q deformations in the interacting boson model for nuclei

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q deformations of the Hamiltonian with SO(6) or U(5) symmetry in the interacting s, d boson model are discussed. The deformations are introduced through q-deformed U(5), SO(3), and the boson pairing algebra SU(1,1). It is shown that the SO(6) or U(5) dynamical symmetry remains after deformation. The deformed Hamiltonian under the two limiting cases can be applied to describe energy spectra of certain isotopes. As examples, energy spectra and some $E2$ transition rates of even-even $^{110-114}$Cd, $^{190-196}$Pt, and $^{124-128}$Xe are fitted and compared with the experimental results. Finally, the physical meaning of q deformation is also discussed.

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I. INTRODUCTION

Quantum algebras [1–3] continue to attract the attention of mathematicians and physicists. Besides their applications to statistical mechanics, and conformal field theory [4–6], there have been several applications in molecular spectroscopy [7–9], the fine structure of the hydrogen atom [10], the *aufbau prinzip* for atoms and monoatomic ions [11], squeezed states [12,13], the nucleon pairing problem [14–16], and rotational spectra of deformed and superdeformed nuclei [17–19]. A q-deformed two-dimensional toy interacting boson model (IBM) was first proposed by Bonatsos *et al.* [20,21], in which they constructed the toy IBM Hamiltonian in terms of q-deformed boson creation and annihilation operators. However, the extension of this construction to the real IBM is rather difficult because the q-deformed s, d boson algebra should satisfy the coproduct rule which does not meet the needs of the properties of the subalgebras of $U_q(6)$ in terms of three well-known chains to $SO_q(3)$. Actually, some dynamical symmetries will be lost if one persists in constructing the q-deformed IBM Hamiltonian in terms of q-deformed s and d boson operators directly. One can construct $U_q(6)$ from s and d q-boson operators. Nevertheless, one cannot get the subalgebra chains corresponding to the three IBM limits. In fact, $SO_q(6)$, $SO_q(5)$, as well as $SO_q(3)$ are not the subalgebra of $U_q(6)$ in this case.

In this paper, it will be shown that the q deformations can be made through another way, namely, the q deformation of Lie algebras rather than of boson creation and annihilation operators, which can be achieved by using deforming functionals initiated by Curtright and Zachos [22,23]. In Sec. II, we will discuss q deformations of U(n), SU(2), and SU(1,1) which are useful for our purpose. In Sec. III, we will construct the q-deformed IBM Hamiltonians for the SO(6) and SO(5) limiting cases. As examples, the energy spectra and some $B(E2)$ values of some even-even Cd, Xe, and Pt isotopes are calculated and compared with the experimental values. Finally, the physical meaning of the q deformations will briefly be discussed.

II. q DEFORMATIONS OF U(n), SU(2), AND SU(1,1)

Before discussion on the q deformation of the IBM, we will give a short introduction to q deformations of U(n), SU(2), and SU(1,1). In the following, we always assume that q is not a root of unity.

A. q deformation of symmetric boson algebra U(n)

The generators of the symmetric boson algebra U(n) can be written as $a_i^\dagger a_j$ with $1 \le i, j \le n$, where $a_i^\dagger$ (a_i) are the boson creation (annihilation) operators. Then, generators of q-deformed algebra $U_q(n)$ can be realized in terms of U(n) generators by using the deforming functionals [24]

$$E_{ij}^q = F_{ij}(g) = \left(\frac{[a_i^\dagger a_i]_q}{a_i^\dagger a_i}\right)^{1/2} a_i^\dagger a_j \left(\frac{[a_j^\dagger a_j]_q}{a_j^\dagger a_j}\right)^{1/2} , \quad (1)$$

for $q \le i, j \le n$, where as usual, for given x,

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} . \quad (2)$$

We always assume that the operators given by (1) are acting on the representations in which $a_i^\dagger a_i$ with $i = 1, 2, \ldots, n$ are integer valued, i.e., $a_i^\dagger a_i \in \mathbf{N}$. In fact, the functionals given by (1) can be realized under any basis of U(n) because any basis of U(n) can be constructed in terms of these boson creation operators.

Because of the commutation relations of U(n), the functionals (1) indeed satisfy the commutation relations and Serre relations of $U_q(n)$. The maps F_{ij} are functionals of the U(n) generators g: $a_i^\dagger a_j$. In our application the generators E_{ij}^q should be Hermitian. One should keep the quantity under the square root real in

(1). This requires that q is real or a phase ($q = e^{i|\tau|}$) with $2k\pi < |\tau| N < (2k+1)\pi$, where $k = 0, 1, 2, \ldots$, and

$$N = \sum_{i=1}^{n} a_i^\dagger a_i \tag{3}$$

is the total number of bosons. Consequently, the functionals (1) are invertible, and their inverses $\mathbf{F}^{-1}$ provide a realization of U(n) in terms of quantum algebra U$_q$(n) generators.

B. q deformation of SU(2) and SU(1,1)

We assume that generators of SU(2)~SO(3), $j_\pm, j_0$, satisfy the commutation relations

$$[j_0, j_\pm] = \pm J_\pm, \quad [j_+, j_-] = j_0 \; . \tag{4}$$

The Casimir operator of SU(2) can be written as

$$\mathbf{C}_2 = 2j_- j_+ + j_0(j_0 + 1) \; . \tag{5}$$

The deforming functionals for SU$_q$(2) were made by Curtright and Zachos [22,23]. In this case, one first rewrites the classical Casimir operator $\mathbf{C}_2$ of SU(2) as $\mathbf{j}(\mathbf{j}+1)$, where $\mathbf{j}$ is the formal operator $[(1+4\mathbf{C}_2)^{1/2}-1]/2$. Then, because of the commutation relations of SU(2) generators $j_\pm$ and j_0, the functionals

$$J_0^q = Q_0(g) = j_0 \; ,$$

$$J_+^q = Q_+(g) = \{[j_0 + \mathbf{j}]_q [j_0 - \mathbf{j} - 1]_q / 2(j_0 + \mathbf{j})(j_0 - \mathbf{j} - 1)\}^{1/2} j_+ \; , \tag{6}$$

$$J_-^q = Q_-(g) = j_- \{[j_0 + \mathbf{j}]_q [j_0 - \mathbf{j} - 1]_q / 2(j_0 + \mathbf{j})(j_0 - \mathbf{j} - 1)\}^{1/2} \; ,$$

where the operators of (6) are acting on the representations in which $\mathbf{j}$ and j_0 are diagonal simultaneously, satisfy the commutation relations of SU$_q$(2). In this case, $J_\pm^q$ are Hermitian when q is real or a phase ($q = e^{i|\tau|}$) with

$$2k\pi < |\tau|(2j_{\max} + 1) < (2k+1)\pi \; ,$$

where $k = 0, 1, 2, \ldots$.

The SU(1,1) algebra is generated by $S_\pm, S_0$, which satisfy

$$[S_0, S_\pm] = \pm S_\pm, [S_+, S_-] = -2S_0 \; . \tag{7}$$

The Casimir operator of SU(1,1) is

$$\mathbf{C}_2(\mathrm{SU}(1,1)) = S_0(S_0 - 1) - S_+ S_- \; . \tag{8}$$

Let $|\kappa\mu\rangle$ be the basis vectors of SU(1,1), where κ can be any positive real number, and $\mu = \kappa, \kappa+1, \ldots$. We have

$$\begin{gathered} \mathbf{C}_2(\mathrm{SU}(1,1))|\kappa\mu\rangle = \kappa(\kappa-1)|\kappa\mu\rangle \; , \\ S_0|\kappa\mu\rangle = \mu|\kappa\mu\rangle \; . \end{gathered} \tag{9}$$

Using the commutations relations given by (7), one can check that the deforming functions

$$S_0^q = S_0 \; ,$$

$$\begin{aligned} S_+^q = \{[S_0 + S - 1]_q [S_0 - S]_q / (S_0 + S - 1) \\ \times (S_0 - S)\}^{1/2} S_+ \; , \end{aligned} \tag{10}$$

$$\begin{aligned} S_-^q = S_- \{[S_0 + S - 1]_q [S_0 - S]_q / (S_0 + S - 1) \\ \times (S_0 - S)\}^{1/2} \end{aligned}$$

satisfy the SU$_q$(1,1) algebra [25,26]

$$[S_0^q, S_\pm^q] = \pm S_\pm^q, \quad [S_+^q, S_-^q] = -[2S_0^q]_q \; . \tag{11}$$

The Casimir operator of SU$_q$(1,1) can then be written as

$$\mathbf{C}_2(\mathrm{SU}_q(1,1)) = [S_0^q]_q [S_0^q - 1]_q - S_+^q S_-^q \tag{12}$$

with eigenvalue

$$C_2(\mathrm{SU}_q(1,1)) = [\kappa]_q [\kappa - 1]_q \; . \tag{13}$$

In this case $S_\pm^q$ is Hermitian only when q is real.

III. q DEFORMATION IN THE SO(6) AND U(5) LIMITING CASES

A. U(5) limiting case

In the IBM U(5) limiting case the Hamiltonian can be expressed in terms of Casimir operators of U(5), SO(5), and SO(3) [27,28]

$$\begin{aligned} H = AC_1(\mathrm{U}(5)) + BC_2(\mathrm{U}(5)) + CC_2(\mathrm{SO}(5)) \\ + DC_2(\mathrm{SO}(3)) \; , \end{aligned} \tag{14}$$

where A, B, C, and D are parameters. The wave functions of this limit are denoted, as usual, by $|Nn_d\nu\alpha LM\rangle$. The q deformations of U(6)⊃U(5)⊃SO(5)⊃SO(3) can be achieved in the following way. First, by using the deforming functionals for SO(3) given by (6), the q-deformed angular momentum operators in the IBM can be written as

$$L_0^{q''} = L_0 = \sqrt{10}(d^\dagger \tilde{d})_0^1 ,$$

$$L_+^{q''} = \sqrt{10}\{[L_0 + L]_{q''}[L_0 - L - 1]_{q''}/(L_0 + L)(L_0 - L - 1)\}^{1/2}(d^\dagger \tilde{d})_1^1 , \tag{15}$$

$$L_-^{q''} = \sqrt{10}(d^\dagger \tilde{d})_{-1}^1\{[L_0 + L]_{q''}[L_0 - L - 1]_{q''}/(L_0 + L)(L_0 - L - 1)\}^{1/2} ,$$

where L is the formal operator $\{(1+4C_2)^{1/2} - 1\}/2$, and C_2 is the Casimir operator of SO(3). In order to keep $L_\pm^{q''}$ Hermitian, q'' should be real or a phase ($q'' = e^{i|\tau''|}$) with $2k\pi < (4N+1)|\tau''| < (2k+1)\pi$, where $k = 0, 1, 2, \ldots$. Secondly, we know that the basis of U(5)$\supset$SO(5) is simultaneously the basis of $\mathrm{SU}^d(1,1) \supset$U(1). Their complementary relation can be expressed as [29,30]

$$|Nn_d\nu\alpha LM\rangle = |N, \kappa^d = \tfrac{1}{2}(\nu + \tfrac{5}{2}) ,$$
$$\mu^d = \tfrac{1}{2}(n_d + \tfrac{5}{2}), \alpha LM\rangle , \tag{16}$$

where κ^d and μ^d are quantum numbers of SU^d(1,1) and U(1), respectively. In the IBM, the generators of SU^d(1,1) are nothing but the d-boson pairing algebra

$$S_+^d = (d^\dagger \cdot d^\dagger)/2, \quad S_-^d(\tilde{d} \cdot \tilde{d})/2 ,$$
$$S_0^d = \tfrac{1}{4}\sum_\nu (d_\nu^\dagger d_\nu + d_\nu d_\nu^\dagger) . \tag{17}$$

Then the Casimir operators of SO(5) can also be expressed in terms of the Casimir operators of SU^d(1,1). In fact,

$$\mathbf{C}_2(\mathrm{SU}^d(1,1)) = S_0^d(S_0^d - 1) - \tfrac{1}{4}(d^\dagger \cdot d^\dagger)(\tilde{d} \cdot \tilde{d}) \tag{18}$$

with

$$\mathbf{C}_2(\mathrm{SU}^d(1,1)) = \tfrac{5}{16} + \tfrac{1}{4}\mathbf{C}_2(\mathrm{SO}(5)) . \tag{19}$$

Thus, the generators of the q-deformed algebra SU_q(1,1) can be expressed in terms of $S_\pm, S_0$ given by (17). They are

$$S_0^{q'}(d) = S_0^d ,$$

$$S_+^{q'}(d) = \{[S_0^d + S^d - 1]_{q'}[S_0^d - S^d]_{q'}/(S_0^d + S^d - 1)(S_0^d - S^d)\}^{1/2} S_+^d , \tag{20}$$

$$S_-^{q'}(d) = S_-^d\{[S_0^d + S^d - 1]_{q'}[S_0^d - S^d]_{q'}/(S_0^d + S^d - 1)(S_0^d - S^d)\}^{1/2} ,$$

where S^d is a formal operator

$$\tfrac{1}{2}\{1 + [4\mathbf{C}_2(\mathrm{SU}^d(1,1)) + 1]^{1/2}\} .$$

Because of the complementary relation (16), the operators $S_\pm^{q'}(d)$ are Hermitian when q' is real or a phase ($q' = e^{i|\tau'|}$) with

$$2k\pi < |\tau'|(N + \tfrac{3}{2}) < (2k+1)\pi ,$$

where $k = 0, 1, 2, \ldots$. The Casimir operator of $\mathrm{SU}_{q'}(1,1)$ acting on the basis vector $|Nn_d\nu\alpha LM\rangle$ gives

$$\mathbf{C}_2(\mathrm{SU}_{q'}(1,1))|Nn_d\nu\alpha LM\rangle$$
$$= [(2\nu+1)/4]_{q'}[(2\nu+5)/4]_{q'}|Nn_q\nu\alpha LM\rangle . \tag{21}$$

Finally, using (1) we can write the Casimir operator of $\mathrm{U}_q(5)$ as [31]

$$C_2(\mathrm{U}_q(5)) = \sum_{i=1}^{5}(q^{2i-6+2E_{ii}^q} - q^{2n_d/5})/(q - q^{-1})^2 + \sum_{i>j} q^{E_{ii}^q + E_{jj}^q + 2i - 7} E_{ji}^q E_{ij}^q , \tag{22}$$

where $d_\mu^\dagger$ (d_μ) with $\mu = -2, -1, 0, 1, 2$ are denoted as $a_i^\dagger$ (a_i) with $i = 1, 2, \ldots, 5$. In this case q should be real because the second-order Casimir operators of $\mathrm{U}_q(5)$ are nonunitary if q is a complex number. The eigenvalue of (22) is

$$C_2(\mathrm{U}_q(5)) = \{[5]_q - 5q^{2n_d/5} + (q^{2n_d} - 1)q^4\}/(q - q^{-1})^2 . \tag{23}$$

When $q \to 1$

$$C_2(\mathrm{U}_q(5))|_{q\to 1} = [2n_d(n_d + 5) + 25]/5$$
$$= 2C_2(\mathrm{U}(5))/5 + 5 . \tag{24}$$

In the following, we assume that the general q-deformed Hamiltonian can be expressed in terms of the Casimir operators of $U_q(5)$, $SU_{q'}(1,1)$, and $SO_{q''}(3)$, where q, q', and q'' are different new deformation parameters which should satisfy the conditions given above. Hence we have

$$\mathbf{H}_{qq'q''} = A[n_d]_q + B\mathbf{C}_2(\mathrm{U}_q(5)) + C\mathbf{C}_2(\mathrm{SU}_{q'}(1,1)) + D\mathbf{C}_2(\mathrm{SO}_{q''}(3)) , \tag{25}$$

where the parameters A, B, C, and D may also be deformation parameter dependent. Under the basis vectors $|Nn_d\nu\alpha LM\rangle$, the eigenvalue of (25) is given by

$$E_{qq'q''}(n_d\nu L) = A[n_d]_q + B\{[5]_q - 5q^{2n_d/5} + (q^{2n_d}-1)q^4\}/(q-q^{-1})^2 + C[(2\nu+5)/4]_{q'}[(2\nu+1)/4]_{q'} + D[L]_{q''}[L+1]_{q''} . \tag{26}$$

It should be noted that the eigenvectors of the deformed IBM Hamiltonian given by (25) are just the basis vectors of U(6)⊃U(5)⊃SO(5)⊃SO(3). That means the U(6)⊃U(5)⊃SO(5)⊃SO(3) dynamical symmetry remains after deformation.

B. SO(6) limit

In this limit the Hamiltonian can be written as

$$\mathbf{H} = \alpha\mathbf{C}_2(\mathrm{SO}(6)) + \beta\mathbf{C}_2(\mathrm{SO}(5)) + \gamma\mathbf{C}_2(\mathrm{SO}(3)) . \tag{27}$$

The basis vectors of U(6)⊃SO(6)⊃SO(5)⊃SO(3) are denoted by $|N\sigma\nu\alpha LM\rangle$. Similarly to the U(5) limit case, the basis vectors of this chain are simultaneously the basis vectors of $SU^{sd}(1,1)$, $SU^{d}(1,1)$, where $SU^{sd}(1,1)$ is the s,d boson pairing algebra generated by

$$\begin{aligned} S_+^{sd} &= (d^\dagger\cdot d^\dagger - s^\dagger s^\dagger)/2 ,\\ S_-^{sd} &= (\tilde d\cdot\tilde d - ss)/2 ,\\ S_0^{sd} &= \tfrac14\sum_\mu(d_\mu^\dagger d_\mu + d_\mu d_\mu^\dagger) + \tfrac14(s^\dagger s + ss^\dagger) , \end{aligned} \tag{28}$$

with the complementary relation

$$\begin{aligned} |\sigma\nu\alpha LM\rangle &= |N, \kappa^{sd} = (\sigma+3)/2 ,\\ \mu^{sd} &= (N+3)/2,\ \ \kappa^d = (\nu+\tfrac52)/2, \alpha LM\rangle . \end{aligned} \tag{29}$$

It should be noted that μ^d is not a good quantum number in this case.

The generators of the q-deformed algebra $SU_q^{sd}(1,1)$ can be expressed in terms of $SU^{sd}(1,1)$ generators through the following q-deforming functionals:

$$\begin{aligned} S_0^q(sd) &= S_0^{sd} ,\\ S_+^q(sd) &= \{[S_0^{sd}+S^{sd}-1]_q[S_0^{sd}-S^{sd}]_q/(S_0^{sd}+S^{sd}-1)(S_0^{sd}-S^{sd})\}^{1/2}S_+^{sd} ,\\ S_-^q(sd) &= S_-^{sd}\{[S_0^{sd}+S^{sd}-1]_q[S_0^{sd}-S^{sd}]_q/(S_0^{sd}+S^{sd}-1)(S_0^{sd}-S^{sd})\}^{1/2} . \end{aligned} \tag{30}$$

In this case, $S_\pm^q(sd)$ is Hermitian when q is real or a phase ($q=e^{i|\tau|}$) with $2k\pi < |\tau|(N+2) < (2k+1)\pi$ for $k=0,1,2,\ldots$.

Similarly to the $SU_{q'}^d(1,1)$ case, the eigenvalue of $\mathbf{C}_2(SU_q^{sd}(1,1))$ is

$$C_2(\mathrm{SU}_q^{sd}(1,1)) = [(\sigma+1)/2]_q[(\sigma+3)/2]_q . \tag{31}$$

When $q\to1$,

$$C_2(\mathrm{SU}^{sd}(1,1))|_{q\to1} = \tfrac14 C_2(\mathrm{SO}(6)) + \tfrac34 . \tag{32}$$

Similarly to the U(5) limit case, the general q-deformed Hamiltonian can then be written as

$$\mathbf{H}_{qq'q''} = \alpha\mathbf{C}_2(\mathrm{SU}_q^{sd}(1,1)) + \beta\mathbf{C}_2(\mathrm{SU}_{q'}^{d}(1,1)) + \gamma\mathbf{C}_2(\mathrm{SO}_{q''}(3)) \tag{33}$$

with eigenvalue

$$\begin{aligned} E_{qq'q''} &= \alpha[(\sigma+1)/2]_q[(\sigma+3)/2]_q\\ &\quad+\beta[(2\nu+1)/4]_{q'}[(2\nu+5)/4]_{q'}\\ &\quad+\gamma[L]_{q''}[L+1]_{q''} , \end{aligned} \tag{34}$$

where q, q', and q'' should satisfy the conditions given before, and α, β, and γ may also be deformation parameter dependent. Because the eigenvectors of Eq. (33) are just the basis vectors of U(6)⊃SO(6)⊃SO(5)⊃SO(3), the U(6)⊃SO(6)⊃SO(5)⊃SO(3) dynamical symmetry still remains after deformation.

C. *E*2 transition rates

In the original IBM, the $E2$ transition operator is usually written as

$$T_\mu(E2) = e_2(s^\dagger\tilde d + d^\dagger s)_\mu^{(2)} . \tag{35}$$

In the q-deformed IBM, one can write the E_2 operator in terms of q-deformed s and d boson operators, namely,

$$T_\mu(E2,q) = e_2(q)[s^\dagger(q)\tilde{d}(q) + d^\dagger(q)s(q)]^2_\mu\ , \tag{36}$$

where $e_2(q)$ is the effective charge which may also be q dependent. Equation (36) can also be expressed in terms of the usual s and d boson operators similarly to Eq. (1),

$$T_\mu(E2,q) = e_2(q)\left[\left(\frac{[N-n_d][n_d+1]_q}{(N-n_d)(n_d+1)}\right)^{1/2} s^\dagger\tilde{d} + \left(\frac{[n_d]_q[N-n_d+1]_q}{n_d(N-n_d+1)}\right)^{1/2} d^\dagger s\right]^2_\mu\ , \tag{37}$$

where the following q deformation for s and d boson operators has been chosen [24,41]:

$$s^\dagger(q) = \left(\frac{[n_s]_q}{n_s}\right)^{1/2} s^\dagger\ ,\qquad d^\dagger_\mu(q) = \left(\frac{[n_d]_q}{n_d}\right)^{1/2} d^\dagger_\mu\ . \tag{38}$$

Other deformations of Eq. (35) are also possible, but the calculation will be rather complicated.

In the U(5) limit case, reduced matrix elements of Eq. (37) can be calculated directly. However, in the SO(6) limit case, the d-boson number operator n_d as well as n_s are not good quantum numbers. In order to calculate the reduced matrix elements of Eq. (37) in this case, the SO(6)$\supset$SO(5)$\supset$SO(3) basis vectors can be expanded in terms of U(5)$\supset$SO(5)$\supset$SO(3) basis vectors [30]. Then the reduced matrix elements in the SO(6) limit can be obtained from those in the U(5) limit case. The results are very complicated because no closed form of the reduced matrix elements exists. For example, we have

$$B(E2;\tau+1,2\tau+2\to\tau,2\tau) = e_2^2(q)\frac{\tau+1}{2\tau+5}F^2(N,q,\tau)\ , \tag{39}$$

and

$$B(E2;\tau+1,2\tau\to\tau,2\tau) = e_2^2(q)\frac{(4\tau+2)}{(4\tau-1)(2\tau+5)}F^2(N,q,\tau)\ , \tag{40}$$

where

$$F(N,q,\tau) = \left(\frac{N+\tau+4}{N-\tau}\right)^{1/2} \times \sum_{n_d=\text{even}}\left(\frac{[n_d+1]_q[N-n_d]_q(N-n_d)}{n_d+1}\right)^{1/2}\frac{(N-\tau)!(N+\tau+3)!(n_d-\tau+1)!!}{2^{N+1}(N+1)!(N-n_d)!(n_d-\tau+1)!(n_d+\tau+3)!!}$$
$$+\left(\frac{N-\tau}{N+\tau+4}\right)^{1/2}\sum_{n_d=\text{odd}}\left(\frac{[n_d+2]_q[N-n_d]_q(N-n_d)}{n_d+2}\right)^{1/2}\frac{(N-\tau-1)!(N+\tau+4)!(n_d-\tau)!!}{2^{N+1}(N+1)!(N-n_d)!(n_d+\tau+4)!!(n_d-\tau)!} \tag{41}$$

In Table I, we list some theoretical $B(E2)$ values in the U(5) limit case, which are calculated by using Eq. (37).

It can be seen that the q-deformed version of $T(E2)$ (37) is analogous to the original IBM so that the selection rules are the same as those in the original model.

TABLE I. Theoretical $B(E2)$ values in the q-deformed U(5) limit.

Transition	$B(E2)$ values
$2_1^+\to 0_1^+$	$e_2^2(q)[N]_q$
$2_2^+\to 0_1^+$	0
$4_1^+\to 2_1^+$	$e_2^2(q)[N-1]_q[2]_q$
$6_1^+\to 4_1^+$	$e_2^2(q)[N-2]_q[3]_q$
$0_3^+\to 2_1^+$	0
$0_3^+\to 2_2^+$	$e_2^2(q)[N-2]_q[3]_q$
$2_2^+\to 2_1^+$	$e_2^2(q)[N-1]_q[2]_q$
$0_2^+\to 2_1^+$	$e_2^2(q)[N-1]_q[2]_q$
$2_3^+\to 0_1^+$	0
$2_3^+\to 0_2^+$	$\frac{7}{15}e_2^2(q)[N-2]_q[3]_q$

Furthermore, the ratios of $B(E2)$ values along the yrast band are close to those in the original model, and the ratios decrease with increase of the deformation parameter $|\tau|$. A comparison of q-deformed $B(E2)$ ratios with those in the original model for both SO(6) and U(5) limits is shown in Fig. 1.

One can also calculate explicitly the following $B(E2)$ ratio:

$$R(N,|\tau|) = \frac{B(E2,4_1^+\to 2_1^+)}{B(E2,2_1^+\to 0_1^+)}\ , \tag{42}$$

where $|\tau|$ is the deformation parameter. In Table II, we list some R values for different $|\tau|$ in both limits when $N\to\infty$. The facts presented above manifest that all features of the original IBM remain after the deformation, while $B(E2)$ ratios decrease with increase of the deformation parameter $|\tau|$.

TABLE II. $B(E2)$ ratio R in SO(6) and U(5) limits when $N \to \infty$.

U(5)			SO(6)		
$\vert\tau\vert = 0$	$\vert\tau\vert = 0.01$	$\vert\tau\vert = 0.1$	$\vert\tau\vert = 0$	$\vert\tau\vert = 0.1$	$\vert\tau\vert = 0.2$
2	1.98	1.82	$\frac{10}{7}$	$0.9991 \times \frac{10}{7}$	$0.9986 \times \frac{10}{7}$

D. Comparison with the experimental results

We take even-even $^{110-114}$Cd as an example for the q-deformed U(5) limit, and even-even $^{190-196}$Pt and $^{124-128}$Xe for the q-deformed SO(6) limit. The parameters A, B, C, and D in the U(5) limit case are all taken to be deformation parameter independent, while γ is chosen to be $\gamma(q) = \gamma q^2$ after the fits to the experimental energy levels. By comparison to the experimental data, we find it is reasonable that deformation parameters q and q' are taken to be real, while q'' is taken as a phase $q'' = e^{i|\tau''|}$, which was shown to be related to the stretching effects in the moment of inertia of nuclei in rotational regions [17]. The operators $L_{\pm}^{q''}$ should be Hermitian, and the eigenvalues of the Hamiltonian should be an increasing function of $\mathbf{L}$. To guarantee this one must have $(L+1)|\tau''| < \pi/2$. In the case of Cd isotopes, we restrict $L_{\max} = 8$, while for Pt isotopes we find $L_{\max} = 13$. These $\mathbf{L}$ values are higher than the observed $\mathbf{L}$ in the lower-lying spectra ($E \sim 3$ MeV) of these isotopes. The real deformation parameters are chosen to be $q = q' = e^{|\tau|}$. Using the least squares fits, we find that the energy levels can best be fitted by using a set of fixed parameters A, B, C, and D for the U(5) limit case, and α, β, and γ for the SO(6) limit case, while $|\tau|$ and $|\tau''|$ differ from nucleus to nucleus. For Cd isotopes there may be two-particle–two-hole excitations created across the proton shell closure at $Z = 50$. A collective band is expected to be built on the mixed intruder plus normal 0^+ states configuration mixing [44]. This will affect both energy spectra and $E2$ transition rates. As a first approximation, this is not taken into account in our calculation. The calculated energy spectra of these isotopes and the corresponding experimental values [32–40] are shown in Figs. 2–4, while the corresponding data are given in Tables III–V, respectively. The quality of the fits is indicated by the quantity

$$\sigma = \left(\frac{1}{N_{\text{total}}} \sum_{i,\text{total}} [E_{\exp}(i) - E_{\text{th}}(i)]^2 \right)^{1/2} , \tag{43}$$

where N_{total} is the number of the energy levels fitted for the isotopes.

The deformation parameter $|\tau|$ is obtained from the fits to the energy spectra. Then we use this $|\tau|$ value to calculate the $E2$ transition rates for the corresponding nucleus by using Eq. (37). The q dependence of the effective charge $e_2(q)$ will be determined by the best fits to the experimental values. In our calculation, we chose $e_2(q) = e_2 q^{-1}$ for Cd isotopes, and $e_2(q) = e_2 q^{-1/2}$ for both Pt and Xe isotopes. The results are given in Tables VI–VIII, respectively. In all cases the quality of the fits is measured by

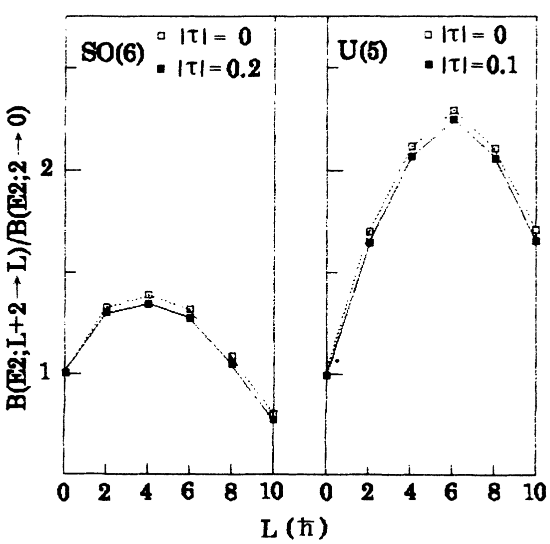

FIG. 1. $B(E2)$ ratios for $N = 7$ along the ground state band in SO(6) and U(5) limits.

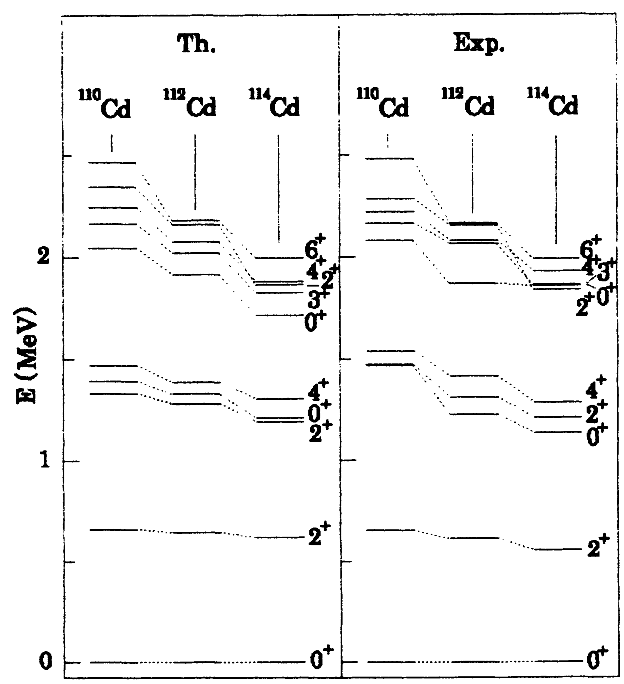

FIG. 2. Energy spectra of even-even $^{110-114}$Cd isotopes. The corresponding data are given in Table III.

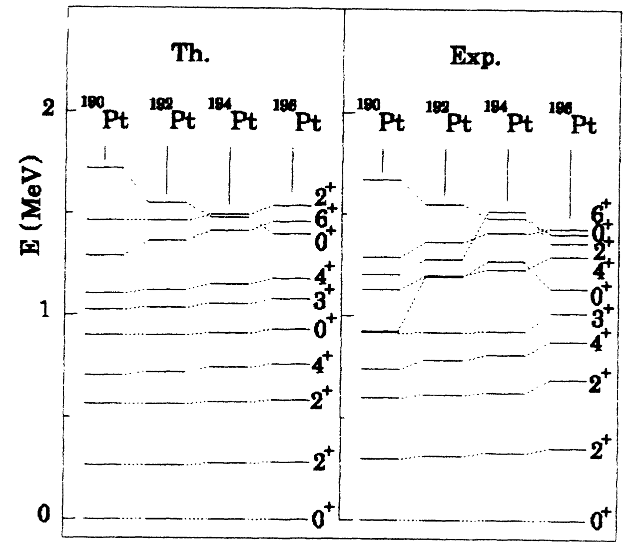

FIG. 3. Energy spectra of even-even $^{190-196}$Pt isotopes. The corresponding data are given in Table IV.

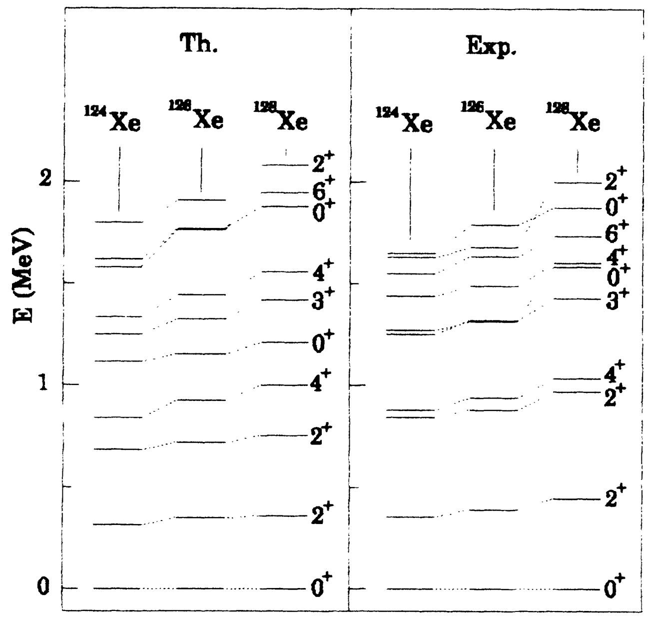

FIG. 4. Energy spectra of even-even $^{124-128}$Xe isotopes. The corresponding data are given in Table V.

$$\sigma = \left(\frac{1}{N_{\text{total}}} \sum_{i,\text{total}} [B(E2,i)_{\text{exp}} - B(E2,i)_{\text{th}} \right)^{1/2}, \quad (44)$$

where N_{total} is the total number of values used in the fits for these isotopes.

IV. SUMMARY AND DISCUSSION

In this paper, the q deformations of the SO(6) and U(5) Hamiltonian in the IBM are discussed. The energy spectra and some $B(E2)$ values of the even-even $^{110-114}$Cd, $^{190-196}$Pt, and $^{124-128}$Xe isotopes are fitted rather well by using the Hamiltonian with two more deformation parameters. From these examples, we have shown that the main characteristics of the deformed U(5) and SO(6) spectra as well as the $B(E2)$ values remain unchanged. In fact, the q deformation does not change the dynamical symmetry any more, but just inputs all high-order terms of a certain type, which preserves the underlying dynamical symmetry. This can clearly be seen from the following facts.

From Eqs. (25) and (33) one knows that the q-deformed Hamiltonians in both U(5) and SO(6) limits are expressed in terms of q-deformed SO(3), SU(1,1), and U(5) Casimir operators. q-deformed SO(3) and SU(1,1) Casimir operators can be expanded in terms of unde-

TABLE III. Lower-lying energy spectra of even-even $^{110-114}$Cd (in keV). The theoretical values are calculated by using Eq. (26) with $q = q' = e^{|\tau|}$ and $q'' = e^{i|\tau''|}$. The parameters are $A = 543$ keV, $B = 22$ keV, $C = -11$ keV, and $D = 10$ keV. The experimental data are taken from Refs. [32–34]. σ is defined by Eq. (43).

	^{110}Cd		^{112}Cd		^{114}Cd	
L	Th.	Exp.	Th.	Exp.	Th.	Exp.
0^+	0	0	0	0	0	0
2_1^+	663.06	657.76	648.34	617.57	625.20	558.45
0_2^+	1391.78	1473.77	1326.77	1224.06	1209.11	1134.53
2_2^+	1329.93	1475.78	1279.34	1312.32	1197.23	1209.71
4_1^+	1469.93	1542.43	1383.98	1415.38	1305.40	1283.74
0_3^+	2042.81	2078.65	1915.52	1870.94	1716.66	1859.69
3_1^+	2162.81	2162.79	2021.29	2064.22	1823.90	1864.26
4_2^+	2242.81	2220.06	2076.67	2081.00	1881.70	1932.07
6_1^+	2462.81	2479.93	2177.50	2167.00	1993.04	1991.10
2_3^+	2347.10	2287.44	2159.82	2156.23	1869.79	1841.94
	$\|\tau\| = 0.225$		$\|\tau\| = 0.17$		$\|\tau\| = 8 \times 10^{-6}$	
	$\|\tau''\| = 0$		$\|\tau''\| = 0.18$		$\|\tau''\| = 0.17$	
	$\sigma = 62.312$		$\sigma = 41.883$		$\sigma = 60.038$	

TABLE IV. Lower-lying energy spectra of even-even $^{190-196}$Pt (in keV). The theoretical values are calculated by using Eq. (34) with $q = q' = e^{|\tau|}$ and $q'' = e^{i|\tau''|}$. The parameters are $\alpha = -172$ keV, $\beta = 200$ keV, $\gamma(q) = 11q^2$ keV. The experimental data are taken from Refs. [35–38]. σ is defined by Eq. (43).

	^{190}Pt		^{192}Pt		^{194}Pt		^{196}Pt	
L	Th.	Exp.	Th.	Exp.	Th.	Exp.	Th.	Exp.
0_1^+	0	0	0	0	0	0	0	0
2_1^+	264.54	295.80	266.00	316.51	272.27	328.45	277.25	355.69
2_2^+	564.55	597.64	566.00	612.47	574.81	621.99	584.25	688.67
4_1^+	703.21	737.04	720.00	784.58	742.24	811.32	761.15	876.86
0_2^+	900.00	920.86	900.00	1195.15	909.73	1267.15	926.89	1135.28
3_1^+	1025.98	916.62	1032.00	920.92	1053.23	922.74	1078.51	1015.03
4_2^+	1103.21	1128.20	1120.00	1201.05	1148.91	1229.54	1179.59	1293.29
6_1^+	1289.58	1287.73	1362.00	1365.40	1412.01	1411.86	1457.57	1430.10
0_3^+	1720.00	1670.50	1548.00	1546.87	1479.78	1479.22	1397.50	1402.73
2_3^+	1464.54	1203.02	1466.00	1278.09	1495.12	1511.94	1540.68	1361.56
	$\|\tau\| = 0$		$\|\tau\| = 0$		$\|\tau\| = 0.0836$		$\|\tau\| = 0.1386$	
	$\|\tau''\| = 0.11$		$\|\tau''\| = 0$		$\|\tau''\| = 0$		$\|\tau''\| = 0$	
	$\sigma = 93.318$		$\sigma = 122.567$		$\sigma = 127.167$		$\sigma = 111.22$	

formed Casimir operators of the same type [24,41]

$$
\begin{aligned}
C_2(\mathrm{SO}_q(3)) &= L_-^q L_+^q + [L_0]_q[L_0+1]_q \\
&= C_2(\mathrm{SO}(3)) - \frac{1}{3!}[C_2(\mathrm{SO}(3))]^2|\tau|^2 \\
&\quad + \tfrac{1}{45}[C_2(\mathrm{SO}(3))]^3|\tau|^4 + \cdots ,
\end{aligned}
\tag{45}
$$

where we assumed that q is a phase $q = e^{i|\tau|}$. For small value and low spin states, the q-deformed SO(3) Casimir operator is equivalent to a higher-order power of the undeformed operator. The situation is very similar in the SU(1,1) case:

$$
\begin{aligned}
C_2(\mathrm{SU}_q(1,1)) &= [\kappa]_q[\kappa-1]_q \\
&= \kappa(\kappa-1) + \frac{1}{3!}[\kappa(\kappa-1)]^2|\tau|^2 \\
&\quad + \tfrac{1}{45}[\kappa(\kappa-1)]^3|\tau|^4 + \cdots ,
\end{aligned}
\tag{46}
$$

where q is real, $\kappa = \frac{1}{2}(\sigma+3)$, and $\sigma = \sqrt{4 + C_2[\mathrm{SO}(6)]} - 2$ is a formal operator for SO(6), and $\kappa = \frac{1}{4}(2\nu+5)$, and $\nu = \frac{1}{2}(\sqrt{9 + 4C_2[\mathrm{SO}(5)]} - 3)$ is a formal operator for the SO(5) case.

The expression for the $\mathrm{U}_q(5)$ Casimir operator is very complicated. We do not even know whether it can be expressed in terms of a combination of the undeformed U(5) Casimir operators. However, the eigenvalue of $C_2(\mathrm{U}_q(5))$ has the following combination

TABLE V. Lower-lying energy spectra of even-even $^{124-128}$Xe (in keV). The theoretical values are calculated by using Eq. (34) with $q = q' = e^{|\tau|}$ and $|\tau''| = 0$. The parameters are $\alpha = -180$ keV, $\beta = 248$ keV, $\gamma = 11q^2$ keV. The experimental data are taken from Refs. [39–40]. σ is defined by Eq. (43).

	^{124}Xe		^{126}Xe		^{128}Xe	
	Th.	Exp.	Th.	Exp.	Th.	Exp.
0_1^+	0	0	0	0	0	0
2_1^+	314.00	354.02	347.62	388.63	358.22	442.91
2_2^+	686.00	846.88	718.66	879.88	754.95	969.58
4_1^+	840.00	879.17	923.24	941.90	1000.36	1033.15
0_2^+	1116.00	1268.73	1151.02	1313.81	1211.80	1582.97
3_1^+	1248.00	1248.30	1326.37	1317.40	1422.16	1429.56
4_2^+	1336.00	1438.30	1443.27	1488.40	1562.39	1603.41
6_1^+	1578.00	1548.71	1764.75	1634.90	1948.05	1737.04
0_3^+	1620.00	1650.40	1764.39	1760.40	1880.75	1877.32
2_3^+	1802.00	1628.38	1908.23	1678.44	2074.99	1999.64
	$\|\tau\| = 0$		$\|\tau\| = 0.142$		$\|\tau\| = 0.233$	
	$\sigma = 97.333$		$\sigma = 112.392$		$\sigma = 156.215$	

TABLE VI. $E2$ transition rates of even-even $^{110-114}$Cd isotopes. Theoretical values are calculated by using Eq. (37). All values are in W.u. The asterisk indicates that the parameter e_2 is fixed by this value. The parameter $|\tau|$ is determined by the fits to the energy spectra. σ is defined by Eq. (44), in which experimentally undetermined values are not included. Experimental data are taken from Ref. [44].

	^{110}Cd		^{112}Cd		^{114}Cd	
Transition	Th.	Exp.	Th.	Exp.	Th.	Exp.
$2_1^+ \to 0_1^+$	27.4*	27.4(3)	32.7576	30.2(3)	39.81	31.09(19)
$2_2^+ \to 0_1^+$	0	1.34(20)	0	0.65(10)	0	0.50(4)
$4_1^+ \to 2_1^+$	43.73	46(16)	54.47	61(7)	70.772	61(5)
$6_1^+ \to 4_1^+$	52.351		67.678		92.886	119(39)
$0_3^+ \to 2_1^+$	0		0	0.01238	0	0.0027
$0_3^+ \to 2_2^+$	52.351		67.678	99(8)	92.886	109(8)
$2_2^+ \to 2_1^+$	43.73	29.59	54.47	45.582	70.772	22.074
$0_2^+ \to 2_1^+$	43.73		54.47	51(13)	70.772	27.3(16)
$2_3^+ \to 0_1^+$	0	0.3	0	0.34	0	0.33(2)
$2_3^+ \to 0_2^+$	27.4	≤ 15	32.7576	55	39.81	16(2)
	$\sigma = 7.753$		$\sigma = 13.402$		$\sigma = 24.376$	

$$
\begin{aligned}
C_2(\mathrm{U}_q(5)) &= \{[5]_q - 5q^{2n_d/5} + (q^{2n_d} - 1)q^4\}/(q - q^{-1})^2 \\
&= 5 + \tfrac{2}{5}C_2(\mathrm{U}(5)) + \tfrac{2}{25}\{5C_2(\mathrm{U}(5)) + 4n_dC_2(\mathrm{U}(5)) - 5n_d^2\}|\tau| + \cdots ,
\end{aligned}
\tag{47}
$$

where n_d is the eigenvalue of the U(5) Casimir operator of the first kind. From this expansion one can easily see that the eigenvalue of the $\mathrm{U}_q(5)$ Casimir operator can equivalently be expressed in terms of the eigenvalues of the undeformed ones though the $\mathrm{U}_q(5)$ Casimir operator but probably cannot be expanded in terms of the undeformed U(5) Casimir operators.

The above discussion shows that the q-deformed Hamiltonian just involves all high-order terms of a certain type that keeps the underlying symmetry unchanged. Furthermore, the q-deformed Hamiltonian is equivalent to a higher-order power of the undeformed Casimir operators as long as the deformation parameters are small. However, in order to sum up the expansions to some or-

TABLE VII. $E2$ transition rates of even-even $^{190-196}$Pt isotopes. Theoretical values are calculated by using Eq. (37). All values are in $e^2\,\mathrm{b}^2$. Experimental data are taken from Refs. [35–38] and [45,46]. Others are the same as Table VI.

	^{190}Pt		^{192}Pt		^{194}Pt		^{196}Pt	
Transition	Th.	Exp.	Th.	Exp.	Th.	Exp.	Th.	Exp.
$2_1^+ \to 0_1^+$	0.5643	0.5915	0.4629	1.893	0.3567	0.374	0.276*	0.264
								0.276
$4_2^+ \to 2_1^+$	0.7716		0.6269		0.4566	0.47	0.3596	0.409
								0.38
$6_1^+ \to 4_1^+$	0.8439		0.6752		0.5002	0.38	0.3647	0.421
								0.40
$2_2^+ \to 0_1^+$	0		0	0.0132	0	1.4×10^{-3}	0	3×10^{-4}
								2×10^{-4}
$2_2^+ \to 2_1^+$	0.7716		0.6269		0.4566	0.58	0.3596	0.34
								0.318
$4_2^+ \to 2_2^+$	0.4421		0.3537		0.2620	0.21	0.191	0.177
								0.17
$0_2^+ \to 2_2^+$	0.8439		0.6752		0.5002	1.2716	0.3647	0.142
								0.14
$0_2^+ \to 2_1^+$	0		0		0	1.2323	0	0.022
								0.021
$4_2^+ \to 4_1^+$	0.402		0.3215		0.2382	0.2059	0.1736	0.193
								0.18
$4_2^+ \to 2_1^+$	0		0		0	0.00227	0	0.003
						$\sigma = 0.463$		$\sigma = 0.072$

TABLE VIII. E_2 transition rates of even-even $^{124-128}$Xe isotopes. Theoretical values are calculated by using Eq. (37). All values are in W.u. Experimental data are taken from Refs. [39,40]. Others are the same as Table VI.

	^{124}Xe		^{126}Xe		^{128}Xe	
Transition	Th.	Exp.	Th.	Exp.	Th.	Exp.
$2_1^+ \to 0_1^+$	47.5*	47.5	38.223	40.414	31.562	36.3

der, the Hamiltonian should include many terms, which leads to energy eigenvalue expressions with a large number of terms and parameters. The situation is similar in the transition operators. For example, the q-deformed s or d boson operators can also be expanded in terms of the undeformed one:

$$\begin{aligned} s^\dagger(q) &= \left(\frac{[N-n_d]_q}{N-n_d}\right)^{1/2} s^\dagger \\ &= \{1+\tfrac{1}{12}[(N-n_d)^2-1]|\tau|^2 \\ &\quad +\tfrac{1}{1440}[(N-n_d)^4-10(N-n_d)^2+9]|\tau|^4 \\ &\quad +\cdots\}s^\dagger \end{aligned} \tag{48}$$

and

$$\begin{aligned} d_\mu^\dagger(q) &= \left(\frac{[n_d]_q}{n_d}\right)^{1/2} d_\mu^\dagger \\ &= \{1+\tfrac{1}{12}(n_d^2-1)|\tau|^2 \\ &\quad +\tfrac{1}{1440}(n_d^4-10n_d^2+9)|\tau|^4+\cdots\}d_\mu^\dagger\ , \end{aligned} \tag{49}$$

where N and n_d are the quantum numbers of the total number of bosons and of d bosons, respectively.

In the original IBM, the neglect of higher-order terms does not represent any fundamental constraint (and indeed has been relaxed in the latter applications of the model), but rather stems from the desire to keep the complexity of the overall Hamiltonian at a manageable level. The q-deformation technique enables us to input all high-order terms of a certain type and only add a few parameters to the Hamiltonian, which can be regarded as a possible extension of the original IBM. The multiparameter deformation given by (25) and (33) are also possible, and need to be further investigated.

In Refs. [42,43], in order to calculate the energy spectra of isotopes in the transitional region, the parameter A in the U(5) limit is written as $A = A_0 e^{\theta(N_\pi N_\nu - N_0)}$. In fact, all parameters in the IBM are N dependent. The deformation parameter q plays a role similar to that of $e^{\theta(N_\pi N_\nu - N_0)}$. A detailed analysis of this relation is necessary although we have not done it here.

In conclusion, the q-deformation technique provides us another way to introduce higher-order terms, which may be possible corrections to the many-body problem, and keeps the complexity of the overall Hamiltonian at a manageable level. This technique was also successfully applied to the description of rotation-vibration spectra of diatomic molecules [47], and to the nucleon pairing problem [16]. q deformation of the SU(3) limit in the IBM is also under consideration.

Note added. After completion of this work, the author became aware of Refs. [48,49], in which the authors discussed the same problem from a somewhat different point of view. In particular, the relations between the generators of quantum algebras and the ordinary s and d boson operators were not mentioned in those papers.

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