Fully isotopic model of fragmentation

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A general model for the fragmentation of a two-component or bipartite system (e.g., protons and neutrons) is proposed and solved exactly. The extension of this model to any number of components is also shown to be exactly solvable. A connection between this model and the permutation group is discussed and used to obtain closed-form solutions. The notion of isotopic equivalence is defined in order to evaluate the equivalence of these models to earlier one-component models. All the one-component models considered in earlier papers are shown to be equivalent to a particular subclass of two-component models. A simplified model applicable to the case of nuclear fragmentation is introduced and analyzed. Modifications to this model to include effects such as pairing and Coulomb interactions are discussed.

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I. INTRODUCTION

The fragmentation of mesoscopic systems, particularly heavy nuclear systems, has attracted considerable interest over the past several years [1–24]. A number of statistical models have been proposed to explain the phenomena. In these models, the final state of the fragmentation of a nucleus of A nucleons is represented by the vector $\mathbf{n} = (n_1, \ldots, n_A)$, where n_j is the number of fragments with j nucleons. The set of all such states is denoted by Π_A, each of which must satisfy $\sum_{j=1}^{A} j n_j = A$. These states should be recognized as partitions of the integer A, a subject in the theory of numbers with a long history beginning with results due to Euler.

In actuality, the final state of a nuclear fragmentation process is a more complicated object. Fragments are formed with varying numbers of neutrons and protons, which are distinguishable objects. For this reason, a representation of the final fragmentation state of such a two-component or isotopic system should minimally contain information on the neutron and proton content of each fragment. In this representation, the final state of the fragmentation of a nucleus containing Z protons and N neutrons is given by a vector of $(Z+1)(N+1)-1$ integers, $\mathbf{n} = (n_{01}, \ldots, n_{0N}, n_{10}, \ldots, n_{1N}, \ldots, n_{ZN})$, where n_{jk} is the number of fragments with j protons and k neutrons. The set of all such states is denoted by Π_{ZN}, each of which is subject to the two constraints $\sum_{jk} j n_{jk} = Z$, $\sum_{jk} k n_{jk} = N$. These states are equivalent to the bipartite partitions of (Z, N), a subject principally investigated by the English mathematician P. A. MacMahon [26] in the early years of this century.

In general the fragmentation of an N-component system of $A = \sum_{j=1}^{N} A_j$ particles where A_j is the number of indistinguishable particles of type j can be described by a vector of $\prod_{j=1}^{N}(A_j + 1) - 1$ integers, $\mathbf{n} = (n_{00\ldots01}, \ldots, n_{A_1 \ldots A_N})$, where $n_{k_1 \ldots k_N}$ is the number of fragments with k_j particles of type j, $j = 1, \ldots, N$. The set of all such states or multi-partite partitions is denoted by $\Pi_{A_1 \ldots A_N}$, each of which is subject to the constraints $\sum_{k_1 \ldots k_N} k_j n_{k_1 \ldots k_N} = A_j$, $j = 1, \ldots, N$.

There are two principal reasons why one-component models are primarily used in the statistical modeling of nuclear fragmentation. The first reason is simplicity. One-component models are fundamentally simpler than two-component models, and the exact solution for a class of one-component models has been known for some time. This simplicity is best seen in the size of the state space (see Table I). For one-component partitions, the number of states is asymptotically given by $p(A) \sim K \exp(aA^{1/2})/A$, with a and K particular constants. Bipartite partitions grow much more quickly, in fact $p(A) \sim K' \exp(a'A^{2/3} + b'A^{1/3})/A^{55/36}$ for $Z = N = A/2$ as demonstrated by Auluck [25]. This increase in the number of states makes it significantly more difficult to model these systems by standard Monte Carlo techniques.

The second reason is sufficiency. A one-component model can always model the physics if the underlying isotopic states are inaccessible to the observer. For example, typically only the charge distribution of the fragments is measured in a fragmentation experiment. The isotopic distribution for any particular charge Z is not recorded. So a one-component model distinguishing protons is sufficient to model the observables measured. The weight for a particular charge fragment distribution is then given by summing the two-component weights of

TABLE I. Number of partitions for Z protons and N neutrons.

Z	N	P_Z	P_{N+Z}	P_{ZN}
10	10	42	627	59521
20	20	627	37338	3026 73029
30	35	5604	20 12558	231 50477 43167
40	50	37338	566 34173	6 00944 62493 61633
50	70	9 66467	18443 49560	26953 04447 12166 28606

all the states which have an equivalent configuration of protons. Since this weight is a complicated sum of two-component weights, there is no guarantee that the one-component weight will be simple. Indeed, this is usually not the case, as we will discover in Sec. IV.

However, there are good reasons to prefer modeling nuclear systems with a two-component model, especially if such a model is amenable to analytic solution. For instance, any choice of statistical weight based on the fundamental statistics of the constituent particles must by necessity be based on a two-component model. Also, unlike one-component models, definite predictions about the isotopic distribution of fragments could be made for a two-component model. Additionally, nuclear effects such as proton-neutron symmetry, pairing, and Coulomb interactions could be modeled, effects which are difficult (if not impossible) to treat in a one-component model.

This paper shows that a particular class of two-component models are in fact exactly solvable by recursion. Realistic models lie within this class, and their solution obviates the need for Monte Carlo and allows for the exact evaluation of all expectation values. A discussion of the relation of bipartitions to permutations provides a closed form solution for a specific two-component model, as well as a proof that all the one-component models of the same form as those in Ref. [4] are equivalent to a particular subclass of two-component models.

II. ISOTOPIC MODELS AND THEIR SOLUTION

Earlier papers [1–4] discussed the one-component model where each state $\mathbf{n} \in \Pi_Z$ was given the weight

$$P_Z(\mathbf{n},\mathbf{x}) = \frac{W_Z(\mathbf{n},\mathbf{x})}{Z_Z(\mathbf{x})} = \frac{1}{Z_Z(\mathbf{x})} \prod_{j=1}^{Z} \frac{x_j^{n_j}}{n_j!}. \tag{1}$$

The partition functions can be obtained by recursion,

$$Z_Z(\mathbf{x}) = \frac{1}{Z} \sum_{j=1}^{Z} j x_j Z_{Z-j}(\mathbf{x}) \tag{2}$$

and all the ensemble averages are simple functions of the partition function.

In analogy with the one-component model, we can define a canonical weight for $\mathbf{n} \in \Pi_{ZN}$ by

$$P_{ZN}(\mathbf{n},\mathbf{x}) = \frac{W_{ZN}(\mathbf{n},\mathbf{x})}{Z_{ZN}(\mathbf{x})} = \frac{1}{Z_{ZN}(\mathbf{x})} \prod_{jk} \frac{x_{jk}^{n_{jk}}}{n_{jk}!}, \tag{3}$$

where $\mathbf{x}$ is a parameter vector which determines the fragmentation behavior. The $1/n_{jk}!$ are the expected Gibbs factors which arise from the exchange statistics of identical fragments. The solution of this system can be derived from the following property of the partition function:

$$\frac{\partial Z_{ZN}}{\partial x_{jk}} = Z_{Z-j,N-k}(\mathbf{x}), \tag{4}$$

where $Z_{jk}(\mathbf{x}) = 0$ when $j < 0$ or $k < 0$. This property can be seen by taking a derivative of the grand canonical partition function $\mathcal{Z}(u,v,\mathbf{x}) = \sum_{ZN} Z_{ZN}(\mathbf{x}) u^Z v^N = \exp(\sum_{jk} x_{jk} u^j v^k)$ and comparing terms.

From this property any moment can be calculated in terms of the partition functions. For example,

$$\langle n_{jk} \rangle = \frac{x_{jk}}{Z_{ZN}(\mathbf{x})} \frac{\partial Z_{ZN}}{\partial x_{jk}} = x_{jk} \frac{Z_{Z-j,N-k}(\mathbf{x})}{Z_{ZN}(\mathbf{x})} \tag{5}$$

or in general,

$$\left\langle \prod_{jk} [n_{jk}]_{p_{jk}} \right\rangle = \frac{1}{Z_{ZN}(\mathbf{x})} \prod_{jk} x_{jk}^{p_{jk}} \frac{\partial^p Z_{ZN}}{\partial x_{10}^{p_{10}} \dots \partial x_{ZN}^{p_{ZN}}}$$
$$= \prod_{jk} x_{jk}^{p_{jk}} \frac{Z_{Z-\sum j p_{jk}, N-\sum k p_{jk}}(\mathbf{x})}{Z_{ZN}(\mathbf{x})}, \tag{6}$$

where $\mathbf{p}$ is a vector of integers, $p = \sum_{jk} p_{jk}$ and the falling factorial $[z]_k$ is given by $[z]_k = (z-k+1)[z]_{k-1}$, with $[z]_0 = 1$.

Of course, the partition functions are needed to compute these expressions. The constraints $\sum_{jk} j n_{jk} = Z$, $\sum_{jk} k n_{jk} = N$ hold for all configurations, so they must also hold for the expectation values. This allows us to express the partition function as a recursively defined function:

$$Z_{ZN}(\mathbf{x}) = \frac{1}{Z} \sum_{jk} j x_{jk} Z_{Z-j,N-k}(\mathbf{x}) ,$$
$$Z_{ZN}(\mathbf{x}) = \frac{1}{N} \sum_{jk} k x_{jk} Z_{Z-j,N-k}(\mathbf{x}) . \tag{7}$$

Since $Z_{00}(\mathbf{x}) = 1$, these two relations can be used to iteratively construct the set of partition functions for any parameter vector $\mathbf{x}$.

It should be noted that the technique used to solve this model applies equally well to any number of components. The N-component system of $A = \sum_{j=1}^{N} A_j$ particles with the weight

$$P_{A_1 \dots A_N}(\mathbf{n},\mathbf{x}) = \frac{1}{Z_{A_1 \dots A_N}(\mathbf{x})} \prod_{k_1 \dots k_N} \frac{x_{k_1 \dots k_N}^{n_{k_1 \dots k_N}}}{n_{k_1 \dots k_N}!} \tag{8}$$

has a recursively defined set of partition functions determined by the N equations

$$Z_{A_1 \dots A_N}(\mathbf{x}) = \frac{1}{A_j} \sum_{k_1 \dots k_N} k_j x_{k_1 \dots k_N} Z_{A_1 - k_1, \dots, A_N - k_N}(\mathbf{x}) \tag{9}$$

for $j = 1, \dots, N$ where $Z_{00\dots00}(\mathbf{x}) = 1$. All the ensemble averages can be determined solely from these partition functions and the parameter vector $\mathbf{x}$.

III. RELATION TO THE PERMUTATION GROUP

It is useful to consider an alternative representation of these models. Consider the set of all permutations on

A objects, the symmetric group S_A. Typically a permutation $p \in S_A$ is represented by a vector of integers $\mathbf{p} = (p_1, \ldots, p_A)$ where p_j is the result of the permutation acting on j, i.e., $j \xrightarrow{p} p_j$. We can also represent p by its cycle decomposition. In this representation, the permutation is a set of disjoint cycles of varying length. Each cycle contains all the numbers which permute among themselves under the action of the permutation.

If we identify each element in the permutation vector with a particle, we can consider a cycle in the cycle decomposition of a permutation as representing a cluster of particles. This defines a many-to-one map from the set of permutations S_A onto the set of partitions Π_A by cycle decomposition. For example, Table II details this for $A = 4$. The number of permutations which map to a particular partition of A was computed by Cauchy, and is given by

$$M_2(\mathbf{n}) = A! \prod_{k=1}^{A} \frac{1}{n_k! k^{n_k}} \tag{10}$$

which satisfies the identity $\sum_{\mathbf{n}\in\Pi_A} M_2(\mathbf{n}) = A!$. Since this is of the same form as the weight given by the one-component partition model, we see that the model given by the permutation weight

TABLE II. The twenty-four permutations of S_4 and their partition representations. The first two columns give the typical representation and cycle decomposition of the permutations. The third column gives the equivalent one-component partition vectors $\mathbf{n} = (n_1, n_2, n_3, n_4)$. The last column gives the equivalent two-component partition vectors $\mathbf{n} = (n_{01}, n_{02}, n_{10}, n_{11}, n_{12}, n_{20}, n_{21}, n_{22})$ for $Z = 2$, $N = 2$.

$\mathbf{p}$	p	$\mathbf{n}_4$	$\mathbf{n}_{22}$
(1,2,3,4)	(1)(2)(3)(4)	(4,0,0,0)	(2,0,2,0,0,0,0,0)
(1,2,4,3)	(1)(2)(3,4)	(2,1,0,0)	(0,0,2,0,0,1,0,0)
(1,3,2,4)	(1)(2,3)(4)	(2,1,0,0)	(1,0,1,1,0,0,0,0)
(1,3,4,2)	(1)(2,3,4)	(1,0,1,0)	(0,0,1,0,1,0,0,0)
(1,4,2,3)	(1)(2,4,3)	(1,0,1,0)	(0,0,1,0,1,0,0,0)
(1,4,3,2)	(1)(2,4)(3)	(2,1,0,0)	(1,0,1,1,0,0,0,0)
(2,1,3,4)	(1,2)(3)(4)	(2,1,0,0)	(2,1,0,0,0,0,0,0)
(2,1,4,3)	(1,2)(3,4)	(0,2,0,0)	(0,1,0,0,0,1,0,0)
(2,3,1,4)	(1,2,3)(4)	(1,0,1,0)	(1,0,0,0,0,0,1,0)
(2,3,4,1)	(1,2,3,4)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(2,4,1,3)	(1,2,4,3)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(2,4,3,1)	(1,2,4)(3)	(1,0,1,0)	(1,0,0,0,0,0,1,0)
(3,1,2,4)	(1,3,2)(4)	(1,0,1,0)	(1,0,0,0,0,0,1,0)
(3,1,4,2)	(1,3,4,2)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(3,2,1,4)	(1,3)(2)(4)	(2,1,0,0)	(1,0,1,1,0,0,0,0)
(3,2,4,1)	(1,3,4)(2)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(3,4,1,2)	(1,3)(2,4)	(0,2,0,0)	(0,0,0,2,0,0,0,0)
(3,4,2,1)	(1,3,2,4)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(4,1,2,3)	(1,4,3,2)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(4,1,3,2)	(1,4,2)(3)	(1,0,1,0)	(1,0,0,0,0,0,1,0)
(4,2,1,3)	(1,4,3)(2)	(1,0,1,0)	(0,0,1,0,1,0,0,0)
(4,2,3,1)	(1,4)(2)(3)	(2,1,0,0)	(1,0,1,1,0,0,0,0)
(4,3,1,2)	(1,4,2,3)	(0,0,0,1)	(0,0,0,0,0,0,0,1)
(4,3,2,1)	(1,4)(2,3)	(0,2,0,0)	(0,0,0,2,0,0,0,0)

$$P_A(p, \mathbf{z}) = \frac{1}{Z_A(\mathbf{z})} \prod_{j=1}^{A} z_j^{n_j(p)} \tag{11}$$

is equivalent to the partition model given by Eq. (1) with x_j given by

$$x_j = \frac{z_j}{j}. \tag{12}$$

For example, if $z_j = x$, each permutation contributes x^m to the partition function, where m is the total number of cycles. There are $|S_A^{(m)}|$ permutations with m cycles, where $S_A^{(m)}$ is the Stirling number of the first kind. Therefore, the partition function is given by $Z_A(x) = \sum_{m=1}^{A} |S_A^{(m)}| x^m = x(x+1)\cdots(x+A-1)$, a result which can be found in Goncarov's study of the distribution of cycles in permutations [28].

We can also map the set of permutations S_A onto the set of partitions Π_{ZN} where $A = Z + N$ in an analogous way. Identify elements $1, \ldots, Z$ in the permutation as protons, and elements $Z+1, \ldots, Z+N$ as neutrons. Each cycle in the permutation is mapped to a cluster by counting the number of "neutrons" and "protons" in that particular cycle. As an example, this is done for $Z = 2$, $N = 2$ in Table II. The number of permutations which map in this way to a particular $\mathbf{n} \in \Pi_{ZN}$ is given by the two-component generalization of Cauchy's formula

$$M_2(\mathbf{n}) = Z!N! \prod_{jk} \frac{1}{n_{jk}!} \left(\frac{(j+k-1)!}{j!k!} \right)^{n_{jk}} \tag{13}$$

which also satisfies the identity $\sum_{\mathbf{n}\in\Pi_{ZN}} M_2(\mathbf{n}) = A!$. Since this number is of the same form as Eq. (3), the permutation model given by

$$P_{ZN}(p, \mathbf{z}) = \frac{1}{Z_{ZN}(\mathbf{z})} \prod_{jk} z_{jk}^{n_{jk}(p)} \tag{14}$$

is equivalent to the model given by Eq. (3) with

$$x_{jk} = z_{jk} \frac{(j+k-1)!}{j!k!}. \tag{15}$$

This can be generalized to any number of components. The N-component generalization of Cauchy's number for $A = \sum_{j=1}^{N} A_j$ is given by

$$M_2(\mathbf{n}) = \prod_{j=1}^{N} A_j! \prod_{k_1 \ldots k_N} \frac{1}{n_{k_1 \ldots k_N}!} \left(\frac{(\sum_j k_j - 1)!}{k_1! \cdots k_N!} \right)^{n_{k_1 \ldots k_N}} \tag{16}$$

and the permutation model

$$P_{A_1 \ldots A_N}(\mathbf{z}) = \frac{1}{Z_{A_1 \ldots A_N}(\mathbf{z})} \prod_{k_1 \ldots k_N} z_{k_1 \ldots k_N}^{n_{k_1 \ldots k_N}(p)} \tag{17}$$

is equivalent to the partition model given by Eq. (8) with $x_{k_1 \ldots k_N}$ given by

$$x_{k_1 \ldots k_N} = z_{k_1 \ldots k_N} \frac{(\sum_j k_j - 1)!}{k_1! \cdots k_N!}. \quad (18)$$

This treatment of permutations of distinguishable groups of indistinguishable particles is similar to the notion of multiset permutations in the mathematical literature, except that we treat the particles in the permutation representation as ultimately completely distinguishable, and only concern ourselves with the indistinguishability when carried over to the partition representation. This is essentially a variation on the idea of representing permutations as colored partitions as presented in Ref. [27].

Using the two-component model as defined over the permutation group allows us to construct a closed-form solution for the partition function of a particular model. Consider Eq. (14) with $z_{jk} = x$. As in the one-component case, each permutation contributes x^m to the partition function, where m is the total number of cycles, and there are $|S_A^{(m)}|$ permutations with m cycles. Therefore, $Z_{ZN}(\mathbf{z}) = \sum_{m=1}^{A} |S_A^{(m)}| x^m = x(x+1)\cdots(x+A-1)$. By Eq. 15, this implies that the partition model with

$$x_{jk} = x\frac{(j+k-1)!}{j!k!} = \frac{x}{j+k}\binom{j+k}{j} \quad (19)$$

has its partition function given by the closed-form expression

$$Z_{ZN}(x) = \frac{1}{Z!N!}\frac{\Gamma(x+Z+N)}{\Gamma(x)}. \quad (20)$$

The expected number of fragments with j protons and k neutrons is

$$\langle n_{jk} \rangle = x\binom{Z}{j}\binom{N}{k}B(j+k, x+A-j-k), \quad (21)$$

where $B(x,y)$ is the Euler Beta function, $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The expected number of fragments of charge j, $n_j = \sum_{k=0}^{N} n_{jk}$ is given by

$$\langle n_j \rangle = x\binom{Z}{j}B(j, x+Z-j), \quad (22)$$

which can be shown by applying the Nörlund formula $[x+y]^n = \sum_{k=0}^{n}\binom{n}{k}[x]^k[y]^{n-k}$ to Eq. (21), where the rising factorial is given by $[x]^k = (x+k-1)[x]^{k-1}$ and $[x]^0 = 1$.

Notice that Eq. (22) is identical to the one-component expectation value for the model given by Eq. (1) with $x_j = x/j$ (see [3]). This is not a coincidence. In fact, the relation of the partition models to the permutation group allows us to prove in the next section that there exist two-component models of the form of Eq. (3) for every one-component model of the form of Eq. (1) such that the results of the one-component model are preserved when a summation over isotopes is performed. In this case, the two-component model given by Eq. (19) is equivalent to the one-component model given by $x_j = x/j$.

IV. ISOTOPIC AND PERMUTATION EQUIVALENCE

As mentioned in Sec. III, two-component models are sometimes equivalent to one-component models. In this section, we make this notion precise as well as show under what conditions this is true.

A two-component model is said to be *isotopically equivalent* to a one-component model if for every function $f(\mathbf{n})$, the one-component ensemble average $\langle f \rangle_Z = \sum_{\mathbf{n}\in\Pi_Z} f(\mathbf{n})P_Z(\mathbf{n})$, is equal to the two-component ensemble average $\langle f \rangle_{ZN} = \sum_{\mathbf{n}'\in\Pi_{ZN}} f(\mathbf{n}(\mathbf{n}'))P'_{ZN}(\mathbf{n}')$, where $n_j(\mathbf{n}') = \sum_{k=0}^{N} n'_{jk}$. For example, $\langle n_j \rangle_Z = \sum_{k=0}^{N}\langle n_{jk}\rangle_{ZN}$ must hold if two models are considered to be isotopically equivalent. As mentioned in the Introduction, this is true if and only if

$$P_Z(\mathbf{n}) = \sum_{\mathbf{n}'\in\Pi_{ZN}(\mathbf{n})} P'_{ZN}(\mathbf{n}'), \quad (23)$$

where $\Pi_{ZN}(\mathbf{n})$ denotes all the partition vectors $\mathbf{n}' \in \Pi_{ZN}$ such that $n_j = \sum_{k=0}^{N} n'_{jk}$.

If we consider deriving the models over the permutation group, we can define the notion of *permutation equivalence* of $P'_{Z'N'}(p')$ to $P_{ZN}(p)$ by the condition

$$P_{ZN}(p) = \sum_{p'\in S_{Z'+N'}(p)} P'_{Z'N'}(p'), \quad (24)$$

where $Z' \geq Z$, $N' \geq N$, and $S_{Z'+N'}(p)$ is the set of all permutations $p' \in S_{Z'+N'}$ which give p by the following construction. Decompose p' into its cycle representation. Eliminate the numbers $Z+1, \ldots, Z'$ and $Z'+N+1, \ldots, Z'+N'$ from the cycles. Renumber the elements $Z'+1, \ldots, Z'+N$ as $Z+1, \ldots, Z+N$ to recover the permutation in S_{Z+N}. In other words, if k is an element to eliminate, find j such that $p_j = k$, and modify the permutation vector so that $p_j = p_k$. This removes the element from the permutation. Do this for all the numbers to eliminate, resulting in a new permutation isomorphic to a permutation in S_{Z+N}. For example, $p' = (1,5)(3)(4,6,2) \in S_6$ becomes $(1,5)(4,2)$ by eliminating $3,6$. After renumbering, this becomes $p = (1,4)(2,3) \in S_4$.

Permutation equivalence of $P'_{ZN}(p')$ to $P_{Z0}(p)$ implies isotopic equivalence of $M_2(\mathbf{n}'(p'))P'_{ZN}(\mathbf{n}'(p'))$ to $M_2(\mathbf{n}(p))P_{Z0}(\mathbf{n}(p))$, assuming the permutation weights are only functions of $\mathbf{n}(p)$, i.e., all permutations which have the same partition configuration or cycle decomposition have the same weight. One particular permutation weight which satisfies this condition is the unnormalized weight $W_{ZN}(p,\mathbf{z}) = \prod_{jk} z_{jk}^{n_{jk}(p)}$. We will now show that this weight with $A = Z+N$ is permutationally equivalent to the $Z, N+1$ weight with the same z_{jk} if $z_{jk} = z_j$. This implies by induction that the $Z0$ model is permutationally equivalent to the ZN model, and therefore the two weights corrected by the Cauchy factor are isotopically equivalent.

For any permutation $p \in S_A$, there are $A+1$ per-

mutations in $S_{A+1}(p)$. One of these permutations has the element $A+1$ in its own cycle. It has weight $W_{Z,N+1}(p',\mathbf{z}) = W_{ZN}(p,\mathbf{z})z_{01}$. A of the permutations are hybrids, with the extra element combined with a cycle from p. Suppose the element is in a cycle of j neutrons, k protons. The weight of that cycle is $W_{Z,N+1}(p',\mathbf{z}) = W_{ZN}(p,\mathbf{z})z_{j,k+1}/z_{jk}$, since the introduction of the additional element has reduced n_{jk} by one, but increased $n_{j,k+1}$ by one. There are $j+k$ possible places for the element to be inserted into that cycle, and there are n_{jk} cycles of that type it can combine with. So the following is true:

$$\sum_{p'\in S_{A+1}(p)} \frac{W_{Z,N+1}(p')}{W_{ZN}(p)} = z_{01} + \sum_{jk} f_{jk}n_{jk}, \tag{25}$$

where $f_{jk} = (j+k)z_{j,k+1}/z_{jk}$. If z_{jk} is independent of k, this reduces to $z_{01}+N+Z$. In this case, we can multiply both sides of the equation by $W_{ZN}(p)$ and sum over $\forall p \in S_A$ to arrive at

$$Z_{Z,N+1}(\mathbf{z}) = Z_{ZN}(\mathbf{z})(z_{01}+Z+N)\,. \tag{26}$$

This implies Eq. (24) when substituted back into Eq. (25). So under these conditions, P_{ZN} is equivalent to $P_{Z,N+1}$. Therefore, P_{Z0} is equivalent to P_{ZN} by induction and transitivity, and Eq. (1) is equivalent to Eq. (3) for x_{jk} given by

$$x_{jk} = x_j \binom{j+k-1}{k}. \tag{27}$$

The partition function for this two-component model is

$$Z_{ZN}(\mathbf{x}) = \frac{\Gamma(z_{01}+Z+N)}{\Gamma(z_{01}+Z)} Z_Z(\mathbf{x}). \tag{28}$$

Notice that at no point was it asserted that these are the only two-component models that are isotopically equivalent to the one-component models. In the previous discussion, the key property was the independence of $\sum_{jk} f_{jk}n_{jk}$ from $\mathbf{n}$. Perhaps we could satisfy this with choices other than $f_{jk} = j+k$. In fact, it can be proven that this condition is satisfied only by $f_{jk} = \alpha Aj/Z + (1-\alpha)Ak/N$ where α is arbitrary. However, $\alpha \in [0,1]$ to guarantee non-negative probabilities in the statistical model. Given z_{j0}, we could then construct permutationally equivalent two-component models specified by α and the recursion relation

$$z_{j,k+1} = z_{jk}\frac{A}{j+k}\left(\alpha\frac{j}{Z} + (1-\alpha)\frac{k}{N}\right) \tag{29}$$

and evaluate the equivalent partition weights using Eqs. (3) and (15).

If we require that the models be isotopically equivalent for any choice of N, these are the only models given by Eqs. (1) and (3) that are isotopically equivalent. This is true because isotopic equivalence for any N implies permutation equivalence of N to $N+1$. Permutation equivalence requires $\sum_{jk} f_{jk}n_{jk}$ to be independent of $\mathbf{n}$, which can be shown to be true only if z_{jk} is of the form given above. Notice that this condition is very similar to Kingman's noninterference condition on one-component partition weights [29]. We require that the neutrons do not "interfere" with the overall proton distribution, just as in one-component models Kingman required that the addition of new objects does not interfere with the probability structure of the original set of objects.

This technique can be generalized to construct an $N+1$ component model that is equivalent to an N-component model. For example, if the N-component weight is given by Eq. (8), then an equivalent $N+1$ component model is given by the parameter vector defined by

$$x_{k_1\ldots k_{N+1}} = x_{k_1\ldots k_N}\binom{\sum_{j=1}^{N+1}k_j - 1}{k_{N+1}}. \tag{30}$$

These results provide a simple way of constructing two-component models from the one-component models explored in earlier papers. For example, one could apply Eqs. (27) and (28) to the chain model, $x_j = x$, to construct a two-component extension of the chain model as developed by Gross *et al.* [10].

V. APPLICATION TO NUCLEAR FRAGMENTATION

The two-component analogs of one-component models given by Eq. (29) could be taken as the starting point for a fully isotopic model of fragmentation. These models would contain no surprises in their one-component expectation values, and would provide predictions for the isotopic behavior.

This is perhaps too strong a requirement. Equation (29) constrains the set of possible partition weights, forbidding a number of physically interesting choices for x_{jk}. For example, if the symmetric term $\exp\{-\alpha_s(j-k)^2/(j+k)\}$ appeared in the parameter z_{jk}, the resulting parameter vector could never satisfy Eq. (29). Since such a symmetry term should appear due to the symmetry term in the binding energy of nuclei, we should not expect realistic models to be isotopically equivalent to the models considered in earlier papers.

With that in mind, we start with a simplified model of nuclear fragmentation specified by the following choice for x_{jk}:

$$x_{jk} = \frac{x}{(j+k)^\tau}\binom{j+k}{j}\exp\left\{-\frac{\alpha_s(j-k)^2}{j+k}\right\}. \tag{31}$$

We will find it appropriate sometimes to modify this when $j+k=1$, i.e., to avoid the exponential suppression factor $e^{-\alpha_s}$ for monomers. Let us specialize to the case $\tau = 1$ and consider the limiting cases. When $\alpha_s \to 0$, the model becomes the two-component analog of the model discussed in Sec. III. The partition function has a closed-form solution, which is given by Eq. (20). When $\alpha_s \to \infty$, x_{jk} is proportional to a Kronecker delta function and the model reduces to the one-component model specified by parameters

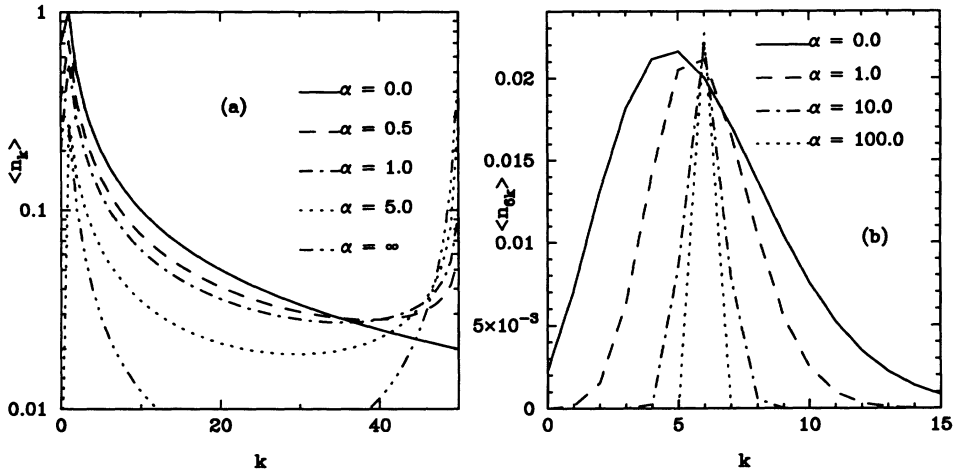

FIG. 1. Overall (a) and isotopic (b) distribution of fragments for $Z = 50$, $N = 50$, $x = 1$, $\tau = 1$ and various α_s. Isotopic distribution is for carbon, $Z = 6$.

$$x_k = \frac{x}{2k}\binom{2k}{k}, \tag{32}$$

where k indexes both proton and neutron number, since only symmetric fragments with $j = k$ are allowed. In this case, the partition function also has a closed-form solution

$$Z_A(x) = \frac{x}{A!}\frac{\Gamma(x+2A)}{\Gamma(x+A+1)}. \tag{33}$$

Besides, note that all models with $x_k = \frac{x}{nk}\binom{nk}{k}$ for $n = 1, 2, 3, \ldots$ have closed-form solutions given by $Z_A(x) = (x/A!)\{\Gamma(x+nA)/\Gamma[x+(n-1)A+1]\}$. Models with intermediate values for α_s are not as simple as these two limits, but can still be computed easily. Figures 1 and 2 show the prediction of this model at $x = 1$ and various α_s for a symmetric case $Z = N = 50$ as well as an asymmetric case $Z = 50$, $N = 75$. These figures display the total charge distribution $\langle n_j \rangle = \sum_{k=0}^{N} \langle n_{jk} \rangle$, as well as the distribution of isotopes for a particular charge, specifically carbon, i.e., $\langle n_{jk} \rangle$ for $j = 6$.

From the total charge distribution figures, we see that increasing α_s increases the number of large fragments. This is due to the exponential suppression of asymmetric fragments, which make up a large fraction of the partition space, especially for smaller fragments. The isotopic distribution figures reveal that the distribution of isotopes is essentially Gaussian. As seen earlier [5], an initial asymmetry in the relative numbers of protons and neutrons yields a similar asymmetry in the distribution of fragments. This asymmetry can be diminished by increasing α_s and removing the exponential suppression from monomer parameters (x_{01}, x_{10}), as was done in the figures. Indeed, as $\alpha_s \to \infty$, such a model would allow only symmetric fragments and monomers.

To this simple model, we can add other effects, eventually including all the terms in the semiempirical mass formula. For example, nucleon pairing interactions can be added using the term $\exp\{\alpha_p p_{jk}/(j+k)\}$ where p_{jk} is 0 for even-odd nuclei, -1 for odd-odd nuclei, and $+1$ for even-even nuclei. This can be done for all intrafragment interactions. Interfragment interactions do not fit the form of Eq. (3) and must be handled in some kind of mean-field manner in this approach. Coulomb interactions are the only persistent long-range interactions in

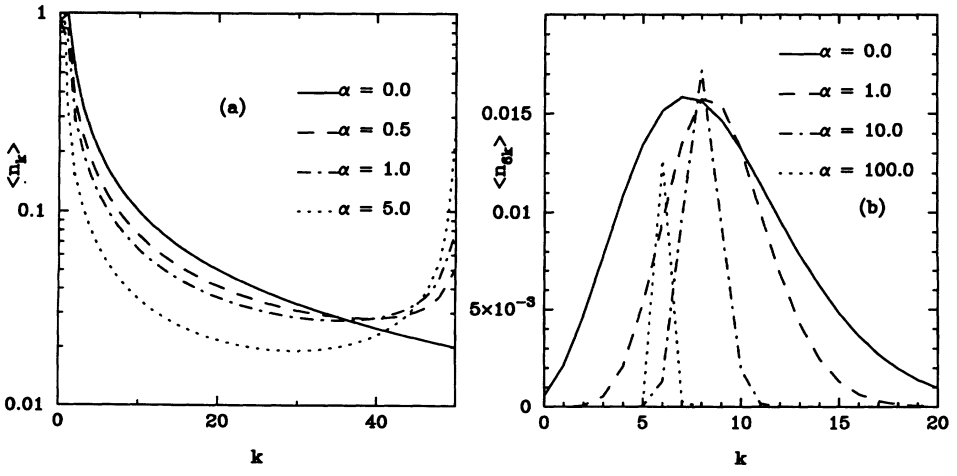

FIG. 2. Overall (a) and isotopic (b) distribution of fragments for $Z = 50$, $N = 75$, $x = 1$, $\tau = 1$ and various α_s. Isotopic distribution is for carbon, $Z = 6$.

the thermalized phase. Suppose each fragment interacts with all the others via the Coulomb interaction. This contributes an interaction energy given by

$$E_I = \frac{1}{2}\sum_{jklm} n_{jk}(n_{lm} - \delta_{jl}\delta_{km})E^I_{jklm}. \tag{34}$$

If we choose $E^I_{jklm} = jle^2/\langle r\rangle$ for the Coulomb interaction, then

$$E_I = \frac{e^2}{2\langle r\rangle}\left(Z^2 - \sum_{jk} j^2 n_{jk}\right), \tag{35}$$

which suggests that we include interfragment Coulomb interactions by appending the term $\exp(\alpha_c j^2)$ to x_{jk}. We can estimate α_c by assuming $\langle r\rangle \approx V^{1/3} \approx r_0 A^{1/3}$.

We postpone investigating such a realistic choice for x_{jk} for a later paper.

VI. CONCLUSION AND SUMMARY

This paper established a nontrivial exactly solvable isotopic model of fragmentation. The model is specified by a set of parameters x_{jk}, where x_{jk} includes all the physical parameters pertinent to the fragmenting behavior of clusters of j protons, k neutrons. This model is related to the permutation group and as a result, it was proven that for certain choices of x_{jk} it directly reduces to a previously developed one-component model when a summation over isotopes is performed.

The usefulness of this model in nuclear fragmentation lies in its simplicity and exact solvability. Expectation values relating to various experimental observables are easily derived from the partition functions of the model. In turn, the partition functions are related to the underlying weight given to each possible partition or fragmentation state of the system of protons and neutrons. For the weights considered in this paper, the partition functions were obtained as either closed form expressions or more generally as recursively defined functions, either of which can be used to quickly generate any observable of interest. The alternative approach of Monte Carlo importance sampling of the system allows for the observables to be built up statistically, with a diminishing error, but at a greater computational price. It has the advantage, however, of being applicable to any weight scheme. Here the approach taken was to restrict attention to a particular class of weight assignments which are amenable to analytic solution, hoping to keep enough of the relevant physics to still be a useful model.

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