Expansion of the Two-Nucleon T Matrix Half Off the Energy Shell*

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Current descriptions of $(p, 2p)$ reactions, nucleon-nucleon bremsstrahlung, and low-energy pion production use the half-off-shell two-nucleon T matrix in the neighborhood of the on-shell point. We present a method for expanding the amplitude $t_0(p, k, ; k^2)$ in powers of $p-k$. After explicitly extracting the contributions arising from the long-range part of the interaction, we obtain a power series with an infinite radius of convergence. The coefficients of this series are expressed in terms of weighted integrals of the difference function, thus permitting us to determine precisely how variations of the interior wave function affect the near off-shell behavior of the half-shell T matrix. This allows treatment of a large class of models for the two-nucleon interaction, including wave-function models as well as local, nonlocal, and energy-dependent potentials. As an example, we apply this method to a wave-function model of the two-nucleon $^1S_0$ T matrix constructed by the authors in a previous paper. We find that the expansion converges rapidly and that the inclusion of more terms extends the representation farther off the energy shell. The expansion coefficients are smooth, rapidly decreasing functions of energy. The even coefficients and the odd coefficients each form a family of similar functions which fall off rapidly with index. The T matrix arising from this model is very similar to those obtained with realistic potentials. Since the coefficient functions can be tabulated or parametrized quite simply, this expansion should provide a compact, useful method of expressing and comparing the near off-shell behavior of the two-nucleon half-shell T matrices arising from different models while permitting one to maintain both a fixed phase shift and the correct long-range behavior of the potential.

I. INTRODUCTION

Certain nuclear reactions are believed to probe the off-shell behavior of the two-nucleon force in a direct manner. These include $(p, 2p)$ reactions,[1] bremsstrahlung,[2] and pion production near threshold.[3] They are described as proceeding primarily via reaction mechanisms which involve two processes: (1) a pair of nucleons interact with each other; and (2) a breakup or production process occurs. (See Fig. 1.) The second step limits the extent of the time taken for step (1) so that the scattering of the nucleons need not conserve energy.

There are three energies involved in a general two-nucleon scattering: the initial relative energy, $k_i^2/2m$, the final relative kinetic energy, $k_f^2/2m$, and the (center-of-mass) energy of propagation, E. We will write the partial-wave T matrix for a scattering as $t_l(k_i, k_f; E)$. If the three energies are all unequal, the T matrix is said to be *fully off shell*.

In the reactions described above, the time for the two-nucleon scattering process is only restricted on one side. After (before) their interaction, the two nucleons proceed to (come from) infinity unscattered. The center-of-mass energy of propagation must therefore be equal to the final (initial) relative kinetic energy. The T matrix applicable in this case, $t_l(k', k; k^2/2m)$, is called a *half-shell T matrix*. We will call a reaction proceeding by such a process a *half-shell reaction*.

In elastic two-nucleon scattering, there are no other particles to provide additional scatterings and thus constrain the time available for the two-nucleon interaction. As a result, the three energies must be equal. The T matrix for the elastic process, $t_l(k, k; k^2/2m)$, is called *on shell* and can be expressed in terms of the elastic scattering phase shifts.

The description of the half-shell reactions requires the calculation of the half-shell T matrices. Neither the potential[4] nor the T-matrix methods[5] currently in use for describing the two-nucleon interaction in a semiphenomenological manner are well adapted to a description of half-shell reactions, since their adjustable parameters are very remote from the T matrix needed. In this paper we present a method for parametrizing the half-shell T directly. This method can serve as a useful and simple intermediary step between the theoretical models and the experimental data, and can greatly simplify the calculation and use of T matrices in the analysis of half-shell experiments.

The region of the half-shell T relevant to the half-shell reactions is crucial for the choice of method. In $(p, 2p)$ reactions the amount that the scattering is off shell is controlled by the binding energy of the struck proton and the extent of its form factor in momentum space.[1] In stable nuclei this restricts the difference between the initial and final relative kinetic energies of the pair to less than 200 MeV (in practice, $|k-k'|< 1\ \mathrm{F}^{-1}$). In bremsstrahlung the amount the scattering is off shell is determined by the energy carried off by the photon. In principle this can be anything, but in practice, experiments have been restricted to incident proton energies of about 200 MeV or less.[6] This implies $|k-k'|<1.5\ \mathrm{F}^{-1}$. The difference of the initial and final energies in pion production is determined primarily by the pion's mass. As a result, the difference is on the order of 150 MeV ($|k-k'|\sim 2\ \mathrm{F}^{-1}$). The reaction mechanism becomes more complicated well above threshold owing to the presence of pion-nucleon resonances.[3]

We are therefore primarily interested in half-shell T matrices for which $|k-k'|\leq 2\ \mathrm{F}^{-1}$. The on-shell momenta may vary from essentially zero (in some bremsstrahlung experiments) up to 4 or 5 F^{-1} [in some $(p, 2p)$ experiments]. In addition, the on-shell value of the T matrix is well determined from elastic scattering experiments. We therefore choose to expand the half-shell T matrix in a power series about the on-shell point. Previous expansions of the half-shell T matrix[7] were carried out about the point $k=p=0$. These methods are not convenient at energies above the region of applicability of the effective-range expansion and can not easily include the experimentally determined on-shell values. The coefficients of our series serve as parameters for a description of the half-shell behavior of the T matrix. Calculations of the half-shell T matrices arising from various models in the regions of k and k' indicated show a slow enough variation that we expect a few terms in the power series to suffice.[8]

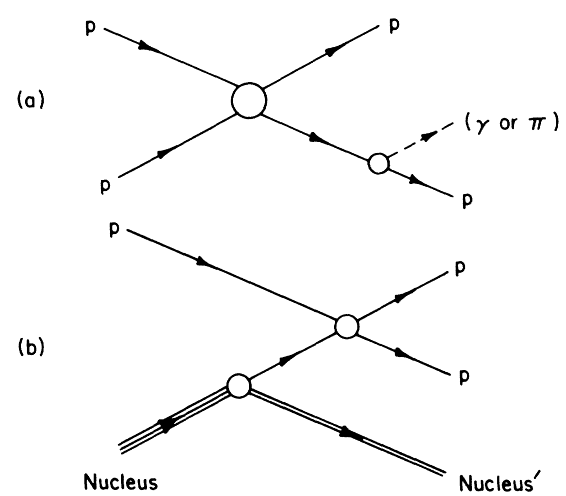

FIG. 1. Diagrams for half-shell reactions: (a) post-emission in a production reaction; (b) prior emission in a production reaction, or knockout.

Since the potential in the interior region is poorly known, we eliminate it in Sec. II using the Schrödinger equation and express the half-shell T matrix in terms of the difference function (the difference between the exact scattering wave function and the phase-shifted free wave function). An expansion of the T matrix about the on-shell point is then carried out, the coefficients being expressed in terms of moments of the difference function. As is well known,[9] the half-shell T matrix has complex singularities in the off-shell momentum for Yukawa or exponential tails. The contributions of the tails are eliminated, resulting in an expansion with an infinite radius of convergence. This also permits the inclusion of our knowledge of the longest-range parts of the interaction. Thus, the known aspects of the two-nucleon interaction, namely, the on-shell T matrix and the long-range tail of the potential, are included exactly. The off-shell variation resulting from changing our off-shell parameters reflects the remaining indeterminacy in the problem, namely, the wave function in the interior.

In order to determine whether one may realistically expect the resulting expansion to provide a good representation of the half-shell T matrix in a sufficiently large region, we apply the expansion to a model of the two-nucleon scattering wave function constructed by the authors in a previous paper.[10] The expansion coefficients can be calculated analytically and the approximate results obtained by truncating the expansion compared with the exact T matrix. We find that a few coefficients suffice to give a good fit quite far off the energy shell, and that the coefficients possess the nice properties of smooth variation with energy and rapid decrease with index.

The results and conclusions are summarized in Sec. IV. The derivation of the expansion coefficients and a discussion of their low-energy behavior is given in the Appendix.

II. OFF-SHELL EXPANSION

In this section we expand the half-shell T matrix in a Taylor series about the on-shell point and relate the coefficients directly to the exact wave function. We restrict ourselves to uncoupled s waves, since the half-shell behavior in the $^1S_0$ state is the most important for the half-shell p-p

reactions. The extension to an arbitrary uncoupled partial wave without bound state is straightforward, although algebraically more complicated.

We choose units such that $\hbar = 2m = 1$, where m is the two-nucleon reduced mass. The s-wave T matrix is then defined by

$$t_0(p,k;k^2) = \int_0^\infty j_0(pr)[V\psi_k^{(+)}](r)r^2 dr\,, \tag{1}$$

where j_0 is the spherical Bessel function, $j_0(x) = x^{-1}\sin(x)$. $\psi_k^{(+)}$ is the exact outgoing scattering wave function normalized to

$$\psi_k^{(+)}(r) \underset{r\to\infty}{\longrightarrow} \frac{1}{kr} e^{i\delta_0(k)} \sin[kr + \delta_0(k)]\,, \tag{2}$$

where δ_0 is the s-wave phase shift. In general, we permit V to contain both local and nonlocal parts. Since V, j_0, and $e^{-i\delta_0(k)}\psi_k^{(+)}$ are real, t_0 has the form

$$t_0(p,k;k^2) = e^{i\delta_0(k)}\tau_0(p,k;k^2)\,, \tag{3}$$

with τ_0 real. The on-shell T matrix is given by

$$\tau_0(k) \equiv \tau_0(k,k;k^2) = -\frac{1}{k}\sin\delta_0(k)\,. \tag{4}$$

Since we wish to deemphasize the potential, we use the Schrödinger equation to eliminate V. The kinetic energy term is then integrated by parts. The surface term at infinity is eliminated by subtracting the phase-shifted free wave function, $e^{i\delta_0(k)}(kr)^{-1}\sin[kr+\delta_0(k)]$, from $\psi_k^{(+)}$. Since the differential operator in the integral annihilates this, the value of the integral is unchanged. As a result, the wave function $\psi_k^{(+)}$ is replaced by the difference function,

$$\Delta_0(k,r) = e^{-i\delta_0(k)}kr\psi_k^{(+)}(r) - \sin[kr+\delta_0(k)]\,. \tag{5}$$

As may be seen from Eq. (2), $\Delta_0(k,r)$ vanishes asymptotically in r. The final equation [derived in Ref. (10)] is

$$\tau_0(p,k;k^2) = \tau_0(k) + \frac{k^2-p^2}{pk}\int_0^\infty dr\,\sin pr\,\Delta_0(k,r)\,. \tag{6}$$

The dependence of τ_0 on the off-shell variable, p, comes only via known analytic functions. This equation therefore provides an analytic continuation of τ_0 to the complex p plane. In order to expand in p about k we introduce the variable $q = p-k$. In terms of q, Eq. (6) becomes

$$\tau_0(k+q,k;k^2) = \tau_0(k) - \frac{q^2+2qk}{k(k+q)} \times \int_0^\infty dr\,\sin[(k+q)r]\,\Delta_0(k,r)\,. \tag{7}$$

Expanding all the functions of q in a power series about the on-shell point, $q=0$, and collecting terms, we find

$$\tau_0(k+q,k;k^2) = \tau_0(k) + \sum_{m=1}^{\infty} \overline{A}_m(k)q^m\,, \tag{8}$$

where

$$\overline{A}_m(k) = (-1)^m k^{-m} \sum_{n=0}^{[m/2]-1} (-1)^n[\overline{D}^{(2n)} - \overline{D}^{(2n+1)}] + (1+\delta_{(1+m)/2,\,[(1+m)/2]})(-1)^{m/2}\overline{D}^{(m-1)}\,. \tag{9}$$

The notation $[n]$ means the largest integer contained in n, and the functions $\overline{D}^{(n)}(k)$ are related to moments of the difference function weighted with a trigonometric function. Explicitly, we define

$$\overline{D}^{(n)}(k) = \frac{k^n}{n!}\int_0^\infty dr\, r^n\,\Delta_0(k,r)\begin{cases}\sin kr & \text{for } n \text{ even}\\ \cos kr & \text{for } n \text{ odd}\,.\end{cases} \tag{10}$$

The derivation of Eqs. (8) and (9) and the behavior of the coefficients $A_m(k)$ for small k are given in the Appendix. Note that we have chosen to expand the simple function $(k+q)^{-1}$ in powers of q. We thereby eliminate the apparent pole at $q=-k$ ($p=0$). This is not a true singularity, since the zero in the denominator is canceled by a zero in the numerator at the same point. It would, however, appear as a singularity in a truncated series if the numerator alone were expanded. [See also the discussion in the Appendix following Eq. (A6).]

This provides the desired relation between the expansion coefficients and the interior wave function (equivalently, the difference function). The region of applicability of the series [Eq. (8)] is somewhat restricted by the singularities of τ_0 in the complex p plane. Since these singularities arise from the long-range parts of the two-nucleon potential, they can be eliminated if we use our knowledge of the potential tail.[6] This has the additional advantage of assuring that all our parametrizations will be consistent with this knowledge.

The singularities of τ_0 are well known for the cases of interest, but they may also be obtained directly from Eq. (6). For example, if the potential vanishes outside the radius R, then so does the function $\Delta_0(k,r)$. The off-shell part of the T matrix, defined by

$$\tau_0^{\text{off}}(p,k;k^2) \equiv \tau_0(p,k;k^2) - \tau_0(k) = \frac{k^2-p^2}{pk}\int_0^\infty dr\,\sin pr\,\Delta_0(k,r)\,, \tag{11}$$

is finite for all complex values of p as long as $kr\psi_k^{(+)}(r)$ is absolutely integrable on the interval

$[0, R]$. (Note that this condition is weaker than those usually encountered in potential treatments[11] and will hold even for hard-core potentials and nonsingular finite-range nonlocal potentials.) The amplitude is then an entire function of p.

For an exponential potential[9] the singularities are poles at $p = \pm k \pm ni\mu$, where μ is the inverse range of the exponential and n is a positive integer. For a Yukawa potential of inverse range μ the singularities are cuts[9] originating at $p = \pm k \pm ni\mu$. These are illustrated in Fig. 2.

For the two-nucleon interaction the longest-range part of the potential is due to one-pion exchange. The range of the resulting Yukawa potential is the pion Compton wave length or about 1.4 F. The radius of convergence of the expansion on Eq. (8) is therefore about 0.7 F^{-1}. This has a number of implications for the use of this expansion.

First, we are interested in seeing the relation between the off-shell coefficients and the interior wave function. Since the coefficients are composed of integrals over the difference functions times powers of r, the long-range parts of the difference function will be weighted more heavily than the short-range parts. Since there will be large cancellations between different terms from those parts due to the tail (indeed, it is the tail contributions which lead to the divergent part of the series), our information concerning the interior is severely masked. Fortunately, the long-range part of the two-nucleon interaction is rather well known. If we extract the contribution of a known tail explicitly, our information concerning the interior wave function is improved.

Second, since the behavior of the long-range part of the interaction is believed known, we wish to include it in order to be certain that our parametrizations are consistent with that behavior.

Third, if one wishes to extract the coefficients from a half-shell experiment and to use the resulting representation in a region *larger* than that in which there are data, the representation will not converge to the exact T matrix as additional terms are added to the series. This is a consequence of the standard theorems of complex analysis.[12] To demonstrate that this has practical consequences, we have taken the exact T matrix from a model of Ref. 10 described in the next section and fitted it with a power series in q. Twelve data points around the on-shell point having $|q| < 1.05$ F^{-1} were taken as input. Fits were obtained with third-, fourth-, fifth-, and sixth-degree polynomials.[13] These functions were then extended beyond $|q| = 1.05$ F^{-1}. The results are displayed in Fig. 3 (a). Large variations occur outside the data region. Figure 3(b) shows the results of fitting the same data after the contribution of the tail has been removed by the method described above. The tail contribution has been added back in after the fitting. The convergence is considerably improved.

For these reasons, we explicitly put in the potential for large distances. It is generally assumed that the two-nucleon interaction is local beyond about one pion Compton wavelength ($R \sim 1.43$ F). We may then write

$$V(r, r') \approx \frac{L(r)\delta(r - r')}{rr'} + N(r, r') \quad r, r' > R, \quad (12)$$

with

$$|L(r)| \gg \int dr'\, r'^2 |N(r, r')|, \quad r > R. \quad (13)$$

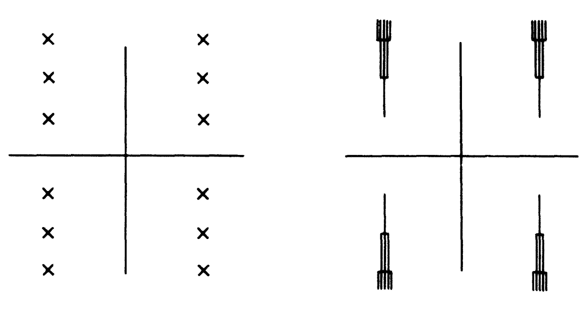

FIG. 2. Singularities of $\tau_0(p, k; k^2)$ in the complex p plane for (a) an exponential potential, (b) a Yukawa potential.

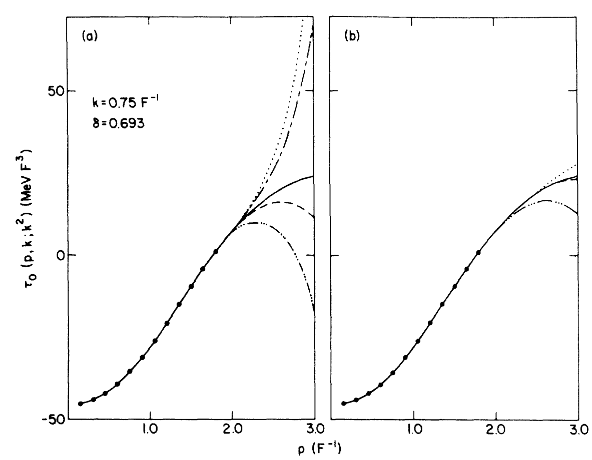

FIG. 3. Analytic continuation of $\tau_0(p, k; k^2)$ as a function of p for $k = 0.75$ F^{-1}. The heavily marked points were taken as input data and fitted with polynomials of third–sixth degree. The curves are indicated as follows: exact, ——; third, – – –; fourth, —·····—; fifth, —·—; sixth, ······. In (a) the value of the full T matrix was taken as input data. In (b) the polynomials were fitted to the interior contribution alone. The contribution of the tail has been added back in after fitting.

The difference function for $r>R$ satisfies the integral equation

$$\Delta_0(k,r)=\Delta_0^{(0)}(k,r)+\frac{1}{k}\int_r^\infty dr'\sin k(r'-r)\times L(r')\Delta_0(k,r')\quad r>R\,, \tag{14}$$

with

$$\Delta_0^{(0)}(k,r)=\frac{1}{k}\int_r^\infty dr'\sin k(r'-r)\times L(r')\sin[kr'+\delta_0(k)]\,. \tag{15}$$

This is derived in Ref. 10. Since this is a Volterra equation, the Neumann series converges to the solution. For the two-nucleon case we are aided considerably in practice by the fact that $\Delta_0^{(0)}$ is an excellent approximation to Δ_0 for $r>R$. (See Appendix B, Ref. 10.)

To include the long-range parts explicitly, we divide τ_0^{off} into a contribution from $r>R$ (the *exterior* region) and a contribution from $r<R$ (the *interior* region). For the purposes of this work, we will choose $R=1.43$ F. Assuming the tail, $L(r)$, is known, the contribution from the exterior to τ_0 may be calculated from Δ_0, which can be found by solving Eq. (14). We then expand the contribution of the interior in a power series. Since the integral for the interior now cuts off at R, it is entire in p. Thus we can write

$$\tau_0(p,k;k^2)=\tau_0(k)+\tau_0^{\text{ext}}(p,k;k^2)+\sum_1^\infty A_m(k)q^m\,, \tag{16}$$

where

$$\tau_0^{\text{ext}}(p,k;k^2)=\frac{k^2-p^2}{pk}\int_R^\infty dr\sin pr\,\Delta_0(k,r)\,. \tag{17}$$

The coefficients $A_m(k)$ are given by Eq. (9) with the moments $\bar{D}^{(n)}$ replaced by the moments $D^{(n)}$ defined by

$$D^{(n)}(k)=\frac{k^n}{n!}\int_0^R dr\,r^n\Delta_0(k,r)\begin{Bmatrix}\sin kr\\ \cos kr\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}n\text{ even}\\ n\text{ odd}\end{Bmatrix}\,. \tag{18}$$

Note that the integrations now only go from 0 to R.

The result of using Eq. (16) in an analytic continuation is considerably superior to that of Eq. (8), as is shown in Fig. 3. (Note that the coefficients were extracted directly from the curves rather than calculated from the difference functions.) In addition, the variations of the coefficients A_m do not conflict with our knowledge of the potential tails. It is therefore preferable to deal with the coefficients for the interior alone after explicit extraction of the tail contribution.

III. APPLICATION TO A PARTICULAR MODEL

As an example, we apply the method of the previous section to a model constructed in Ref. 10. This model reproduces the wave function and half-shell T matrix of a soft-core potential quite well, and has the advantage of extreme ease of calculation. We therefore expect the results obtained here to by typical of those which would be obtained for realistic potentials.

In the region beyond $r=R=1.43$ F the model considered takes the difference function as equal to the zeroth-order difference function, $\Delta_0^{(0)}$, given in Eq. (15). The phase shift and the exterior potential are taken from the soft-core potential of Reid.[4] This is done to simplify the caclulations. Ideally, the phase shift should be taken from experiment and the potential tail from theory. In the exterior region, the model difference function implies an exact wave function which agrees with the wave function produced by the Reid soft-core (RSC)[14] potential to better than one percent at energies corresponding to $k\gtrsim 0.15$ F^{-1}. Lower energies were not considered, because of the importance of the Coulomb potential at these energies. Coulomb corrections have not been included.

In the interior region, $r<R$, the difference function is represented by a polynomial in r. The coefficients of the polynomial are restricted by the condition that the value and the first two derivatives of the difference function match those of the exterior difference function, and by the condition that the model wave function and its first two derivatives vanish at the origin. The latter condition results in a suppression of the wave function at the origin similar to that due to a soft-core potential. The smoothest polynomial fitting the end-point conditions is fifth order, and we choose this polynomial for this paper. The interior difference function can be written

$$\Delta_0(k,r)=\sum_0^5 C_m r^m\,. \tag{19}$$

The equations for the coefficients C_m are given in Ref. 10. This reference also shows that the wave functions and the T matrices arising from this model are very similar to those of the RSC.

The RSC potential consists of a sum of Yukawa potentials. For such a potential $\Delta_0^{(0)}$ can be found in closed form in terms of exponential integrals. [See Ref. 10, Eq. (30).] These determine the coefficients $C_m(k)$. Since the interior wave functions are polynomials, $D^n(k)$ may be calculated analytically using standard integrals.[15] Explicitly, we find

$$D^{2n}=\frac{k^{2n}}{(2n)!}\sum_{m=1}^{5} C_m\left[\frac{(m+2n)!}{k^{m+2n+1}}(-1)^n\cos\frac{m\pi}{2}-\sum_{l=0}^{m+2n} l!\binom{m+2n}{l}\frac{R^{m+2n-l}}{k^{l+1}}\cos\left(kR+\frac{l\pi}{2}\right)\right], \tag{20}$$

$$D^{2n+1}=\frac{k^{2n+1}}{(2n+1)!}\sum_{m=0}^{5} C_m\left[\frac{(m+2n+1)!}{k^{m+2n+2}}(-1)^n\sin\frac{(m+1)\pi}{2}-\sum_{l=0}^{m+2n+1} l!\binom{m+2n+1}{l}\frac{R^{m+2n+1-l}}{k^{l+1}}\sin\left(kR+\frac{l\pi}{2}\right)\right].$$

Calculations of the difference function, half-shell T matrix, and expansion coefficients were carried out for various values of k. The fits to the T matrix at three typical momenta ($k=0.75\ \text{F}^{-1}$, $1.2\ \text{F}^{-1}$, and $1.8\ \text{F}^{-1}$) are displayed in Fig. 4.

Note that the retention of additional terms provides a good fit increasingly far off the energy shell. The third order fits well up to about $1.5\ \text{F}^{-1}$ off shell, while the fifth order extends the fit to almost $2\ \text{F}^{-1}$ off shell. This is considerably farther than the radius of convergence of the original series ($\sim 0.7\ \text{F}^{-1}$). The fit of a given order generally improves as the on-shell energy increases.

The expansion will provide a useful representation of the half-shell T matrix if two conditions are satisfied: (1) the coefficients decrease rapidly as the index increases, so that a few terms in the series suffice to describe the relevant off-shell behavior at each energy; and (2) each coefficient is a smoothly varying function of energy. The latter condition guarantees that the coefficients can be tabulated at a small number of points and the remaining values obtained by interpolation.

The even coefficients A_2, A_4, and A_6 are displayed in Fig. 5(b), and the odd coefficients A_1, A_3, and A_5 are displayed in Fig. 5(a). Owing to the evenness of $j_0(x)$, the odd coefficients must vanish at $k=0$, while the even ones do not.[16] Note that each set shows the following features: (1) the shapes of the curves are similar; (2) except for the rapid rise of the odd coefficients at very low energies, the coefficients fall rapidly and smoothly as a function of energy; and (3) the coefficients change sign and drop by an order of magnitude as the index is increased by two. Although the variation of both sets looks rapid at low energies, the behavior is actually quite simple, as shown in the Appendix [Eq. (A29)]. In order to display the similarity of these functions at low k we have plotted logarithmically the even coefficients and the odd coefficients divided by k in Fig. 6. The similarity is striking.

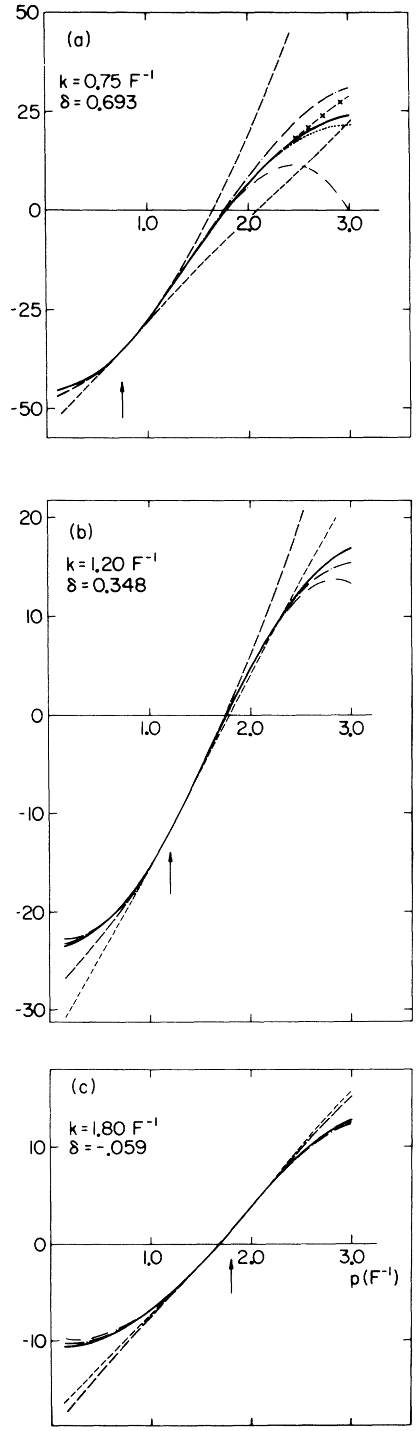

FIG. 4. Calculated polynomial approximations to $\tau_0(p, k; k^2)$ for (a) $k=0.75\ \text{F}^{-1}$, (b) $k=1.2\ \text{F}^{-1}$, and (c) $k=1.8\ \text{F}^{-1}$. The curves are indicated as follows: exact, ———; first-degree polynomial, -----; second, — — —; third, —·—; fourth, —···—; fifth, ······; sixth, —x— . The tail contribution has been included in all the curves. The arrow marks the on-shell point.

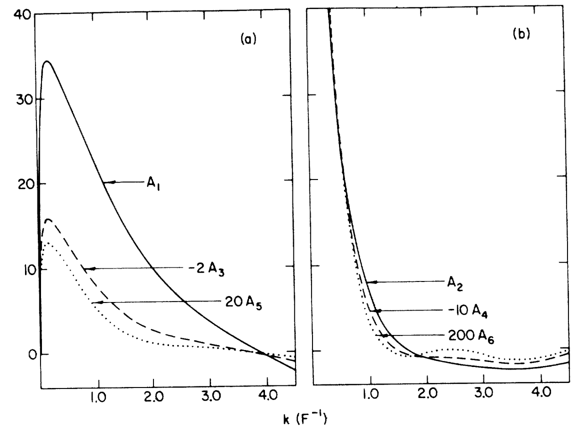

FIG. 5. (a), (b) Dependence of the off-shell coefficients on the on-shell momentum k.

The coefficients therefore possess the very desirable properties of rapid convergence and ease of parametrization and should be useful for expressing half-shell T matrices. Note that the quadratic term, A_2, is dominant for very small k ($k<0.4$ F^{-1}), while for larger k ($k>0.6$ F^{-1}) the linear term, A_1, is strongly dominant.

In order to show that the coefficients can be easily extracted from a few values of the half-shell T matrix, even using low-order polynomials, we have compared the coefficients determined for the analytic continuation discussed in conjunction with Fig. 3(b) with the calculated coefficients. Recall that the exact tail contribution was subtracted from the exact T matrix at the 12 heavily marked points. The resulting values were used as input data and fitted by a polynomial in q.[13] The values of the coefficients thus determined and the values calculated from the exact wave functions are displayed in Table I. The coefficients determined by the fit and the calculated coefficients agree to an accuracy of better than a few percent even when a low-order polynomial is used in the extraction routine. The three-term fit gives accurate values for all three coefficients determined. Since these coefficients are related to low-order sine and cosine moments of the difference function, information about the interior wave function may thus be extracted from the near off-shell behavior of the T matrix.[17] Since the half-shell T matrices in this model are very similar to those of realistic potentials, the same conclusion should hold for potential models.

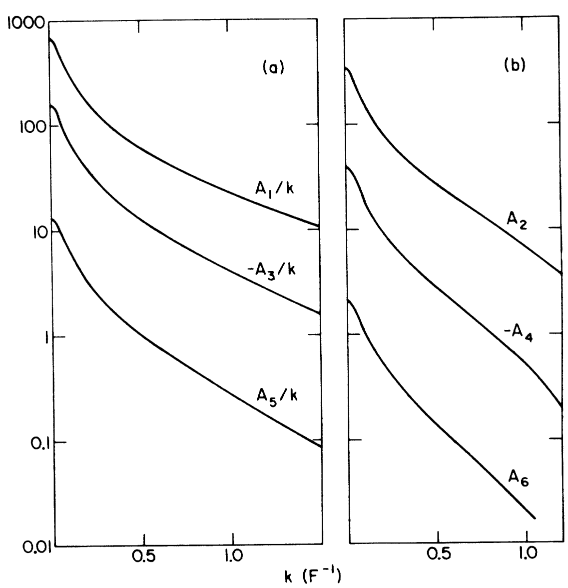

FIG. 6. (a), (b) Behavior of the off-shell coefficients for small values of k. Note the logarithmic scale.

The exact tail was used to calculate and extract τ^{ext}. In order to determine the sensitivity of the result to the tail, we have calculated the value of τ^{ext} for various potential tails. The tail used is

$$L(r)=\sum_{i=1}^{3} G_i \frac{e^{-\mu_i \mu r}}{\mu r}, \tag{21}$$

$$\mu_1=1.0,\ G_1=-10.463\ \text{MeV},\ \mu=0.7\ F^{-1},$$

$$\mu_2=4.0,\ G_2=-1650.6\ \text{MeV},$$

$$\mu_3=7.0,\ G_3=6484.2\ \text{MeV}.$$

Values of τ^{ext} for various strengths and ranges of the three terms are shown in Fig. 7 for $k=0.75$ F^{-1}. Only one factor was varied at a time. The inverse ranges μ_1 and μ_2 are the most important parameters. The strengths and μ_3 are less critical. The longest ranges are determined by the mass of the pion and the effective mass of the two-pion contribution. The pion mass is well known, but an accurate determination of the two-pion part may require some further investigation.[18]

TABLE I. Fitted and calculated coefficients. $k=0.75$ F^{-1}.

	Extracted coefficients $N=3$	$N=4$	$N=5$	Calculated coefficients
C_0 [a]	−35.30	−35.32	−35.32	−35.32
C_1	26.57	26.57	26.58	26.59
C_2	13.11	13.57	13.57	13.56
C_3	−5.06	−5.06	−5.16	−5.24
C_4		−1.23	−1.23	−1.24
C_5			0.21	0.38

[a] C_0 is the on-shell value of τ.

IV. SUMMARY AND CONCLUSIONS

We have presented an expansion of the half-shell T matrix around the on-shell point and related the coefficients to the exact wave function. The coefficients are sums of moments of a trigonometric function times the difference of the exact and phase-shifted free wave functions. The contribution of the long-range part of the potential is extracted explicitly from the T matrix. This has a number of advantages: (1) The contributions of the singularities are then calculated exactly, yielding a series with an infinite radius of convergence. (2) Most field-theoretic calculations and phenomenological potentials give essentially the same potential beyond about 1.5 F. If the contribution of this part of the potential is extracted explicitly, our resulting parametrizations will always be consistent with this knowledge. (3) We wish to focus on the short-range behavior of the wave functions. Because of the divergence of the power series for τ^{ext}, it is easier to extract the contributions of the exterior region before the expansion is made rather than after.

As a test of this method, we have applied it to a simple wave-function model of the two-nucleon $^1S_0$ state. This model has wave functions and T matrices closely resembling those of soft-core potentials, but is sufficiently simple that the off-shell coefficients and the contribution of the tail can be calculated in closed form. In this model, we find that the coefficients possess very nice properties, namely: (1) The even and the odd coefficients form families of similar functions. The odd coefficients vanish at $k=0$, while the even coefficients go to a constant. However, the behavior of the even coefficients near $k=0$ is similar to that of the odd coefficients divided by k. (2) Within each family the coefficients change sign and drop by an order of magnitude as the index is increased. (3) The variation of the coefficient functions with energy is simple and usually quite smooth. Since most phenomenological potentials have qualitatively similar T matrices, their off-shell coefficients should show similar behavior.

We have replaced a function of two variables, namely, the half-shell T matrix, by a discrete set of functions of one variable, namely, the energy-dependent coefficients of the power series, and an (almost) universal function of two variables which can be written in closed form, namely, the contribution of the long-range tail. The convergence rate of the power series and the simple form of the energy dependence suggest that these coefficients are a simple and useful way of parametrizing the off-shell behavior of the half-shell T matrix. The fact that the expansion is made in the difference between the on- and the off-shell momenta, $q=p-k$, rather than in the off-shell momentum, p, permits the inclusion of the on-shell

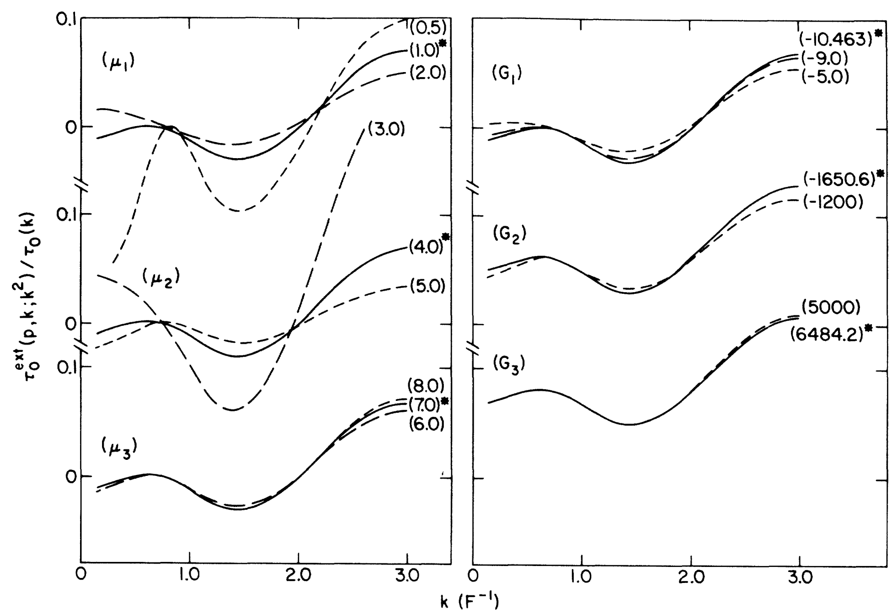

FIG. 7. Variation of the contribution of the potential tail to τ^{off} as its individual parameters are varied. The ratio of the contribution to $\tau_0(k)$ is presented for $k=0.75\ \text{F}^{-1}$. For each set of curves the parameter varied is indicated above to the left. The value of that parameter associated with each curve is given in parentheses to the right. The standard value is indicated with an asterisk.

T matrix exactly, and makes this the appropriate expansion for the treatment of the half-shell reactions. Since, in practice, the on-shell T matrix is much better known than the off-shell T matrices, this is a highly desirable feature. Furthermore, because the tail has been extracted, any parametrization by this method will be consistent with the correct long-range behavior of the two-nucleon potential.

Since we have related the off-shell coefficients to the difference function, we are able to determine how the interior wave function affects the near off-shell behavior. In currently accessible half-shell reactions, only two or three coefficients are relevant, so only a small number of integrals of the difference function weighted by simple functions are probed by these reactions [Eqs. (9) and (10)].

It is therefore possible to express realistic half-off-shell behavior in a simple form with two or three parameters, facilitating the inclusion of off-shell T matrices in the analysis of half-shell experiments with a minimum of computational effort. Furthermore, comparison of the coefficients arising from different models would indicate whether half-shell experiments could be expected to distinguish between the potentials in question.

APPENDIX

In this Appendix the expansion coefficients for the off-shell part of the T matrix are derived. We define the off-shell part of the T matrix by Eq. (11) with the upper limit of the integral replaced by a. Here, a is infinity for the coefficients $\bar{A}_m$ [Eq. (8)] and equals the cutoff radius R for the coefficients A_m [Eq. (16)]. When expressed in terms of the distance off shell, $q=p-k$, the off-shell part of the T matrix becomes

$$\tau^{\text{off}}(k+q,k;k^2)=-\frac{q^2+2kq}{k(k+q)}\int_0^a dr\,\Delta_0(k,r)\times(\sin kr\cos qr+\cos kr\sin qr)\,. \tag{A1}$$

If the trigonometric functions of qr are expanded in powers of q, we obtain

$$\tau^{\text{off}}=-\frac{q^2+2kq}{k(k+q)}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{q}{k}\right)^{2n}D^{(2n)}+\sum_{n=0}^{\infty}(-1)^n\left(\frac{q}{k}\right)^{2n+1}D^{(2n+1)}\right], \tag{A2}$$

where the functions $D^{(n)}$ are sine and cosine moments of the difference function:

$$D^{(2n)}(k)=\frac{k^{2n}}{(2n)!}\int_0^a dr\,\Delta_0(k,r)r^{2n}\sin kr\,,$$
$$D^{(2n+1)}(k)=\frac{k^{2n+1}}{(2n+1)!}\int_0^a dr\,\Delta_0(k,r)r^{2n+1}\cos kr\,. \tag{A3}$$

If we temporarily introduce the variable $z=q/k$, this becomes

$$\tau^{\text{off}}=-\frac{z^2+2z}{1+z}\left[\sum_0^{\infty}(-1)^n z^{2n}D^{(2n)}+\sum_0^{\infty}(-1)^n z^{2n+1}D^{(2n+1)}\right]. \tag{A4}$$

Defining the negative moments $D^{(-n)}$ ($n=1,2,\ldots$) as zero and incorporating the numerator into the brackets gives

$$\tau^{\text{off}}=\frac{1}{1+z}\left\{\sum_1^{\infty}(-1)^n z^{2n}[2D^{(2n-1)}+D^{(2n-2)}]-\sum_0^{\infty}(-1)^n z^{2n+1}[2D^{(2n)}-D^{(2n-1)}]\right\}. \tag{A5}$$

We now apply the following:

Lemma 1

$$\frac{1}{1+z}\sum_{n_0}^{\infty}(-1)^n z^{2n}a_n=\sum_{j=2n_0}^{\infty}z^j\left\{\sum_{n=n_0}^{[j/2]}(-1)^{j-n}a_n\right\},$$

where $[j/2]$ is the largest integer contained in $j/2$. (This and the subsequent lemmas are easily proven by explicating the terms on both sides.)

Applying this to Eq. (A5) gives

$$\tau^{\text{off}}=\sum_{j=2}^{\infty}z^j\left\{\sum_{n=1}^{[j/2]}(-1)^{j-n}[2D^{(2n-1)}+D^{(2n-2)}]\right\}-\sum_{j=0}^{\infty}z^{j+1}\left\{\sum_{n=0}^{[j/2]}(-1)^{j-n}[2D^{(2n)}-D^{(2n-1)}]\right\}. \tag{A6}$$

A word about the convergence of the series is in order here. It may appear that by expanding the term $(1+z)^{-1}$ we are unnecessarily introducing additional singularities into the expansion. This is not correct, as long as both series have nonzero radii of convergence. Within the minimum of the two radii both series converge absolutely.[19] Therefore, inside this radius we may rearrange them to obtain the indicated product series. These also converge absolutely. The expansion therefore provides an *element* of the analytic function τ^{off}. The region of convergence of this series will be determined only by the singularities of τ^{off} and not by those of the product functions. Since τ^{off} has no singularity at $q=-k$, the series obtained will converge there. The region of convergence will be as discussed in Sec. II in the body of the paper. (See Forsyth,[20] Sec. 34.)

We displace the limits and collect powers of z using the conventions: (1) negative moments vanish; (2) $[m] \leq m$; and (3) if the upper limit of a summation is negative, the sum vanishes. We then obtain the expansion

$$\tau^{\text{off}} = \sum_{m=0}^{\infty} z^m (-1)^m B_m = \sum_{m=0}^{\infty} A_m q^m , \tag{A7}$$

$$A_m = (-1)^m k^{-m} B_m , \tag{A7a}$$

with

$$A_m = \frac{1}{k^m}\left\{ \sum_{n=1}^{[m/2]} (-1)^{m-n}[2D^{(2n-1)} - D^{(2n-2)} + D^{(2n-3)}] + \delta_{(m+1)/2,\,[(m+1)/2]}(-1)^{(m+1)/2}[2D^{(m-1)} - D^{(m-2)}]\right\} . \tag{A8}$$

The function $\delta_{\alpha\beta}$ is 1 if its arguments agree, and 0 otherwise. In this case, it is 1 if m is odd, and 0 if m is even. For calculational purposes, a rearrangement of this series is useful. The even and odd D's may be collected separately and the limits on the summation altered somewhat to yield

$$B_m = \sum_{n=0}^{[m/2]-1} (-1)^n [D^{(2n)} - D^{(2n+1)}] + (1 + \delta_{(m+1)/2,\,[(m+1)/2]})(-1)^{[m/2]} D^{(m-1)} . \tag{A9}$$

This is the desired result. The first few coefficients are

$$B_0 = 0 , \qquad B_3 = D^{(0)} - D^{(1)} - 2D^{(2)} ,$$

$$B_1 = 2D^{(0)} , \qquad B_4 = D^{(0)} - D^{(1)} - D^{(2)} + 2D^{(3)} ,$$

$$B_2 = D^{(0)} - 2D^{(1)} , \quad B_5 = D^{(0)} - D^{(1)} - D^{(2)} + D^{(3)} + 2D^{(4)} . \tag{A10}$$

The coefficients may also be expressed recursively beginning with Eq. (A7). One easily obtains

$$B_{2m+1} = B_{2m} + (-1)^m [2D^{(2m)} - D^{(2m-1)}] ,$$
$$B_{2m+2} = B_{2m+1} - (-1)^m [2D^{(2m+1)} + D^{(2m)}] . \tag{A11}$$

One point remains to be considered. From a first glance at Eq. (A9), it appears as if the coefficients A_m, $m \geq 1$, become infinite as k approaches zero. This conflicts with what one would have obtained from expanding Eq. (A2) in powers of q after the limit k goes to zero had been taken. [Note that $\Delta_0(0, r) = 0$ from Eq. (5).] There is really no conflict, since the coefficients of all the divergent powers cancel, as we now show by explicitly expanding the coefficients B_m in powers of k.

We require the quantities

$$\Delta^{(l)}(0, r) \equiv \left.\frac{d^l \Delta_0(k, r)}{dk^l}\right|_{k=0} , \tag{A12}$$

and their moments

$$\Delta_n^{(l)} \equiv \frac{1}{n!\,l!} \int_0^a dr\, r^n \Delta^{(l)}(0, r) . \tag{A13}$$

The full $\Delta_0(k, r)$ may now be expanded in the Taylor series

$$\Delta_0(k, r) = \sum_{l=0}^{\infty} \frac{k^l}{l!} \Delta^{(l)}(0, r) . \tag{A14}$$

Using this and expanding the trigonometric function in Eq. (A3) in a power series gives

$$D^{(2n)} = k^{2n+1} \sum_{l,j=0}^{\infty} (-1)^j k^{2j+l} \binom{2n+2j+1}{2n} \Delta_{2n+2j+1}^{(l)} ,$$

$$D^{(2n+1)} = k^{2n+1} \sum_{l,j=0}^{\infty} (-1)^j k^{2j+l} \binom{2n+2j+1}{2n+1} \Delta_{2n+2j+1}^{(l)} . \tag{A15}$$

The symbol $\binom{m}{n}$ is the standard binomial coefficient. We now group powers of k using the following:

Lemma 2

$$\sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j k^{2j+l} g(l, j) = \sum_{m=0}^{\infty} k^{2m} \sum_{l=0}^{m} (-1)^l g(2m-2l, l) + \sum_{m=0}^{\infty} k^{2m+1} \sum_{l=0}^{m} (-1)^l g(2m-2l+1, l) .$$

This gives

$$D^{(2n)} = k^{2n+1} \sum_{m=0}^{\infty} k^{2m} \sum_{l=0}^{m} (-1)^l \binom{2n+2l+1}{2n} [\Delta_{2n+2l+1}^{(2m-2l)} + k\Delta_{2n+2l+1}^{(2m-2l+1)}] ,$$

$$D^{(2n+1)} = k^{2n+1} \sum_{m=0}^{\infty} k^{2m} \sum_{l=0}^{m} (-1)^l \binom{2n+2l+1}{2n+1} [\Delta_{2n+2l+1}^{(2m-2l)} + k\Delta_{2n+2l+1}^{(2m-2l+1)}] . \tag{A16}$$

We now write

$$B_\mu = B_\mu^\Sigma + B_\mu^\delta . \tag{A17}$$

The term B_μ^Σ is the summation term in Eq. (A9), while B_μ^δ is the term proportional to $(1+\delta)$. Inserting Eq. (A16) gives

$$B_\mu^\Sigma = \sum_{n=0}^{[\mu/2]-1} (-1)^n k^{2n+1} \sum_{m=0}^{\infty} k^{2m} \sum_{l=0}^{m} (-1)^l \left[\binom{2n+2l+1}{2n} - \binom{2n+2l+1}{2n+1} \right] [\Delta_{2n+2l+1}^{(2m-2l)} + k\Delta_{2n+2l+1}^{(2m-2l+1)}] . \tag{A18}$$

We now perform the following manipulations on this form: (1) interchange the m and l sums, (2) redefine the m variable by $\nu=m-l$, (3) redefine the l variable by $\lambda=n+l$, and (4) interchange the n and λ sums. For steps (1) and (4) we need the following:

Lemma 3

$$\sum_{m=0}^{\infty} \sum_{l=0}^{m} g(m,l) = \sum_{l=0}^{\infty} \sum_{m=l}^{\infty} g(m,l) ;$$

Lemma 4

$$\sum_{n=0}^{n_0} \sum_{\lambda=n}^{\infty} g(n,\lambda) = \sum_{\lambda=0}^{\infty} \sum_{n=0}^{\min(\lambda,n_0)} g(n,\lambda) .$$

The resulting expression is

$$B_\mu^\Sigma = \sum_{\nu=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{n=0}^{\min(\lambda,n_0)} (-1)^\lambda k^{2\lambda+2\nu+1} \left[\binom{2\lambda+1}{2n} - \binom{2\lambda+1}{2n+1} \right] \{\Delta_{2\lambda+1}^{(2\nu)} + k\Delta_{2\lambda+1}^{(2\nu+1)}\} , \tag{A19}$$

with

$$n_0 = \left[\frac{\mu}{2}\right] - 1 . \tag{A20}$$

The only part of the summand depending on n is the square brackets containing the binomial coefficients. Let us look at this sum in detail. We have

$$\sum_{n=0}^{\min(\lambda,n_0)} \left[\binom{2\lambda+1}{2n} - \binom{2\lambda+1}{2n+1} \right] = \sum_{n=0}^{2\min(\lambda,n_0)+1} (-1)^n \binom{2\lambda+1}{n} .$$

If $\lambda \le n_0$, then $\min(n_0,\lambda)=\lambda$ and

$$\sum_{n=0}^{\min(\lambda,n_0)} \left[\binom{2\lambda+1}{2n} - \binom{2\lambda+1}{2n+1} \right] = \sum_{n=0}^{2\lambda+1} (-1)^n \binom{2\lambda+1}{n} = (1-1)^{2\lambda+1} = 0 . \tag{A21}$$

Therefore, all the terms with $\lambda \le n_0$ cancel. Equation (A19) may then be written (relabeling $l=\lambda-n_0-1$)

$$B_\mu^\Sigma = (-1)^{n_0+1} k^{2n_0+3} \sum_{\nu=0}^{\infty} k^{2\nu} \sum_{l=0}^{\infty} (-1)^l k^{2l} [\Delta_{2l+2n_0+3}^{(2\nu)} + k\Delta_{2l+2n_0+3}^{(2\nu+1)}] \sum_{n=0}^{2n_0+1} (-1)^n \binom{2l+2n_0+3}{n} . \tag{A22}$$

Using the fact, obtained from Eq. (5), that

$$\Delta_m^{(0)} = 0 , \tag{A23}$$

we find as $k \to 0$

$$B_\mu^\Sigma \to (-1)^{n_0} k^{2n_0+4} \Delta_{2n_0+3}^{(1)} 2(n_0+1) ; \tag{A24}$$

but

$$n_0 = \begin{cases} \frac{1}{2}(\mu-2), & \mu \text{ even} \\ \frac{1}{2}(\mu-3), & \mu \text{ odd} \end{cases} , \tag{A25}$$

so

$$B_{2j}^\Sigma \xrightarrow[k\to 0]{} k^{2j+2}(-1)^{j-1}\Delta_{2j+1}^{(1)} 2j ,$$

$$B_{2j+1}^\Sigma \xrightarrow[k\to 0]{} k^{2j+2}(-1)^{j-1}\Delta_{2j+1}^{(1)} 2j . \tag{A26}$$

The low-k behavior of B_μ^δ may be obtained directly from Eqs. (A9) and (A15):

$$B_{2j}^\delta = (-1)^j k^{2j-1} \sum_{i,l=0}^{\infty} (-1)^i k^{2i+l} \binom{2j+2i-1}{2j-1} \Delta_{2i+2j-1}^{(l)} ,$$

$$B_{2j+1}^\delta = 2(-1)^j k^{2j+1} \sum_{i,l=0}^{\infty} (-1)^i k^{2i+l} \binom{2j+2i+1}{2j} \Delta_{2i+2j+1}^{(l)} . \tag{A27}$$

These give the limits

$$B_{2j}^\delta \xrightarrow[k\to 0]{} (-1)^j k^{2j} \Delta_{2j-1}^{(1)} ,$$

$$B_{2j+1}^\delta \xrightarrow[k\to 0]{} 2(-1)^j k^{2j+2} \Delta_{2j+1}^{(1)} (2j+1) . \tag{A28}$$

Putting together Eqs. (A7a), (A26), and (A28) gives

$$A_{2j} \underset{k\to 0}{\longrightarrow} (-1)^j \Delta^{(1)}_{2j-1}\,,$$
$$A_{2j+1} \underset{k\to 0}{\longrightarrow} k(-1)^j 2(j+1)\Delta^{(1)}_{2j+1}\,. \qquad \text{(A29)}$$

This leads to the equation

$$\tau^{\text{off}}(q,0;0)=\sum_{j=1}^{\infty} q^{2j}(-1)^j\Delta^{(1)}_{2j-1}\,, \qquad \text{(A30)}$$

which agrees with the result obtained when Eq. (A2) is expanded about $q=0$ *after* the limit $k\to 0$ has been taken using $\Delta(k,r)/k\to\Delta^{(1)}(0,r)$.

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