\Phi}_{b'}\rangle$ is the column vector denoting the channel state with the three-nucleon bound state. With varying b, $|\Phi_{b';b}\rangle$ are the Faddeev components of the target bound state, while $\langle\phi_0|$ is the three-nucleon plane-wave state.

In conclusion, we have obtained a connected-kernel scheme for the coupled NNN-πNNN problem. Our equations are formally similar to the Yakubovskiĭ-Grassberger-Sandhas equations for the standard four-body problem. However, our equations couple the four-body (πNNN) sector to the underlying three-nucleon system, thereby allowing a consistent evaluation of pion production/absorption amplitudes. Admittedly, the nucleon renormalization problem, and the associated question of the inclusion of intermediate multipion states are left unresolved by the present investigation. Nevertheless, the development herein illustrated represents an important step forward, since for the first time it reconciles in a simple way the requirements of connectedness with the effects due to the presence of an absorption/emission vertex in a four-body system. Under this point of view, the present approach may have for the π-trinucleon problem the same stimulating role played by the AB approach in the πNN case; any future (and more comprehensive) theory of the πNNN system is bound to contain a connected structure such as the one here depicted. Needless to say, the inclusion of more complex intermediate states and of the proper nucleon renormalization requires further theoretical efforts, in the quest for a satisfactory approximation to the underlying field-theoretic many-body problem.

APPENDIX

Here, we briefly outline the proof of connectedness for the kernel of the basic equation (18), referring the interested reader to Ref. [19] for the details.

As is usual in few-body literature we limit ourselves to the lowest-order approximation K to the full πNNN-NNN kernel, namely, we approximate the subamplitudes appearing in the "Green's function" $\mathsf{G}^{(2)}_0$ with the driving terms of the corresponding dynamical equations (10)–(12). If K can be shown to be connected, then the same is true for the full kernel $\mathsf{V}^{(2)}\mathsf{G}^{(2)}_0$. One has

$$\mathsf{K} = \begin{vmatrix} K_{(a'a|b'b)} & K_{(a'a|b'b_1)} \\ K_{(a'a_1|b'b)} & K_{(a'a_1|b'b_1)} \end{vmatrix} = \begin{vmatrix} \bar{\delta}_{a'b'}\bar{\delta}_{ab}\delta_{a\subset b'} t_b G_0 & \bar{\delta}_{a'b'}(f_{b'})_a g_0 \\ \bar{\delta}_{a'b'}(f^\dagger_{b'})_b G_0 t_b G_0 & \bar{\delta}_{a'b'}\mathcal{V}_{b'} g_0 \end{vmatrix} \equiv \begin{vmatrix} K_{(1|1)} & K_{(1|0)} \\ K_{(0|1)} & K_{(0|0)} \end{vmatrix}. \tag{A1}$$

Here, to make the discussion of the various iterations of K easier, we have resorted to the simplified notation introduced for the transition amplitudes at the beginning of this paper. This notation emphasizes the presence or absence of the pion, independently from the nature of the operator.

Connectedness is proved through a stepwise procedure, namely, the connected terms of K^2 are isolated first, then the connected pieces of K^3 are identified, and so on, until full connectedness is exhibited for K^4. In so doing, a crucial role is played by the classification of two-cluster partitions into I- , II-, and III-types, as well as by the topological structure of the "internal vertices" $(f_{a'})_a$ and $(f^\dagger_{a'})_a$, as defined by Eq. (9). After one iteration one has

$$\mathsf{K}^2 = \begin{vmatrix} K_{(1|1)}K_{(1|1)} + K_{(1|0)}K_{(0|1)} & K_{(1|1)}K_{(1|0)} + K_{(1|0)}K_{(0|0)} \\ K_{(0|1)}K_{(1|1)} + K_{(0|0)}K_{(0|1)} & K_{(0|1)}K_{(1|0)} + K_{(0|0)}K_{(0|0)} \end{vmatrix} \equiv \begin{vmatrix} K^2_{(1|1)} & K^2_{(1|0)} \\ K^2_{(0|1)} & K^2_{(0|0)} \end{vmatrix}, \tag{A2}$$

where the explicit dependence upon the chain labels can be easily restored, when necessary, owing to Eq. (A1).

The analysis of the topological structure of the various graphs associated to K^2 shows that the term $K^2_{(0|0)}$ is already connected; actually, it gives rise to three-nucleon forces in the NNN space, due to intermediate π exchange in the four-body sector. Also the term $K_{(0|1)}K_{(1|1)}$ can be shown to be connected by direct inspection, all other terms in K^2 containing dangerous, disconnected graphs. One is forced therefore to go up to the next iteration K^3 of the kernel K. Most of the contributions to this operator can be shown to be connected by resorting to results obtained in the first step of the proof. The only disconnected terms one finds in K^3 are $K_{(1|0)}K_{(0|0)}K_{(0|1)}$ and $K_{(1|1)}K_{(1|0)}K_{(0|0)}$. The former expresses scattering in the full four-body space, passing through pure three-nucleon propagation, the latter represents a higher-order production process. Here, disconnected graphs occur, describing multiple rescatterings inside a nucleon pair, while the pion is produced or absorbed by the third nucleon (a typical 2 + 2 situation). One can directly verify that both these terms give rise to connected contributions after a third iteration (i.e., when K^4 is considered).

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