Boson analyses in the Ge isotopes

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The strong variation in energy of the first excited 0^+ state in the even-even Ge isotopes is investigated with pairing type interactions in various model spaces. Although it had been possible in previous work to describe this variation in a BCS–random-phase-approximation (RPA) calculation in the neutron configuration only, the corresponding exact diagonalization disagrees with the BCS-RPA results and experiment. It is furthermore shown that the behavior of the first excited 0^+ state cannot be obtained by a reasonable variation of parameters in the exact neutron pair calculations. This suggests that the model space be enlarged to include proton, neutron, and proton-neutron pairing. The construction of the corresponding basis is then simplified by performing a boson mapping and employing the ideal collective boson basis. This, however, may lead to the occurrence of spurious states, even at fairly low energies, and three separate methods by means of which they can be identified are discussed. Although the enlarged model space gives a desired lowering of excitation energy of the first excited 0^+ states in an exact diagonalization, the strong experimental variation could not be achieved with this simple interaction. Nevertheless, the analysis yields further and new insight into applications of boson mappings, specifically about how and when spurious states may be identified in calculations which are performed in a truncated space not invariant under a given collective algebra.

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I. INTRODUCTION

Boson mapping procedures are usually associated with two types of application. First, they are used to investigate the microscopic basis of phenomenological boson models and to estimate, qualitatively and quantitatively, various features of the boson model parameters which are used to fit nuclear spectra. Second, they may be used to simplify calculations in fermion model spaces and may thus aid in introducing truncation schemes for calculations in large model spaces.

In this article, attention is paid to the second type of application. The specific physical problem we focus on concerns the behavior of the first excited 0^+ state as a function of neutron number in the sequence of nuclei $^{68-76}_{32}$Ge (see Fig. 1). Its energy varies strongly over the range of isotopes, being lowest for ^{72}Ge where it is also the lowest excited state overall.

This behavior suggests the presence of an intruder state. If one wants to explain this feature in an interacting boson model (IBM-) type [1, 2] or other boson expansion [3] method, one invariably needs an additional mechanism which introduces the variation of energy of the intruder 0^+ state. The strong variation in energy of the first excited 0^+ state as well as the rest of the low-lying states shown in Fig. 1 can be described [1] by introducing an s' boson, with an appropriate variation of its single-particle energy $\epsilon_{s'}$, into the interacting boson model which usually involves only s and d bosons. (In the IBM fit presented in Ref. [1] the intruder state was considered to be the dominant component in the first excited 0^+ state in ^{70}Ge and ^{72}Ge and in the second excited 0^+ state in ^{68}Ge and ^{74}Ge.) The motivation for this is that ground state of each nucleus is regarded (to lowest order) as comprised of N s bosons (N is the number of fermion pairs in the valence shell) and the intruder 0^+ state as $(N-1)$ s bosons and one s' boson. Both the s and s' bosons represent correlated nucleon pairs coupled

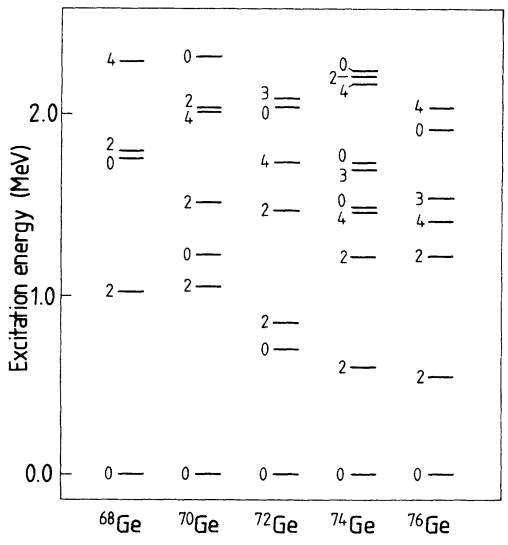

FIG. 1. Energy spectra of even Ge isotopes.

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to angular momentum 0. In an attempt to understand the content of the s and s' bosons, the monopole pairing interaction was considered in Ref. [1].

Calculations with the monopole pairing interaction could, independently of the latter consideration, also be regarded as a direct way to search for a lowest order description of the intruder 0^+ state.

As the variation in energy of the intruder 0^+ state occurs as a function of the number of neutron pairs, one is naturally led to investigate the effect of the neutron monopole pairing interaction. In Sec. II we recall that the observed variation could be obtained from a BCS–random-phase-approximation (RPA) calculation in the neutron configuration only. However, it is also shown that results from the corresponding exact diagonalization of the monopole neutron pairing Hamiltonian disagree with the BCS-RPA calculation and experiment.

This large disagreement casts doubt about the validity of BCS-RPA theory for some nuclei, at least in the mass region of germanium. We mention three instances where BCS-RPA theory was in fact applied to such nuclei. In a study of the energy spectra of germanium isotopes, Weeks *et al.* [3] coupled a BCS-RPA solution of the pairing vibration in the neutron configuration to a calculated collective quadrupole oscillation. Pedrocchi and Tamura [4] obtained significant lowering of the first excited 0^+ state in Se and Kr isotopes by taking into account the monopole pairing vibrational mode from BCS-RPA theory. Tazaki *et al.* [5], too, made use of BCS-RPA theory in a study of, amongst others, the nuclei, Zn, Ge, Se, Kr, and Sr.

In Sec. II it is also shown that the variation of the first excited 0^+ state cannot be obtained by varying parameters in the exact neutron pair calculations within reason. This led us to enlarge the model space to include proton, neutron, and proton-neutron monopole pairing. The construction of a basis in which to diagonalize the Hamiltonian is then simplified by performing a boson mapping and employing the ideal collective boson basis. It is well known, however, that this may lead to the occurrence of spurious states, even at fairly low energies [8]. Methods by means of which they can be identified are discussed in Sec. III A.

In Sec. IV we shift our focus from the search for a description and fit of the intruder 0^+ state to a study of the effects of truncation on the boson basis used for such a calculation. We use our present model to investigate how and when spurious states may be identified in calculations which are performed in a truncated space not invariant under some collective algebra.

II. THE MONOPOLE PAIRING INTERACTION IN THE NEUTRON CONFIGURATION

In this section, where we consider the pairing interaction in the neutron configuration only, the presence of only four single-particle energy levels in the valence shell (see Fig. 2) presents us with the opportunity to perform both BCS-RPA and exact calculations and to compare results.

In the BCS-RPA formalism the configurations obtained in an initial BCS calculation are introduced in a subsequent RPA calculation. The RPA part of the calculations should be compared and contrasted with the RPA formalism used in the context of particle-hole excitations. There the RPA formalism typically describes nuclei with closed shells and the relatively large energy gap between the closed and valence shells plays an important role in the applicability of the theory. However, throughout the sequence of Ge isotopes under consideration there are, in both the proton and neutron configurations, only open shells, so that the RPA only becomes applicable after some additional considerations provided by BCS theory.

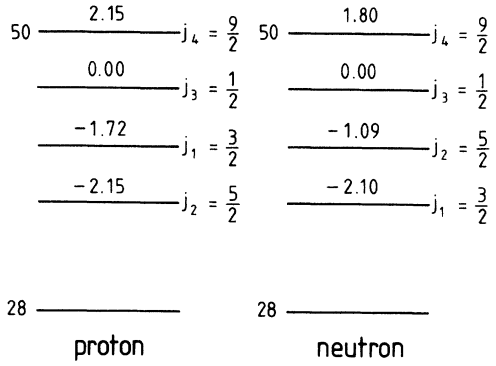

FIG. 2. Valence single-particle energy levels in $^{68-76}_{32}$Ge nuclei. The energy values shown (in MeV) are for ^{72}Ge as given in Ref. [3].

In the BCS formalism one transforms, by means of the Bogoliubov transformation, from the nucleon representation to a quasiparticle representation. Whereas the Hartree-Fock procedure treats interactions between particles mainly in terms of a single-particle potential, the BCS method goes further by incorporating part of the residual pairing interaction in the single-particle energies of the quasiparticles.

The BCS formalism gives an expression for the lower bound (denoted by 2Δ) of the energy gap between the ground and first excited states in even-even nuclei. Under certain conditions a sufficiently large energy gap guarantees (as in the case of closed-shell nuclei) the applicability of a subsequent RPA calculation in which the results of the foregoing BCS calculation are then used.

Results of a BCS-RPA calculation [1] are shown in Fig. 3. One indeed finds that the gap Δ has a minimum value at ^{72}Ge and furthermore that the pairing vibration calculated in BCS-RPA has a minimum energy at ^{72}Ge. The value of the energy of the first excited 0^+ state in this calculation is found to be at the right value, using very acceptable model parameters. (Single-particle energy levels are from Ref. [3]—see also Fig. 2.) A satisfactory description therefore seems to have been found using only a very schematic interaction. However, if one diagonalizes the pairing Hamiltonian in the neutron valence space exactly, the nice agreement disappears (see Fig. 3), and in particular one finds that the BCS-RPA approximation is in fact at its worst where the excitation level is lowest, i.e., for ^{72}Ge.

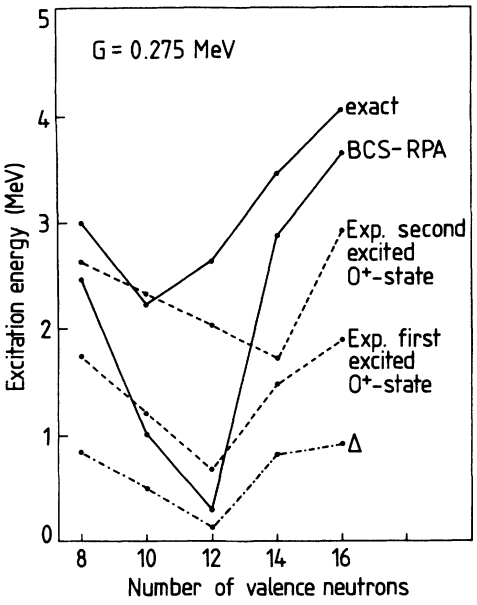

FIG. 3. First excited 0^+ state from BCS-RPA and exact neutron configuration calculations.

It is not too difficult to find reasons for this breakdown of the BCS-RPA approximation. In the BCS part of the BCS-RPA calculations the *neutron* quasispin operators $S_+(j), S_-(j)$, and $S_z(j)$ become, under the Bogoliubov transformation,

$$S'_+(j) = \sum_{m>0} a^\dagger_\alpha a^\dagger_{\tilde{\alpha}} , \tag{2.1}$$

$$S'_-(j) = \sum_{m>0} a_{\tilde{\alpha}} a_\alpha , \tag{2.2}$$

$$S'_z(j) = \tfrac{1}{2}\left(\sum_m a^\dagger_\alpha a_\alpha - \Omega_j\right) , \tag{2.3}$$

where $a^\dagger_\alpha$ and a_α are quasiparticle creation and annihilation operators, respectively. [The label α represents jm, while $a^\dagger_{\tilde{\alpha}} = (-1)^{j-m} a^\dagger_{j-m}$.]

In the RPA part of the calculations [7, 1], a Holstein-Primakoff mapping of the operators in expressions (2.1)–(2.3) is performed:

$$S^H_+(j) = B^\dagger_j(\Omega_j - B^\dagger_j B_j)^{1/2} , \tag{2.4}$$

$$S^H_-(j) = (\Omega_j - B^\dagger_j B_j)^{1/2} B_j , \tag{2.5}$$

$$S^H_z(j) = \tfrac{1}{2}(2B^\dagger_j B_j - \Omega_j) , \tag{2.6}$$

where $\Omega_j = \frac{1}{2}(2j+1)$, while the operators $B^\dagger_j$ and B_j are standard boson operators satisfying $[B_j, B^\dagger_{j'}] = \delta_{jj'}$ and $[B_j, B_{j'}] = [B^\dagger_j, B^\dagger_{j'}] = 0$. The expression $2B^\dagger_j B_j$ is to be associated with the number operator for quasiparticles in single-particle level j.

Everywhere where $(\Omega_j - B^\dagger_j B_j)^{1/2}$ from Eqs. (2.4) and (2.5) appears in the Holstein-Primakoff mapped Hamiltonian, only the lowest order term, namely $\sqrt{\Omega_j}$, is retained in the expansion of the square root. This is done under the assumption that the number of quasiparticles in any subshell of the valence shell is small compared to the multiplicity of that subshell. This assumption is supported by the fact that the ground state in the BCS formalism is represented by the quasiparticle vacuum and the first excited state by a distribution of two quasiparticles over the valence shell. However, the approximation might become poor when there is a subshell present with $j = \frac{1}{2}(\Omega_j = 1)$ as, indeed, there is in our case (see Fig. 2).

Another reason for the disagreement may be found in Fig. 3 from the graph for Δ (which is one-half of the lower bound of the energy gap in the BCS solution). The largest disagreement between the exact and the BCS-RPA results is obtained when Δ takes on small values. This is to be expected since, as pointed out already earlier in this section, the RPA part of the calculation is regarded as reliable only for relatively large values of the energy gap.

We now show that several attempts to describe the intruder 0^+ state in the exact neutron pair calculation (using the exact boson method) by only varying the model parameters within reason were unsuccessful.

The first model parameter to be varied in the exact diagonalization of the neutron pairing Hamiltonian was the pairing interaction strength G. Figure 4 shows the variation in the calculated energy of the first excited 0^+ state as a function of G for the interesting case of ^{72}Ge. The neutron single-particle levels were kept fixed as given in Fig. 2. It is clear that the variation of G brings us nowhere near the experimental first excited 0^+ energy. The behavior of the full series of isotopes as a function of G can be seen in Fig. 5.

As a next attempt to describe the first excited 0^+ state of ^{72}Ge within the neutron configuration only, we varied the single-particle energy of the $j = \frac{1}{2}$ level between its limits -1.09 MeV and 1.80 MeV as given in Fig. 2. The $j = \frac{1}{2}$ level was varied because it is the last filled subshell in an unperturbed picture and we refrained from changing the range of valence single-particle energies by keeping the $j = \frac{3}{2}$ and $j = \frac{9}{2}$ levels fixed. We also kept

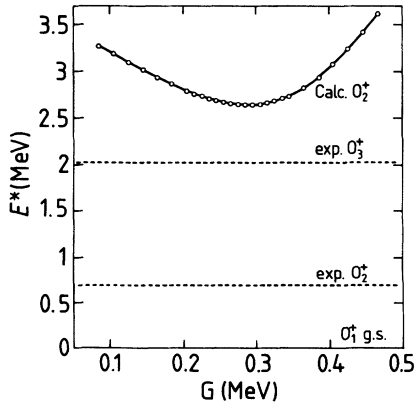

FIG. 4. First excited 0^+ state of ^{72}Ge as a function of the neutron pairing interaction strength G in exact neutron configuration calculations.

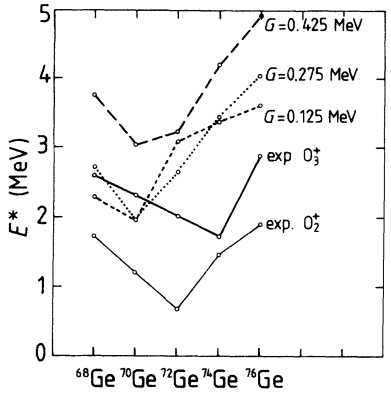

FIG. 5. First excited 0^+ state from exact neutron configuration calculations for different values of the neutron pairing interaction strength G.

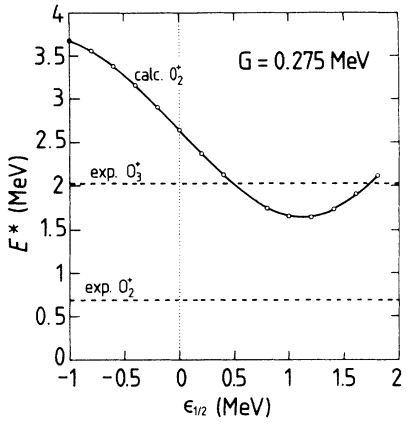

FIG. 6. First excited 0^+ state from exact neutron configuration calculations as a function of the $j = \frac{1}{2}$ single-particle energy.

G fixed at the value which gave the minimum in Fig. 4.

The results in Fig. 6 show that the variation of the $j = \frac{1}{2}$ single-particle energy does bring us closer to the experimental first excited 0^+ state of ^{72}Ge, but not sufficiently so. The behavior of the full series of isotopes as a function of $\epsilon_{1/2}$ is shown in Fig. 7.

We conclude that it is not possible to describe the intruder 0^+ state retaining a neutron pairing interaction only, not even by varying the model parameters within reason.

III. THE PAIRING INTERACTION WITH PROTON, NEUTRON, AND PROTON-NEUTRON PAIRING

Keeping the interaction as simple as possible, we enlarged as a next step the model space to include both proton and neutron as well as proton-neutron pairing. The Hamiltonian considered is an extended monopole pairing interaction,

$$H = \sum_i \varepsilon_p^i n_p^i + \sum_i \varepsilon_n^i n_n^i - G_p \sum_{ij} S_+^p(i) S_-^p(i) - G_n \sum_{ij} S_+^n(i) S_-^n(i) - \tfrac{1}{2} G_f \sum_{ij} S_+^f(i) S_-^f(i) \ , \quad (3.1)$$

where the indices i and j run over the different single-particle levels in the shell, $p(n)$ refers to proton (neutron) degrees of freedom, and f denotes the proton-neutron pairing part. $S_+^\theta(i)$ and $S_-^\theta(i)$ are creation and annihilation operators for pairs of type θ in single-particle level i. A charge-independent interaction is obtained for $G_p = G_n = G_f = G$, with the operators S_+ and S_- constructed in such a way that the interaction terms, taken together in Eq. (3.1), constitute a scalar with respect to isospin [8].

The operators appearing in the extended pairing model Hamiltonian (3.1), are generators of SO(5)⊗SO(5)⊗··· (enumerated by the single-particle levels) [6]. Although this allows an analysis with methods of group theory, one still encounters the standard difficulty of defining state labels in addition to those which simply follow from the chain of subalgebras. A simpler computational framework is obtained by first representing bifermion excitations in terms of bosons, mapping the Hamiltonian, and then diagonalizing it in a boson Fock space spanned by all those bosons appearing in the collective realization [6].

Here each SO(5) is realized in terms of collective bosons of the proton, neutron, and proton-neutron pairing type, the creation (annihilation) operators of which we refer to by $B^p(B_p)$, $B^n(B_n)$, and $B^f(B_f)$, respectively, with a further label for each single-particle level. In terms of these bosons the collective part of the Hamiltonian (3.1) takes the form [6]

$$H_B = \left(\sum_i \varepsilon_p^i (2B_i^p B_p^i + B_i^f B_f^i) \right) + (p \leftrightarrow n)$$
$$- \left(G_p \sum_{ij} (\Omega_i B_i^p B_p^j - B_i^p B_i^p B_p^i B_p^j - B_i^p B_i^f B_f^i B_p^j - B_i^f B_i^f B_n^i B_p^j) \right) - (p \leftrightarrow n)$$
$$- \tfrac{1}{2} G_f \sum_{ij} (2\Omega_i B_i^f B_f^j - B_i^f B_i^f B_f^i B_f^j - 2B_i^f B_i^p B_p^i B_f^j - 2B_i^f B_i^n B_n^i B_f^j - B_i^p B_i^n B_f^i B_f^j) \ . \quad (3.2)$$

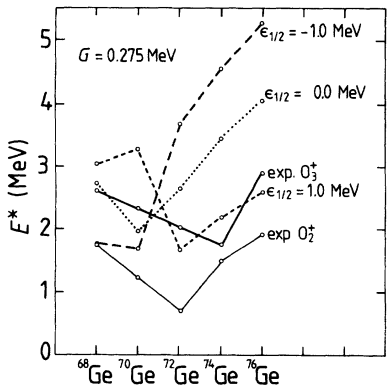

FIG. 7. First excited 0^+ state from exact neutron configuration calculations for different values of the $j = \frac{1}{2}$ single-particle energy.

The number of bosons of the various types are denoted by $N_p(i), N_n(i)$, and $N_f(i)$. It is a simple matter to construct the so-called ideal boson basis

$$\prod_i \mid N_p(i)N_n(i)N_f(i)) \tag{3.3}$$

with the constraint that $2\sum_i N_p(i) + \sum_i N_f(i)$ equals the total number of protons and similarly for neutrons. Furthermore, $2N_p(i) + N_f(i)$ is constrained by the degeneracy of level i, and similar for neutrons. It is also a simple matter to diagonalize the non-Hermitian Hamiltonian (3.2) and to obtain the spectrum and left- and right-hand eigenstates.

Although the above constraints lead to the correct enumeration of ideal boson basis states, one problem remains, namely that, as discussed by Geyer *et al.* [6], the boson basis is generally overcomplete in as much as it may contain (independent) boson states with fermion counterpart states which are linearly dependent. Although physical states and energies emerge unharmed from the diagonalization [6], it may lead to spurious states and, since some of these states may occur in the lower part of the spectrum, they must be properly identified and eliminated.

A. Identification of spurious states

As already stated, it is well known that for a closed collective subalgebra, physical eigenstates and eigenvalues are not contaminated when a boson mapped Hamiltonian is diagonalized in the complete boson Fock space [6]. Spurious states can, however appear in a nontrivial way and their identification is thus of utmost importance in applications of the boson method. In this section we consider the application of three methods to deal with this problem.

In Ref. [1], where only the lowest few eigenvalues and eigenstates were calculated, a rather cumbersome method was used to identify spurious states. First, the physical eigenvalues were determined for separate proton (with $G_p = G$) and neutron (with $G_n = G$) configuration calculations. [Spurious states are very simply avoided in these separate calculations by just taking care that the boson numbers do not exceed the maximum number dictated by the corresponding Ω_j—no implicit linear dependence is then possible among the simplified states (3.3) which have only one type of boson.] These physical eigenvalues were then located among the eigenvalues which were obtained from the diagonalization of H_B [Eq. (3.2)] with $G_p = G_n = G$ and $G_f = 0$. Of the latter ones, several were then seen to be spurious. By increasing the value of G_f in small steps until it reached $G_f = G$ in successive diagonalizations, the physical energies could be followed to the point where they correspond to the charge-independent pairing interaction.

This method of identifying spurious states becomes impracticable when physical and spurious states lie close together, and especially so when one wants to do several calculations in order to move through parameter space in search for a fit to experimental data. We now implement two simpler methods which have been discussed recently [6, 9].

In the first method [6] a physical operator Θ is introduced with the requirement that it can be expressed in terms of elements of the collective algebra—otherwise it is arbitrary. A typical choice which we make is

$$\Theta = \lambda_p \sum_i S_+^p(i)S_-^p(i) + \lambda_n \sum_i S_+^n(i)S_-^n(i) + \lambda_f \sum_i S_+^f(i)S_-^f(i) , \tag{3.4}$$

with λ_p, λ_n, and λ_f arbitrary.

As is shown in Ref. [6], it follows from the invariance of the physical subspace of the ideal boson basis under the action of arbitrary physical operators that the mapped operator Θ_B has the property that, for any spurious bra-eigenstate $(\tilde{\psi}_{\text{spur}} \mid$ of H_B,

$$(\tilde{\psi}_{\text{spur}} \mid \Theta_B \mid \psi_{\text{phys}}) = 0 \tag{3.5}$$

while

$$(\tilde{\psi}_{\text{phys}} \mid \Theta_B \mid \psi_{\text{spur}}) \neq 0 \tag{3.6}$$

where $\mid \psi_{\text{phys}})$ is any nonspurious, i.e., physical state. This provides us with a method to identify the spurious states.

The second method due to Dobaczewski [10], applied to and discussed for SO(5) in Ref. [9], is referred to as $\mathcal{R}$ projection. It amounts to replacing each boson creation operator appearing in a particular eigenstate by the boson image in which that boson operator appears as the leading order term, e.g.,

$$B^p \overset{\text{replace}}{\longrightarrow} (S_+^p)_D = B^p(\Omega - N_p - N_f) - B^f B^f B_n . \tag{3.7}$$

Here $(S_+^p)_D$ denotes the Dyson image of the pairing operator S_+^p in fermion space. As shown in Refs. [10, 9], the $\mathcal{R}$ projection of a spurious state gives zero, namely,

$$\mathcal{R} \mid \tilde{\psi}_{\text{spur}}) = 0 , \tag{3.8}$$

where $\mid \tilde{\psi}_{\text{spur}})$ is the state conjugate to a spurious bra-

eigenstate, while it gives nonzero when applied to a physical state.

B. Exact diagonalization of the charge-independent pairing interaction

Results for the first *physical* excited 0^+ state from the exact diagonalization of the pairing interaction in both the proton and neutron configurations are given in Fig. 8. They show that the increased model space leads to much lower first excited energies than those attainable in the neutron configuration only. A strong energy dependence with a minimum at ^{72}Ge is, however, not achieved. Further improvement might be achieved when single-particle energies and interaction strengths are varied, although this is unclear.

To illustrate, finally, that the appearance of spurious states in these calculations is a very real possibility, we mention some results obtained for ^{72}Ge with $G = 0.275$ MeV. The dimension of the ideal boson basis constructed according to expression (3.3) is 1033. Among the ten lowest eigenvalues obtained from the diagonalization of H_B [given in Eq. (3.2)] in this basis, the third, fifth, seventh, and tenth are in fact spurious.

IV. EFFECTS OF TRUNCATION OF BOSON BASIS

As discussed, the diagonalization for the (multi) SO(5) model and the resulting identification of spurious states can be accomplished in an exact fashion, while taking full advantage of the relative simplicity of a boson calculation. However, general shell-model calculations are usually characterized by the necessity to truncate the basis. As in the (generalized) pairing model, it would be ideal to truncate to some collective subspace. Since it is generally not possible to find such a closed collective subspace, one is mostly content to find a subspace which is weakly coupled to the truncated part. In similar circumstances on the boson level many questions arise about the identification and the effect of spurious states. Applications of this nature are, e.g. [11], where the question of spurious states is mostly dismissed. We now pursue these questions further, capitalizing on a model calculation where different aspects still remain rather transparent.

We first simplify the model to an SO(5)$_4\otimes$ SO(5)$_6$ structure, where the indices 4 and 6 indicate the $(2j+1)$ values of the single-particle levels retained. The Hamiltonian (3.2) is again diagonalized under different conditions, while monitoring the success with which spurious states can be identified.

If $G_f = 0$, the space can be decomposed into

$$(\mathrm{SO}(3)_4 \otimes \mathrm{SO}(3)_6)_{\mathrm{proton}} \otimes (\mathrm{SO}(3)_4 \otimes \mathrm{SO}(3)_6)_{\mathrm{neutron}}$$

and in this space the usual pairing calculations can be carried out exactly. In this case there are no spurious states associated with linear dependence, as one need not introduce bosons of type f. For relatively small values of G_f one may want to introduce f bosons but restrict N_f to some small value and, by doing so, truncate the SO(5)$_4\otimes$ SO(5)$_6$ basis. This represents a truncation to a space with no underlying algebraic structure, and it is of interest to see whether the spurious states, which it will contain in general, can be identified.

We show two sets of calculations of this type. In Fig. 9 the results are shown for the case of four protons and eight neutrons for various strengths G_f. The possible values of $N_f = \sum_i N_f(i)$ in this case are 0, 2, and 4. If we allow maximum values of 0 or 4, we deal with the SO(3) and SO(5) cases discussed above, respectively. When re-

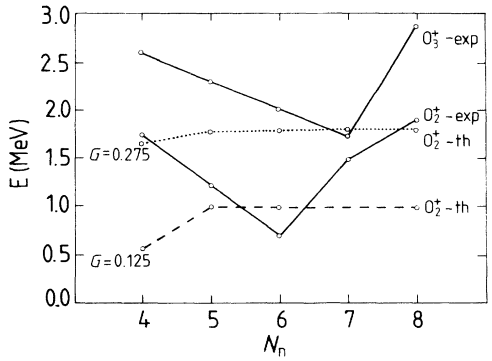

FIG. 8. Proton-neutron configuration calculations with $G = G_p = G_n = G_f$ in MeV.

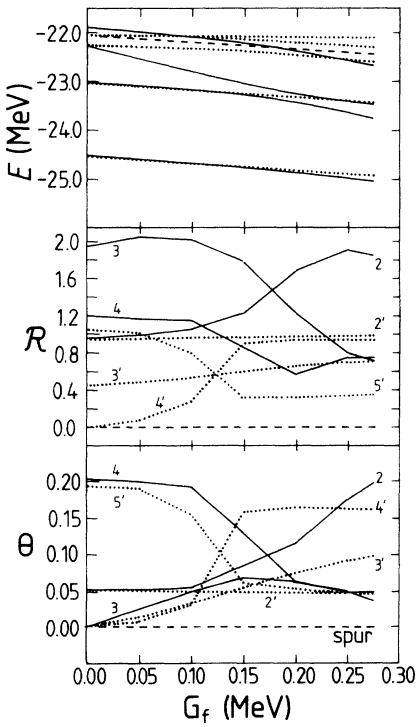

FIG. 9. The dependence on G_f in the SO(5)$_4\otimes$ SO(5)$_6$ model calculation. The full lines connect exact physical state results, the dotted lines, $N_f \leq 2$ truncation results, and the dashed line, the exact spurious state. The parameters of the Hamiltonian [Eq. (8)] are $n_p = 4, n_n = 8, G_p = G_n = 0.275$ MeV, $\epsilon_p^1 = -1.69$ MeV, $\epsilon_n^1 = -2.06$ MeV, $\epsilon_p^2 = -2.11$ MeV, and $\epsilon_n^2 = -1.07$ MeV.

stricting the maximum of N_f to 2 we have an intermediate case which does not correspond to an underlying algebra. It is this case which is shown in Fig. 9, together with the exact SO(5) result. In the upper part of Fig. 9 the lowest five energy levels are shown as functions of G_f. The full lines are from exact calculations and the dotted lines from those restricted by $N_f \leq 2$. The dashed line connects a spurious state of the exact calculation. The middle part of Fig. 9 shows the norm of the vector $\mathcal{R}|\tilde{\psi}_i)$ divided by the norm of the $\mathcal{R}$-projected ground state for the corresponding four excited states.

In the bottom part of Fig. 9 the ratios $|(\tilde{i}|\Theta_B|1)|/|(\tilde{1}|\Theta_B|1)|$ are given. Here λ_f is taken to be zero in the expression for Θ_B. From the exact calculation one clearly identifies the fourth state to be spurious because both $\mathcal{R}|\tilde{\psi}_4)$ and $(\tilde{4}|\Theta_B|1)$ vanish. In the truncated calculation this is no longer true, but we do observe that these quantities remain small for this state in the case of small values of G_f and one concludes that weakly coupled modes may be handled in such an approximate way. It is important to note that when the operator Θ_B is used, one still has to construct it having the strongly coupled subspace in mind. If, for instance, in this case one chose $\lambda_f \neq 0$, then $(\tilde{4}|\Theta_B|1)$ would not tend to zero with G_f in the truncated calculation and state 4 would not appear spurious, although it is.

In passing we note that, as originally envisaged [6], conclusions about the status of a state should not be drawn from a single matrix element calculated with the test operator Θ_B. This could, e.g., mislead us to believe that state 3 is spurious because $(\tilde{3}|\Theta_B|1)$ tends to zero with G_f, while the true state of affairs is only revealed if one looks at $(\tilde{3}|\Theta_B|i)$ for several i, from which it becomes clear that the state is in fact physical.

In the second example shown in Fig. 10 we introduce another truncation scheme, typically employed in IBM-type calculations. The single-particle model space is the same as above, but we consider six protons and six neutrons. The possible numbers of f bosons in the upper level are $N_f(2) = 0,1,2,3,4,5,6$. In the calculations we restrict the value of $N_f(2)$ to $N_f(2) \leq 6 - N_{f2}^{\rm tr}$ and plot the various results for the truncation parameter $N_{f2}^{\rm tr}$ ranging from 0 to 6. Again, the extreme values $N_{f2}^{\rm tr} = 0$ and 6 correspond to exact model results and in between we have a truncated space with no underlying algebra. In general, we again expect that spurious states can be identified for $N_{f2}^{\rm tr}$ ranging from 1 to 5 only approximately, and then only for small G_f. Among the lowest nine states shown in Fig. 10, three are spurious, as can be seen from the $\mathcal{R}$ projection or Θ method. These are the states 4, 7, and 9. It is interesting to note that they behave differently in the truncated calculations. States 4 and 9 retain their spurious character and can be identified exactly even in the truncated calculation, while state 7 mixes with the physical states so that $\mathcal{R}|\tilde{\psi}_7) \neq 0$ for nonvanishing G_f. However, this mixing is again small for small G_f, we can trace it for all $N_{f2}^{\rm tr}$ considered, and the weak-coupling arguments again hold.

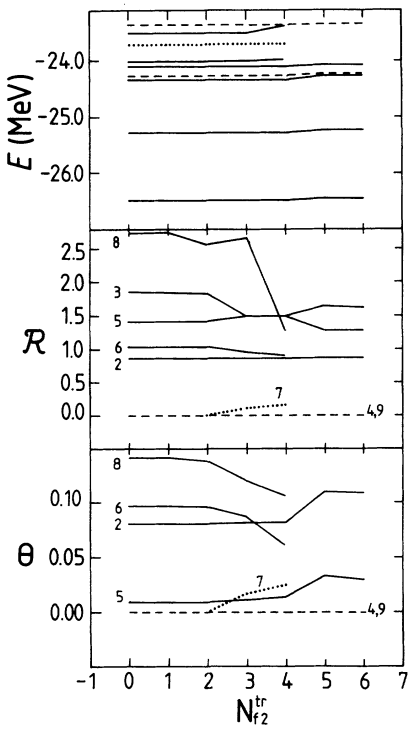

FIG. 10. The same as in Fig. 9 but for the dependence on $N_{f2}^{\rm tr}$. The parameters of the Hamiltonian [Eq. (8)] are the same as in Fig. 9, but $n_p = n_n = 6$. The choice of $G_f = 0.05$ MeV guarantees the weak coupling. For the details, see the text.

V. CONCLUSIONS

The large discrepancy we found between BCS-RPA and exact calculations of neutron monopole pairing vibrations for even-even Ge nuclei indicates that caution needs to be exercised when BCS-RPA theory is applied in studies of nuclei in at least the mass region of germanium. The presence of a $j = 1/2$ level seems to play an important role in the breakdown of BCS-RPA.

It was furthermore established that the variation of parameters within reason in exact diagonalizations of the monopole pairing Hamiltonian in the neutron configuration only is insufficient for a first order description of the intruder 0^+ state in even-even Ge nuclei.

As a next step, the model space was increased to include both proton and neutron pairing as well as proton-neutron pairing. In this extended pairing model, the first excited 0^+ state has indeed been found at a much lower energy than was possible in the neutron pairing case. However, the strong experimental variation in energy with a minimum at ^{72}Ge could not be achieved with this simple schematic interaction.

The extended model calculations, carried out in a boson space, were simultaneously used to investigate the identification of spurious states. It was first illustrated how spurious states, associated with the use of a simple but complete ideal boson basis (for the extended pairing model), could easily be identified. An independent

aspect studied in this context is the feasibility of identifying spurious states in truncated spaces. For weakly coupled states, spurious states can still be identified. For this the physical operator method is the simplest, but it appears that one has to be careful in order to avoid misinterpretation. The $\mathcal{R}$ projection is somewhat more cumbersome, but it appears that the interpretation remains relatively straightforward.

At this stage we may summarize the problem of appearance of spurious states in the Dyson boson mapped systems as follows. In the noncollective SO(4Ω) algebra, spurious states already appear when the boson number for a particular kind of boson exceeds 1. For this case the number of different kinds of bosons grows with Ω. On the other hand, in the collective models like SO(5), SO(8), Sp(6), etc., the number of different kinds of collective bosons is fixed. It is then apparent that for sufficiently large Ω, which characterizes the total dimension of the fermion space, a stage is reached where no spurious states appear.

In this regard it is useful to study the coherent state overlap for a particular model [9]. From it one may deduce that, e.g., for the SO(8) model spurious states appear for boson number N when $N > \Omega/2, \Omega = 2(2k+1)$, for the Sp(6) model when $N > \Omega/3$, with $\Omega = 3(2i+1)/2$, and for the SO(5) model when $N > \Omega$ with $\Omega = j + \frac{1}{2}$. Generally speaking, such results seem to confirm the natural expectation that for a "sufficiently large" space (or "sufficiently small" boson number) spurious states are avoided. However, if a multi-j configuration space is used which includes a relatively small shell or, alternatively, one deals with a direct product of algebras including one pertaining to such a shell, the situation may not be that simple. If only the total boson number is then taken into consideration, spurious states are likely to appear, but can be identified in the full boson space as discussed in the paper. On the other hand, if a truncation is carried out, spurious states may mix strongly and obscurely with physical ones. When the truncation involves only one algebra appearing in a product, spurious states stemming from the unharmed algebra(s) may, however, still be identified, as also discussed. Otherwise, and especially when collective bosons are determined dynamically, truncation leads to spurious states which are not easily identified by the rather simple adaptions of the exact identification procedure considered here. In such instances an effective operator approach has been developed and used [12, 13]. Further investigations are still required to establish whether approximations can then be found which lead to simplified practical identification methods for these cases.

When a simple basis truncation is introduced in a boson calculation, an overoptimistic assessment is that, merely because of the possible large size of the fermion space, one need not be concerned about spurious states. Our investigation shows that this view is unwarranted in general. Nevertheless, a large model space can be helpful.

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