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Coulomb Correction in Nuclear β Decay: Elementary-Particle Treatment

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The "elementary-particle" treatment of nuclear β decay is extended to include the effects of final-state Coulomb interactions. The resulting formalism is valid for both allowed and forbidden decays. The Coulomb correction in allowed transitions is considered in detail and compared with the usual impulse-approximation calculation. The spectrum shape factor for the transition $B^{12} \rightarrow C^{12} + e^- + \bar{\nu}_e$ is evaluated as an example.

I. INTRODUCTION

In the "elementary-particle" treatment of nuclear β decay,[1] initial and final nuclei are treated as elementary, rather than composite, particles. The composite nature of the nucleus, which is usually taken into account through the use of a model-inspired nuclear wave function, manifests itself in the elementary-particle treatment through nuclear form factors. The present elementary-particle approach does not, however, include a systematic formulation of the Coulomb correction. Thus, this approach is limited to the treatment of β decay of light nuclei where Coulomb corrections, which are of order αZ, can be neglected. Of course, even for light nuclei, high-accuracy analyses of β-decay spectra such as those required to test the conserved-vector-current (CVC) hypothesis or the existence of second-class currents require knowledge of the Coulomb correction.

In this paper, we extend the applicability of the elementary-particle treatment of nuclear β decay by including the Coulomb correction in the basic formulation. We define the Coulomb correction as the one due to the static Coulomb final-state interaction between the emitted electron (or positron) and the final nucleus. In the nuclear case this is the dominant electromagnetic correction.

In the customary impulse approximation, the Coulomb correction is introduced by replacing the plane-wave electron wave function by the Coulomb-distorted wave function, properly averaged over the initial and final nuclear wave functions.[2-4] Hence, the Coulomb correction in this approximation is, strictly speaking, model-dependent, though in practice some approximations are used to make it model-independent in order to simplify the calculation.

In the elementary-particle treatment the initial and final states are described in terms of form factors or structure functions characteristic of the nuclei as a whole, the numerical values of these form factors reflecting the complexity of the internal nuclear structure. Thus, the form factors play roles of nuclear wave functions and the Coulomb-distorted electron wave function is averaged over the nuclear form factors. The form factors can be obtained directly from the corresponding electron scattering data with the help of the CVC hypothesis. Hence, a model-indepent calculation of the Coulomb correction to high accuracy is, in principle, possible when this treatment is used.

We restrict our discussion in this paper to the case of allowed β transitions ($|\Delta J|=0,1$ and no parity change). In this case the nuclear form factors contributing to the hadron transition matrix element have about the same momentum-transfer-squared dependence within a few percent,[5] and as a result the formulation is relatively simple. This simplicity facilitates comparison of this approach with that of the usual impulse approximation. A generalization to the case of forbidden β transitions is under investigation.

In Sec. II we first review the usual impulse-approximation formulation of the Coulomb correction. We then develop the formulation in the elementary-particle treatment. The prescription of taking proper averages of the Coulomb-distorted electron wave functions over the nuclear form factors is given.

We present in Sec. III approximate expressions for the Coulomb-distorted electron wave functions for a point nucleus and for a finite size nucleus. Their averages according to the prescription given in Sec. II are discussed.

In Sec. IV, expressions for the square of the transition matrix element including the Coulomb correction are given. As an example, the spectrum shape factor for the transition $B^{12} \to C^{12} + e^- + \bar{\nu}_e$ is discussed.

II. FORMULATION

The conventional effective weak Hamiltonian for strangeness-conserving semileptonic weak processes may be given by

$$H_w = \int d\vec{x}\,\mathcal{H}_w(\vec{x}), \tag{1a}$$

$$\mathcal{H}_w(\vec{x}) = \frac{G}{\sqrt{2}} J_\alpha^{(+)}(\vec{x},0)\, l_\alpha(\vec{x},0) + \text{H.c.}, \tag{1b}$$

$$J_\alpha^{(+)}(\vec{x},0) = V_\alpha^{(+)}(\vec{x},0) + A_\alpha^{(+)}(\vec{x},0), \tag{1c}$$

$$l_\alpha(\vec{x},0) = \bar{\Psi}_e(\vec{x},0)\gamma_\alpha(1+\gamma_5)\Psi_{\nu_e}(\vec{x},0), \tag{1d}$$

where $G = 10^{-5}/m_p^2$; and $l_\alpha(\vec{x},0)$, $V_\alpha^{(+)}(\vec{x},0)$, and $A_\alpha^{(+)}(\vec{x},0)$ are, respectively, the lepton weak current, the charge-raising vector, and axial-vector hadron weak currents. The transition matrix element for the process $i \to f + e^- + \bar{\nu}_e$ is, from Eq. (1),

$$\mathfrak{M} = \langle f(\vec{p}_f), e^-(\vec{p}_e), \bar{\nu}_e(\vec{p}_\nu)|H_w|i(\vec{p}_i)\rangle = \frac{G}{\sqrt{2}} \langle f(\vec{p}_f), e^-(\vec{p}_e), \bar{\nu}_e(\vec{p}_\nu)| \int d\vec{x}\, J_\alpha^{(+)}(\vec{x},0)\, l_\alpha(\vec{x},0)|i(\vec{p}_i)\rangle, \tag{2}$$

where $\vec{p}_i$, $\vec{p}_f$, $\vec{p}_e$, and $\vec{p}_\nu$ are, respectively, the momenta of the particles denoted by subscripts.

A. Impulse-Approximation Treatment

First we discuss the transition matrix element of Eq. (2) in the traditional impulse approximation. In the impulse approximation, one replaces Eq. (1b) by

$$\mathcal{H}_w(\vec{x}) = \frac{G}{\sqrt{2}} \sum_{a=1}^{A} \Gamma_\alpha^{(a)} l_\alpha(\vec{x})\,\delta^{(3)}(\vec{x} - \vec{r}^{(a)}) + \text{H.c.} \tag{3}$$

and replaces the initial and final nuclear states by appropriate nuclear wave functions. In Eq. (3) the sum over a extends over the A nucleons in the nucleus, $\vec{r}^{(a)}$ is the position of the ath nucleon, and $\Gamma_\alpha^{(a)}$ is a single-nucleon operator to be determined from the nucleon case. It is customary to take (neglecting off-shell contributions and second-class currents)

$$\Gamma_\alpha^{(a)} = i\gamma_4^{(a)}\left[\gamma_\alpha^{(a)} g_V - \frac{g_M}{2m_P}\sigma_{\alpha\beta}^{(a)} q_\beta + \gamma_\alpha^{(a)}\gamma_5^{(a)} g_A + \frac{g_P}{m_\pi}\gamma_5^{(a)} q_\alpha\right]\tau_+^{(a)}, \tag{4}$$

where

$$q_\alpha = (p_f - p_i)_\alpha$$

and g_V, g_M, g_A, and g_P are the vector, weak-magnetism, axial-vector, and induced pseudoscalar form factors.

Substituting Eq. (3) into Eq. (2) and carrying out the integration over $\vec{x}$ leads to

$$\mathfrak{M} = \frac{G}{\sqrt{2}} \langle f, e^-, \bar{\nu}_e| \sum_{a=1}^{A} \Gamma_\alpha^{(a)} l_\alpha(\vec{r}^{(a)})|i\rangle. \tag{5}$$

In the absence of Coulomb interaction between the final nucleus and the electron, both the electron and antineutrino may be described by plane waves. In this case, the final state can be written as a product of leptonic and hadronic states. Then, using translational invariance, we can evaluate the leptonic matrix element:

$$\begin{aligned}\langle e^-, \bar{\nu}_e| l_\alpha(\vec{r}^{(a)})|0\rangle &= e^{-i(\vec{p}_e+\vec{p}_\nu)\cdot\vec{r}^{(a)}}\langle e^-, \bar{\nu}_e| l_\alpha(0)|0\rangle \\ &= e^{-i(\vec{p}_e+\vec{p}_\nu)\cdot\vec{r}^{(a)}}\bar{u}_e(\vec{p}_e)\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu).\end{aligned} \tag{6}$$

Equation (5) then becomes

$$\mathfrak{M}=\frac{G}{\sqrt{2}}\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}e^{i\vec{q}\cdot\vec{r}^{(a)}}|\Psi_i\rangle \times[\bar{u}_e(\vec{p}_e)\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu)], \tag{7}$$

where

$$\vec{q}=-(\vec{p}_e+\vec{p}_\nu)=\vec{p}_f-\vec{p}_i .$$

This is the well-known impulse-approximation expression in the absence of Coulomb interactions. In the allowed approximation (which is appropriate for transitions with $\Delta J\equiv|J_i-J_f|=0, 1$ and no parity change), only the leading terms in $\Gamma_\alpha^{(a)}$ are kept and $e^{i\vec{q}\cdot\vec{r}^{(a)}}$ is replaced by unity, implying that the lepton pair does not carry any angular momentum. (Note that the angular momenta of the lepton pair are measured from the center of the daughter nucleus.) In the case of forbidden transitions, one must expand $e^{i\vec{q}\cdot\vec{r}^{(a)}}$ in terms of spherical components, keeping the leading terms in this expansion for which $\mathfrak{M}$ does not vanish.

In the presence of final-state Coulomb interactions the outgoing electron can no longer be described by a plane-wave function, although the antineutrino is still described by a plane-wave function. Since the final nucleus is considerably heavier than the electron, the usual practice is to assume that the net effect of the final-state interaction is approximately equal to the replacement of the plane-wave function of the electron by a Coulomb-distorted one. Hence, one can still write the final-state wave function as a product of leptonic and hadronic parts. Then

$$\mathfrak{M}\cong\frac{G}{\sqrt{2}}\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}\langle e^-,\bar{\nu}_e|l_\alpha(\vec{r}^{(a)})|0\rangle|\Psi_i\rangle , \tag{8}$$

where the lepton part is now given by

$$\langle e^-(\vec{p}_e),\bar{\nu}_e(\vec{p}_\nu)|l_\alpha(\vec{r}^{(a)})|0\rangle \cong\bar{\Psi}_e(\vec{r}^{(a)},\vec{p}_e)\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu)e^{-i\vec{p}_\nu\cdot\vec{r}^{(a)}}; \tag{9}$$

$\Psi_e(\vec{r}^{(a)},\vec{p}_e)$ is the Coulomb-distorted electron wave function, which will be discussed below.

Combining Eqs. (8) and (9), we have the following result

$$\mathfrak{M}\cong\frac{G}{\sqrt{2}}\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}e^{-\vec{p}_\nu\cdot\vec{r}^{(a)}}\bar{\psi}_e(\vec{r}^{(a)},\vec{p}_e)|\Psi_i\rangle \times[\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu)] . \tag{10}$$

This is the well-known result for the transition matrix element in the presence of the final-state Coulomb interactions. As mentioned already, the usual practice is to expand $\bar{\psi}_e(\vec{r}^{(a)},\vec{p}_e)$ as well as $e^{-i\vec{p}_\nu\cdot\vec{r}^{(a)}}$ in terms of spherical components, and to retain the leading terms which satisfy the selection rules for nuclear transitions.

In the case of allowed transitions, we can rewrite Eq. (10) in a form which is particularly useful for comparison with the results of the elementary-particle treatment. We define the average

$$\langle\bar{\psi}_e e^{-i\vec{r}^{(a)}\cdot\vec{p}_\nu}\rangle\equiv\frac{\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}e^{-i\vec{p}_\nu\cdot\vec{r}^{(a)}}\bar{\psi}_e(\vec{r}^{(a)},\vec{p}_e)|\Psi_i\rangle}{\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}|\Psi_i\rangle} .$$

Equation (10) then can be written

$$\mathfrak{M}\cong\frac{G}{\sqrt{2}}\langle\Psi_f|\sum_{a=1}^{A}\Gamma_\alpha^{(a)}|\Psi_i\rangle \times\langle\bar{\psi}_e e^{-i\vec{r}^{(a)}\cdot\vec{p}_\nu}\rangle\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu) . \tag{11}$$

Note that the hadronic matrix element in this expression corresponds to the hadronic matrix element of Eq. (7) with $\vec{q}=0$.

B. Elementary-Particle Treatment

We now consider β decay by treating the initial and final nuclear states involved as "elementary" particles in the sense that we do not consider nuclei as explicitly being composed of protons and neutrons.

Using translational invariance, we can rewrite Eq. (2) as

$$\mathfrak{M}=\frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_i-\vec{p}_f-\vec{p}_e-\vec{p}_\nu) \times\langle f(\vec{p}_f),e^-(\vec{p}_e),\bar{\nu}_e(\vec{p}_\nu)|J_\alpha^{(+)}(0)l_\alpha(0)|i(\vec{p}_i)\rangle . \tag{12}$$

In the absence of final-state Coulomb interaction, the lepton part is independent of the hadron part and can be factored out; Eq. (12) then reduces to

$$\mathfrak{M}=\frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_i-\vec{p}_f-\vec{p}_e-\vec{p}_\nu) \times\langle f(\vec{p}_f)|J_\alpha^{(+)}(0)|i(\vec{p}_i)\rangle\bar{u}_e(\vec{p}_e)\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu) . \tag{13}$$

[Compare Eq. (7). Note that momentum conservation is not explicit in the usual formulation of the impulse approximation.] The exact form of the hadronic matrix element used in the evaluation of Eq. (13) depends on the spins and parities of the final and initial nuclear states. For example, for $\frac{1}{2}^\pm\to\frac{1}{2}^\pm$ transitions

$$\langle f|J_\alpha^{(+)}(0)|i\rangle=\langle f|V_\alpha^{(+)}(0)|i\rangle+\langle f|A_\alpha^{(+)}(0)|i\rangle ,$$

where

$$\langle f|V_\alpha^{(+)}(0)|i\rangle=\bar{u}_f\left[\gamma_\alpha F_V(q^2,i\to f)-\frac{\sigma_{\alpha\beta}q_\beta}{2m_p}F_M(q^2,i\to f)\right]u_i\,,$$
$$\langle f|A_\alpha^{(+)}(0)|i\rangle=\bar{u}_f\left[\gamma_\alpha\gamma_5 F_A(q^2,i\to f)+\frac{i(m_i+m_f)}{m_\pi^2}q_\alpha\gamma_5 F_P(q^2,i\to f)\right]u_i\,. \tag{14}$$

Here, $q=p_f-p_i$; u_i and u_f are spinors describing the motion of the initial and final nuclei (viewed as single particles of spin $\frac{1}{2}$); and F_V, F_M, F_A, and F_P are, respectively, vector, weak-magnetism, axial-vector, and induced pseudoscalar form factors. We assume that weak currents are of first-class nature, so that scalar and tensor form factors do not appear. Likewise, the hadronic matrix elements for $1^+\to 0^+$ transitions are

$$\langle f|V_\alpha^{(+)}(0)|i\rangle=\sqrt{2}\,m\epsilon_{\alpha\beta\gamma\delta}q_\beta\xi_\gamma\frac{Q_\delta}{2m}\frac{F_M(q^2,i\to f)}{2m_p}\,,$$
$$\langle f|A_\alpha^{(+)}(0)|i\rangle=\sqrt{2}\,m\left[\xi_\alpha F_A(q^2,i\to f)+q_\alpha\xi\cdot q\frac{F_P(q^2,i\to f)}{m_\pi^2}\right], \tag{15}$$

where ξ is the polarization vector for the spin-one nucleus, m is the nuclear mass ($m_i\approx m_f\approx m$), and $Q=p_i+p_f$. Form factors are defined as above.

The vector and weak-magentism form factors, F_V and F_M, are uniquely determined from the corresponding electron scattering data through use of the CVC hypothesis. The contribution of the induced pseudoscalar form factor F_P is very small ($\approx m_e$) and can be neglected for the purposes of this work.

Let us now discuss how the Coulomb interaction modifies the expression for $\mathfrak{M}$. In the final state, as mentioned already, the antineutrino has no interaction with the rest of the state, while the electron and the final nucleus are interacting with each other through a static Coulomb potential. Thus, we shall expand the interacting state $\langle f(\vec{p}_f),e^-(\vec{p}_e)|$ in terms of noninteracting electron and nucleus plane waves. For the sake of simplicity, we consider the case in which the initial and final nuclei are spinless. (This simplification enables us to suppress the spin indices of the nuclei, but does not otherwise affect the formulation.)

In order to carry out this expansion, we must utilize completeness relations for both the electronic and final nuclear states. For the electrons, we simply use

$$\sum_\mu[u^{(\mu)}(\vec{p})\bar{u}^{(\mu)}(\vec{p})-v^{(\mu)}(\vec{p})\bar{v}^{(\mu)}(\vec{p})]=1\,, \tag{16}$$

where u and v describe positive- and negative-energy electrons, respectively. In the estimate of Coulomb corrections, we can neglect the strong interactions of nuclei. That is, we need not consider intermediate states in which a proton has been replaced by a π^+ plus a neutron, etc. Furthermore we treat the nuclei nonrelativistically. Thus,

$$\sum_f|f\rangle\langle f|=1\,,$$

where the sum over f runs over all plane-wave states of the bare final nucleus. This does not imply, however, that strong-interaction effects are entirely neglected in this formulation of β decay. Most of the effect is already included in nuclear form factors. Then

$$\langle f(\vec{p}_f),e^-(\vec{p}_e)|=\int d\vec{k}_e\int d\vec{k}_f\sum_\mu[\langle f(\vec{p}_f),e^-(\vec{p}_e)|f(\vec{k}_f),e(\vec{k}_e,\mu,E>0)\rangle\langle f(\vec{k}_f),e(\vec{k}_e,\mu,E>0)|$$
$$-\langle f(\vec{p}_f),e^-(\vec{p}_e)|f(\vec{k}_f),e(\vec{k}_e,\mu,E<0)\rangle\langle f(\vec{k}_f),e(\vec{k}_e,\mu,E<0)|]\,, \tag{17}$$

where μ is the spin index of the intermediate electron [the spin index of the final (physical) electron is suppressed] and the electron with $E>0$ ($E<0$) is a positive- (negative-) energy electron. We express the amplitudes in Eq. (17) in the coordinate representation, i.e.,

$$\langle f(\vec{p}_f),e^-(\vec{p}_e)|f(\vec{k}_f),e(\vec{k}_e,\mu,E\gtrless 0)\rangle=\int d\vec{r}_e\int d\vec{r}_f\,\langle f(\vec{p}_f),e^-(\vec{p}_e)|\,\vec{r}_f,\vec{r}_e\rangle\langle\vec{r}_f,\vec{r}_e|f(\vec{k}_f),e(\vec{k}_e),\mu,E\gtrless 0)\rangle\,, \tag{18}$$

where

$$\langle\vec{r}_f,\vec{r}_e|f(\vec{k}_f),e(\vec{k}_e,\mu,E>0)\rangle=\frac{e^{i\vec{r}_f\cdot\vec{k}_f}}{(2\pi)^{3/2}}\frac{e^{i\vec{r}_e\cdot\vec{k}_e}}{(2\pi)^{3/2}}u^{(\mu)}(\vec{k}_e)\,,$$
$$\langle\vec{r}_f,\vec{r}_e|f(\vec{k}_f),e(\vec{k}_e,\mu,E<0)\rangle=\frac{e^{i\vec{r}_f\cdot\vec{k}_f}}{(2\pi)^{3/2}}\frac{e^{i\vec{r}_e\cdot\vec{k}_e}}{(2\pi)^{3/2}}v^{(\mu)}(\vec{k}_e)\,, \tag{19}$$

and

$$\langle f(\vec{p}_f), e^-(\vec{p}_e)|\vec{r}_f, \vec{r}_e\rangle = \left[\frac{e^{-i\vec{P}\cdot\vec{R}}}{(2\pi)^{3/2}}\right]\left[\frac{1}{(2\pi)^{3/2}}\bar{\Psi}_e(\vec{r}, \vec{p})\right], \tag{20}$$

$$\vec{P} = \vec{p}_f + \vec{p}_e, \quad \vec{R} = \frac{m_e\vec{r}_e + m_f\vec{r}_f}{m_f + m_e}, \quad \vec{r} = \vec{r}_e - \vec{r}_f, \quad \vec{p} = \frac{m_f\vec{p}_e - m_e\vec{p}_f}{m_e + m_f}.$$

The first factor on the right-hand side of Eq. (20) represents the plane wave describing the motion of the center of mass and the last factor represents the relative motion of the electron with respect to the nucleus, $\Psi_e(\vec{r}, \vec{p})$ being the Coulomb-distorted electron wave function. This expression is, of course, not exact, but corresponds to the approximation made in the impulse approximation. Since we have $m_f \gg m_e$, Eq. (20) can be approximated as

$$\langle f(\vec{p}_f), e^-(\vec{p}_e)|\vec{r}_f, \vec{r}_e\rangle \cong \frac{1}{(2\pi)^3} e^{-i(\vec{p}_f + \vec{p}_e)\cdot\vec{r}_f}\bar{\Psi}_e(\vec{r}, \vec{p}_e). \tag{21}$$

After substituting Eqs. (18), (19), and (21) into Eq. (17), we change the integration variable $\vec{r}_e$ into $\vec{r}$ and then carry out the integration over $\vec{r}_f$ and $\vec{k}_f$. The result is

$$\langle f(\vec{p}_f), e^-(\vec{p}_e)| \cong \frac{1}{(2\pi)^3}\int d\vec{k}_e \int d\vec{r}\, \bar{\Psi}_e(\vec{r}, \vec{p}_e)e^{i\vec{r}\cdot\vec{k}_e}$$
$$\times\sum_\mu [u^{(\mu)}(\vec{k}_e)\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e), e(\vec{k}_e, \mu, E>0)| - v^{(\mu)}(\vec{k}_e)\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e), e(\vec{k}_e, \mu, E<0)|], \tag{22}$$

or, after substituting Eq. (22) into Eq. (12),

$$\mathfrak{M} \cong \frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu)\int d\vec{k}_e \int d\vec{r}\,\frac{1}{(2\pi)^3}\bar{\Psi}_e(\vec{r}, \vec{p}_e)e^{i\vec{r}\cdot\vec{k}_e}$$
$$\times\sum_\mu [u^{(\mu)}(\vec{k}_e)\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e), e(\vec{k}_e, \mu, E>0), \bar{\nu}_e(\vec{p}_\nu)|J_\alpha^{(+)}(0)l_\alpha(0)|i(\vec{p}_i)\rangle$$
$$- v^{(\mu)}(\vec{k}_e)\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e), e(\vec{k}_e, \mu, E<0), \bar{\nu}_e(\vec{p}_\nu)|J_\alpha^{(+)}(0)l_\alpha(0)|i(\vec{p}_i)\rangle]. \tag{23}$$

In the matrix element in the square bracket on the right-hand side of Eq. (23), the electron, final nucleus, and antineutrino are now all noninteracting free states; hence, the hadron and lepton parts can be separated, e.g.,

$$\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e), e(\vec{k}_e, \mu, E>0), \bar{\nu}_e(\vec{p}_\nu)|J_\alpha^{(+)}(0)l_\alpha(0)|i(\vec{p}_i)\rangle = \langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e)|J_\alpha^{(+)}(0)|i(\vec{p}_i)\rangle\bar{u}^{(\mu)}(\vec{k}_e)\gamma_\alpha(1+\gamma_5)v(\vec{p}_\nu). \tag{24}$$

Substituting Eq. (24) and the corresponding expression for the negative-energy electron into Eq. (23) and using the completeness relation, we finally obtain

$$\mathfrak{M} \cong \frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu)\left\{\int d\vec{k}_e\left[\int d\vec{r}\,\frac{1}{(2\pi)^3}\bar{\Psi}_e(\vec{r}, \vec{p}_e)e^{i\vec{r}\cdot\vec{k}_e}\right]\right.$$
$$\left.\times\langle f(\vec{p}_f + \vec{p}_e - \vec{k}_e)|J_\alpha^{(+)}(0)|i(\vec{p}_i)\rangle\right\}\gamma_\alpha(1+\gamma_5)v(\vec{p}_\nu). \tag{25}$$

Equation (25) is the elementary-particle-treatment version of Eq. (10) and is fundamental in the investigation which follows. We note that the final result (25) holds for the case of arbitrary nuclear spin and parity.

In the absence of Coulomb corrections, we have

$$\bar{\Psi}_e(\vec{r}, \vec{p}_e) = \bar{u}(\vec{p}_e)e^{-i\vec{r}\cdot\vec{p}_e}, \tag{26}$$

so that the integration over $\vec{r}$ yields $(2\pi)^3\delta^{(3)}(\vec{k}_e - \vec{p}_e)$ and the integration over $\vec{k}_e$ reduces Eq. (25) to Eq. (13), as it should.

As can be seen from Eq. (25), the effects of the final-state Coulomb interaction are present in both lepton and hadron parts, destroying the locality of a lepton-pair production. The hadron part is modified in three ways: (1) Nuclear form factors which characterize the hadron matrix element will be functions of the momentum transfer $q' = [(p_f + p_e - k_e) - p_i]$ instead of $q = (p_f - p_i)$. (2) There will be additional kinematic terms due to the replacement of q by q' in the matrix element. (3) Final nuclear spinors which describe nuclei with nonzero spin are modified; i.e., the momentum of the final nucleus is $\vec{p}_f + \vec{p}_e - \vec{k}_e$ instead of $\vec{p}_f$. The last modification can be neglected in our discussion, for the nuclei involved are always nonrelativistic, so that it does not make any difference whether the momentum is $\vec{p}_f + \vec{p}_e - \vec{k}_e$ or $\vec{p}_f$, since they are negligibly small compared to the mass m ($|\vec{p}_e - \vec{k}_e| \sim \alpha Z m_e$).

We now must evaluate the following two types of quantities which appear in Eq. (25) when the explicit form of the hadronic matrix element is introduced:

$$A_i \equiv \frac{1}{(2\pi)^3}\int d\vec{k}_e \int d\vec{r}\,\Psi_e(\vec{r},\vec{p}_e)e^{i\vec{r}\cdot\vec{k}_e} \times F_i((\vec{p}_f - \vec{p}_i + \vec{p}_e - \vec{k}_e)^2), \tag{27}$$

$$\vec{B}_i \equiv \frac{1}{(2\pi)^3}\int d\vec{k}_e \int d\vec{r}\,\Psi_e(\vec{r},\vec{p}_e)\,e^{i\vec{r}\cdot\vec{k}_e} \times F_i((\vec{p}_f - \vec{p}_i + \vec{p}_e - \vec{k}_e)^2)\,(\vec{p}_e - \vec{k}_e), \tag{28}$$

where $F_i((\vec{p}_f - \vec{p}_i + \vec{p}_e - \vec{k}_e)^2)$ are the nuclear form factors, and i denotes vector (V), weak-magnetism (M), axial-vector (A), and pseudoscalar (P) form factors. The quantity $\vec{B}_i$ arises from the modification (2) listed above. Equations (27) and (28) can be rewritten using structure functions ρ_i as defined by

$$F_i(\vec{q}^2) = \int \rho_i(\vec{x})\,e^{i\vec{q}\cdot\vec{x}}\,d\vec{x}. \tag{29}$$

In the impulse-approximation picture $\rho_i(\vec{x})$ come from the matrix elements of certain operators between the initial wave function of the decaying nucleus at rest and the final wave function of the daughter nucleus after it has absorbed the recoil momentum $\vec{q}$.

First, the expression for A_i of Eq. (27), when Eq. (29) is substituted, becomes

$$A_i = \frac{1}{(2\pi)^3}\int d\vec{k}_e \int d\vec{r}\,\Psi_e(\vec{r},\vec{p}_e)\,e^{i\vec{r}\cdot\vec{k}_e} \times \int d\vec{x}\,\rho_i(\vec{x})\,e^{i(\vec{p}_f - \vec{p}_i + \vec{p}_e - \vec{k}_e)\cdot\vec{x}} = \int d\vec{r}\,\Psi_e(\vec{r},\vec{p}_e)\,e^{i\vec{r}\cdot\vec{p}_\nu}\rho_i(\vec{r}) \equiv \langle\Psi_e\rangle_i, \tag{30}$$

where we have used the momentum conservation $\vec{p}_i = \vec{p}_f + \vec{p}_e + \vec{p}_\nu$, and $\langle\Psi_e\rangle_i$ is the indicated average of the electron wave functions.

Similarly, the quantity $\vec{B}_i$ becomes

$$\vec{B}_i = \frac{1}{(2\pi)^3}\int d\vec{r}\int d\vec{x}\,\Psi_e(\vec{r},\vec{p}_e)\,e^{-i\vec{x}\cdot\vec{p}_\nu}\rho_i(\vec{x}) \times \int d\vec{k}_e(\vec{p}_e - \vec{k}_e)\,e^{i\vec{k}_e\cdot(\vec{r}-\vec{x})}. \tag{31}$$

Using the relation

$$\int d\vec{k}_e\vec{k}_e\,e^{i\vec{k}_e\cdot(\vec{r}-\vec{x})} = (2\pi)^3\frac{\vec{\nabla}_r}{i}\delta^{(3)}(\vec{r}-\vec{x}), \tag{32}$$

and carrying out integration by parts for the variable $\vec{r}$ in Eq. (31), we obtain

$$\vec{B}_i = \int d\vec{r}\,\rho(\vec{r})\,e^{-i\vec{r}\cdot\vec{p}_\nu}\left(\vec{p}_e + \frac{\vec{\nabla}_r}{i}\right)\Psi_e(\vec{r},\vec{p}_e). \tag{33}$$

In the absence of Coulomb corrections, we have

$$\frac{\vec{\nabla}_r}{i}\Psi_e(\vec{r},\vec{p}_e) = -\vec{p}_e\Psi_e(\vec{r},\vec{p}_e), \tag{34}$$

in which case $\vec{B}_i = 0$, as it must. In fact, even in the presence of Coulomb interaction, we have (using the wave functions of Sec. III)

$$\left|\left(\vec{p}_e + \frac{\vec{\nabla}_r}{i}\right)\Psi_e(\vec{r},\vec{p}_e)\right| \approx \vec{p}_e(\alpha Z)|\Psi_e|. \tag{35}$$

From Eq. (35), the quantity $\vec{B}_i$ can be neglected compared with other usual kinematic factors, especially in the nonrelativistic treatment of nuclei. We conclude, therefore, that the most important correction, in the approximations mentioned above, is the modification (1) and the corresponding average of the electron wave functions.

In allowed transitions we have $F_i(0) \neq 0$, so that we can write

$$F_i(q^2) \equiv F_i(0)\mathfrak{F}_i(q^2). \tag{36}$$

In nuclear β decays, most nuclear-structure information is contained in $F_i(0)$, and the transition rates are insensitive to $\mathfrak{F}_i(q^2)$ simply because $\mathfrak{F}_i(q^2) \approx 1$ for $|q^2| \approx m_e^2 \ll m_\pi^2$.

It has been shown,[5] with the help of the impulse approximation, that the nuclear form factors $\mathfrak{F}_i(q^2)$ ($i = V, M, A$) have approximately the same q^2 dependence, i.e.,

$$\mathfrak{F}_V(q^2) \cong \mathfrak{F}_M(q^2) \cong \mathfrak{F}_A(q^2) \equiv \mathfrak{F}(q^2). \tag{37}$$

(F_P does not have this q^2 dependence, but, as mentioned already, the contribution of the F_P term is negligible in nuclear β decay and so will not be considered.) Defining again

$$\mathfrak{F}(q^2) = \int d\vec{x}\,\varphi(\vec{x})e^{i\vec{x}\cdot\vec{q}}, \qquad \varphi(\vec{x}) \cong \rho_i(\vec{x})\Big/\int d\vec{r}\,\rho_i(\vec{r}), \tag{38}$$

we can write, from Eq. (30),

$$A_i = F_i(0)\int d\vec{r}\,\bar{\psi}_e(\vec{r},\vec{p}_e)e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r}) \equiv F_i(0)\langle\bar{\psi}_e\rangle. \tag{39}$$

In contrast to the previous average $\langle\bar{\psi}_e\rangle_i$, $\langle\bar{\psi}_e\rangle$ is *common* for all form factors, and depends upon

$\mathcal{F}(q^2)$ only. This is not the case for the forbidden β transition. The net Coulomb effect is, then, to replace the contributing form factors ($i = V, M, A$) in the nuclear matrix element, $F_i(q^2)$, by $F_i(0)$ and $\bar{u}_e(\vec{p}_e)$ by $\langle\bar{\psi}_e\rangle$:

$$\mathfrak{M} = \frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu) \times \langle f(\vec{p}_f)|\, J_\alpha^{(+)}(0)\,|i(\vec{p}_i)\rangle_0 \langle\bar{\psi}_e\rangle\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu) \tag{40}$$

[compare with Eq. (13)], where the subscript 0 to the hadron part denotes that form factors should be evaluated at $q^2 = 0$, i.e., replaced by the coupling constants. In Eq. (40), the lepton and hadron parts are now formally separated, but $\langle\bar{\psi}_e\rangle$ contains information about nuclear structure through φ. We also remark that Eq. (40) is precisely the elementary-particle-treatment version of Eq. (11) for the case of allowed transitions.

III. ELECTRON WAVE FUNCTION

In this section we shall discuss the Coulomb-distorted electron wave function and its average discussed in the previous section. To describe the emergence at infinity of an electron with a specific momentum $\vec{p}_e$, it is necessary to construct superpositions of the Coulomb-distorted spherical waves, which will represent a plane wave at least at infinity. Such a superposition is given by[6]

$$\psi_e(\vec{r}, \vec{p}_e) = \left[\frac{(2\pi)^3}{m_e p_e}\right]^{1/2} \sum_{\kappa,\mu} i^l \langle l(\mu-\rho)\tfrac{1}{2}(\rho)|\, j(\mu)\rangle \times Y^*_{l,\mu-\rho}(\hat{p}_e)e^{-i\delta''}\psi_{\kappa\mu}(\hat{r}), \tag{41}$$

$$\delta'' \equiv \tfrac{1}{2}[l(\kappa)+1]\pi - \arg\Gamma(\gamma + i\nu) + \eta - \tfrac{1}{2}\pi\gamma, \tag{42}$$

$$\psi_{\kappa,\mu}(r) \equiv \begin{pmatrix} g_\kappa(r) & \chi_{\kappa,\mu}(r) \\ if_\kappa(r) & \chi_{-\kappa,\mu}(r)\end{pmatrix}, \tag{43}$$

where the notation is standard.[7] For a point nucleus the radial functions, $g_\kappa(r)$ and $f_\kappa(r)$ are given, *when normalized per unit energy*, by

$$g_\kappa(r) = \left[\frac{p_e(E_e+m_e)}{4\pi}\right]^{1/2}(Q+Q^*), \qquad f_\kappa(r) = i\left[\frac{p_e(E_e+m_e)}{4\pi}\right]^{1/2}(Q-Q^*), \tag{44}$$

with

$$Q = 2e^{\pi\nu/2}\frac{|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)}(\gamma+i\nu)(2p_e r)^{\gamma-1}e^{-ip_e r + i\eta_\kappa} \times F(\gamma+1+i\nu, 2\gamma+1; 2ip_e r),$$

$$\gamma = (\kappa^2 - \alpha^2 Z^2)^{1/2}, \quad \nu = \frac{\alpha Z E_e}{p_e},$$

$$e^{2i\eta_\kappa} = -\frac{\kappa - i\alpha Z/p_e}{\gamma + i\nu},$$

and $F(a, b; z)$ is the confluent hypergeometric function.

For a finite-size nucleus the radial functions $g_\kappa(r)$ and $f_\kappa(r)$ must be modified inside the nucleus. If the nucleus is represented, for example, by a uniformly charged sphere of radius R, the radial functions for $r \le R$ are given by[8]

$$\begin{aligned}
g_{-1}(r) &= \left[\frac{p_e(E_e+m_e)}{\pi}F(E_e, Z)\right]^{1/2} \times \left\{1 - \left[\frac{13}{30} + \frac{1}{2}\frac{r^2}{R^2}\right]\alpha Z E_e R - \tfrac{1}{6}(p_e r)^2 + \cdots\right\}, \\
f_{+1}(r) &= -\left(\frac{p_e}{E_e+m_e}\right)\left[\frac{p_e(E_e+m_e)}{\pi}F(E_e, Z)\right]^{1/2} \times \left\{1 - \left[\frac{13}{30} + \frac{1}{2}\frac{r^2}{R^2}\right]\alpha Z E_e R - \tfrac{1}{6}(p_e r)^2 + \cdots\right\}, \\
g_{+1}(r) &= -\left[\frac{p_e(E_e+m_e)}{\pi}F(E_e, Z)\right]^{1/2} \times \left\{\tfrac{1}{3}p_e r + \frac{1}{2}\frac{E_e\alpha Z}{p_e}\frac{r}{R} \times \left[1 - \frac{1}{5}\left(\frac{r}{R}\right)^2\right]\left(1 - \frac{m_e}{E_e}\right) + \cdots\right\}, \\
f_{-1}(r) &= -\left(\frac{p_e}{E_e+m_e}\right)\left[\frac{p_e(E_e+m_e)}{\pi}F(E_e, Z)\right]^{1/2} \times \left\{\tfrac{1}{3}p_e r + \frac{1}{2}\frac{E_e\alpha Z}{p_e}\frac{r}{R} \times \left[1 - \frac{1}{5}\left(\frac{r}{R}\right)^2\right]\left(1 + \frac{m_e}{E_e}\right) + \cdots\right\},
\end{aligned} \tag{45}$$

where $F(E_e, Z)$ is the standard Fermi function. For the positron decay, Z should be replaced by $-Z$ in Eqs. (44) and (45) with appropriate changes in the phase η_κ.[8]

For our purposes, it is sufficient to retain only the $\kappa = +1$ and -1 terms from Eq. (41). Then

$$\psi_e(\vec{r}) \cong \left[\frac{(2\pi)^3}{p_e m_e}\right]^{1/2}\frac{1}{4\pi}e^{-i\delta'}\begin{pmatrix} g_{-1} + i\vec{\sigma}\cdot\hat{r}\vec{\sigma}\cdot\hat{p}_e g_{+1}e^{-i\delta} \\ -i\vec{\sigma}\cdot\hat{r}f_{-1} + \vec{\sigma}\cdot\hat{p}_e f_{+1}e^{-i\delta}\end{pmatrix}\chi_\rho, \tag{46}$$

where

$$\delta = \eta_{+1} - \eta_{-1} + \tfrac{1}{2}\pi, \qquad \delta' = \tfrac{1}{2}\pi - \arg\Gamma(\gamma + i\nu) - \tfrac{1}{2}\pi\gamma + \eta_{-1}. \tag{47}$$

In deriving Eq. (46) we have used the relations

$$\chi^\dagger_{\kappa,\mu}(\hat{p})\cdot\chi_\rho = \langle l(\mu-p)\tfrac{1}{2}(\rho)|\, j(\mu)\rangle Y^*_{l,\mu-\rho}(\hat{p}), \qquad (\vec{\sigma}\cdot\hat{p}_e\chi_\mu)^\dagger\cdot\chi_\rho\vec{\sigma}\cdot\hat{r}\chi_\mu = \vec{\sigma}\cdot\hat{r}\,\vec{\sigma}\cdot\hat{p}_e\chi_\rho. \tag{48}$$

We rewrite Eq. (46) using Dirac spinors instead of Pauli spinors; the result is

$$\psi_e(\vec{r},\vec{p}_e)\cong\frac{1}{4\pi}\left[\frac{(2\pi)^3}{m_e p_e}\right]^{1/2}e^{-i\delta'}\times(a+b\gamma_4+c\vec{\gamma}\cdot\hat{r}+d\vec{\gamma}\cdot\hat{r}\gamma_4)u_e\,,$$

or (49)

$$\bar{\psi}_e(\vec{r},p_e)\cong\frac{1}{4\pi}\left[\frac{(2\pi)^3}{m_e p_e}\right]^{1/2}e^{i\delta'}\times\bar{u}_e(a^*+b^*\gamma_4-c^*\vec{\gamma}\cdot\hat{r}+d^*\vec{\gamma}\cdot\hat{r}\gamma_4)\,,$$

where

$$\begin{aligned}a&=\frac{1}{2}\left(\frac{2m_e}{E_e+m_e}\right)^{1/2}\left(g_{-1}+\frac{E_e+m_e}{p_e}f_{+1}e^{-i\delta}\right),\\ b&=\frac{1}{2}\left(\frac{2m_e}{E_e+m_e}\right)^{1/2}\left(g_{-1}-\frac{E_e+m_e}{p_e}f_{+1}e^{-i\delta}\right),\\ c&=-\frac{1}{2}\left(\frac{2m_e}{E_e+m_e}\right)^{1/2}\left(f_{-1}+\frac{E_e+m_e}{p_e}g_{+1}e^{-i\delta}\right),\\ d&=-\frac{1}{2}\left(\frac{2m_e}{E_e+m_e}\right)^{1/2}\left(f_{-1}-\frac{E_e+m_e}{p_e}g_{+1}e^{-i\delta}\right).\end{aligned}\qquad(50)$$

The average value of Eq. (49) is, from Eq. (39),

$$\begin{aligned}\langle\bar{\psi}_e\rangle\cong\bar{u}_e\Big\{&\int d\vec{r}\,Na^*e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\\ &+\gamma_4\int d\vec{r}\,Nb^*e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\\ &+\vec{\gamma}\cdot\int d\vec{r}\,[-Nc^*\hat{r}e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})]\\ &+\vec{\gamma}\cdot\int d\vec{r}\,Nd^*\hat{r}e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\gamma_4\Big\}\\ =\bar{u}_e(&A+B\gamma_4+C\vec{\gamma}\cdot\hat{p}_\nu+D\vec{\gamma}\cdot\hat{p}_\nu\gamma_4)\,,\end{aligned}\qquad(51)$$

where

$$\begin{aligned}A&=\int d\vec{r}\,Na^*e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\,,\\ B&=\int d\vec{r}\,Nb^*e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\,,\\ C&=\hat{p}_\nu\cdot\int d\vec{r}\,[-Nc^*\hat{r}e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})]\,,\\ D&=\hat{p}_\nu\cdot\int d\vec{r}\,Nd^*\hat{r}e^{-i\vec{r}\cdot\vec{p}_\nu}\varphi(\vec{r})\,,\\ N&=\frac{1}{4\pi}\left[\frac{(2\pi)^3}{m_e p_e}\right]^{1/2}e^{i\delta'}\,.\end{aligned}\qquad(52)$$

The lepton contribution to $\mathfrak{M}$ is now given by

$$\begin{aligned}l_\alpha&\equiv\langle\bar{\psi}_e\rangle\gamma_\alpha(1+\gamma_5)v_\nu\\ &\cong\bar{u}_e(A+B\gamma_4+C\vec{\gamma}\cdot\hat{p}_\nu+D\vec{\gamma}\cdot\hat{p}_\nu\gamma_4)\gamma_\alpha(1+\gamma_5)v_\nu\,.\end{aligned}\qquad(53)$$

In the case when the structure function $\varphi(\vec{r})$ is spherically symmetric, i.e., $\varphi(\vec{r})=\varphi(r)$, Eq. (52) can be rewritten as

$$\begin{aligned}A&=\frac{4\pi N}{p_\nu}\int_0^\infty dr\,r\sin(rp_\nu)a^*\varphi(r)\,,\\ B&=\frac{4\pi N}{p_\nu}\int_0^\infty dr\,r\sin(rp_\nu)b^*\varphi(r)\,,\\ C&=\frac{-4\pi iN}{p_\nu^2}\int_0^\infty dr[(rp_\nu)\cos(rp_\nu)-\sin(rp_\nu)]c^*\varphi(r)\,,\\ D&=\frac{4\pi iN}{p_\nu^2}\int_0^\infty dr[(rp_\nu)\cos(rp_\nu)-\sin(rp_\nu)]d^*\varphi(r)\,.\end{aligned}\qquad(54)$$

When nuclear form factors are not known, $\varphi(\vec{r})$ is, of course, unknown. In this case, in view of the fact that $\langle\bar{\psi}_e\rangle$ is not very sensitive to details of nuclear structure, one may use some simple models for $\varphi(\vec{r})$. When the nucleons near the spherical nuclear surface participate in the β transition [i.e., shell distribution with $\varphi(\vec{r})=\varphi(r)=(4\pi R^2)^{-1}\delta(r-R)$], we have

$$\begin{aligned}A&=\frac{N\sin(Rp_\nu)}{Rp_\nu}a^*(r=R)\,,\\ B&=\frac{N\sin(Rp_\nu)}{Rp_\nu}b^*(r=R)\,,\\ C&=\frac{-iN}{(Rp_\nu)^2}[(Rp_\nu)\cos(Rp_\nu)-\sin(Rp_\nu)]c^*(r=R)\,,\\ D&=\frac{iN}{(Rp_\nu)^2}[(Rp_\nu)\cos(Rp_\nu)-\sin(Rp_\nu)]d^*(r=R)\,.\end{aligned}\qquad(55)$$

For the case of a uniform distribution of the transforming nucleons, we have

$$\begin{aligned}A&=\frac{3N}{R^3p_\nu}\int_0^R dr\,r\sin(rp_\nu)a^*\,,\\ B&=\frac{3N}{R^3p_\nu}\int_0^R dr\,r\sin(rp_\nu)b^*\,,\\ C&=\frac{-3iN}{R^3p_\nu^2}\int_0^R dr[(rp_\nu)\cos(rp_\nu)-\sin(rp_\nu)]c^*\,,\\ D&=\frac{3iN}{R^3p_\nu^2}\int_0^R dr[(rp_\nu)\cos(rp_\nu)-\sin(rp_\nu)]d^*\,.\end{aligned}\qquad(56)$$

IV. TRANSITION MATRIX ELEMENT

The transition rate of the process $i\to f+e^-+\bar{\nu}$ is proportional to the square of the matrix element of Eq. (40); in particular, proportional to

$$\sum_{s_e s_\nu}|l_\alpha h_\alpha^{(0)}|^2;\quad h_\alpha^{(0)}\equiv\langle f(\vec{p}_f)|J_\alpha^{(+)}(0)|i(\vec{p}_i)\rangle_0\,,\qquad(57)$$

where l_α is defined in Eq. (53). We sum over the lepton spins s_e and s_ν, since we are interested in the energy spectrum. Evaluation of Eq. (57) is straightforward; the result is

$$\sum_{s_e s_\nu} |l_\alpha h_\alpha^{(0)}|^2 = h_\alpha^{(0)} h_\beta^{(0)*} \mathfrak{L}_{\alpha\beta}\,;$$

$$\begin{aligned}\mathfrak{L}_{\alpha\beta} = \frac{2(-)^{\delta_{\beta 4}}}{m_e m_\nu}\{ &|A|^2 l_{\alpha\beta}(p_e,p_\nu) - |B|^2 l_{\alpha\beta}(p_e - p_e^*, p_\nu) + |C|^2[l_{\alpha\beta}(p_e,p_\nu) - 2\vec{p}_e\cdot\hat{p}_\nu l_{\alpha\beta}(p_e - \hat{p}_\nu, p_\nu)] \\ &+ |D|^2[2\vec{p}_e\cdot\hat{p}_\nu l_{\alpha\beta}(p_e-\hat{p}_\nu,p_\nu) - l_{\alpha\beta}(p_e - p_e^*, p_\nu)] + 2\,\mathrm{Re}(A^*B)\, l_{\alpha\beta}(p_e - p_e', p_\nu) - 2\,\mathrm{Im}(A^*C) m_e l_{\alpha\beta}(p_e - \hat{p}_\nu, p_\nu) \\ &- 2\,\mathrm{Im}(A^*D)[(-)^{\delta_{\alpha 4}} l_{\alpha\beta}(p_e,p_\nu) + 2i(\hat{p}_\nu)_\alpha l_{\alpha=4,\beta}(p_e,p_\nu)] - 2\,\mathrm{Im}(B^*C)[(-)^{\delta_{\alpha 4}} l_{\alpha\beta}(p_e - p_e^*, p_\nu) \\ &+ 2i(\hat{p}_\nu)_\alpha l_{\alpha=4,\beta}(p_e - p_e^*, p_\nu)] + 2\,\mathrm{Im}(B^*D) m_e l_{\alpha\beta}(p_e - \hat{p}_\nu, p_\nu) - 2\,\mathrm{Re}(C^*D)\, l_{\alpha\beta}(p_e - p_e', p_\nu)\}\,,\end{aligned} \tag{58}$$

where

$$l_{\alpha\beta}(p_e,p_\nu) = (p_e)_\alpha (p_\nu)_\beta + (p_e)_\beta (p_\nu)_\alpha - \delta_{\alpha\beta}(p_e\cdot p_\nu) + \epsilon_{\alpha\beta\gamma\delta}(p_\nu)_\gamma (p_e)_\delta\,,$$

and (59)

$$p_e^* = (\vec{p}_e, -iE_e), \quad \hat{p}_e = (\hat{p}_e, 0), \quad p_e' = (0, im_e)\,.$$

The integration of Eq. (58) over the lepton and neutrino angles yields the shape factor for the transition.

To illustrate the use of the result in Eq. (58), we derive the spectrum shape correction factor $S(E_e, Z)$ for the allowed process $1^+ \to 0^+ + e^- + \bar{\nu}_e$; e.g., $B^{12} \to C^{12} + e^- + \bar{\nu}_e$. The hadron matrix element is given in Eq. (15), the nonrelativistic expression being

$$h_\alpha^{(0)} \cong \sqrt{2}\, m \left(F_A \vec{\xi} - i\frac{F_M}{2m_p}\vec{\xi}\times\vec{q}, 0\right), \tag{60}$$

where F_A and F_M are now the nuclear coupling constants

$$F_A \equiv F_A(0; B^{12} \to C^{12}), \quad F_M \equiv F_M(0; B^{12} \to C^{12})\,. \tag{61}$$

The shape factor obtained from Eqs. (58) and (60) is, keeping the terms up to E_e/m_p,

$$\begin{aligned}S(E_e,Z) = \frac{1}{F(E_e,Z)}\Big(&|A|^2 + |B|^2 + 2\frac{m_e}{E_e}\mathrm{Re}(A^*B) + |C|^2 + |D|^2 - 2\frac{m_e}{E_e}\mathrm{Re}(C^*D) + \frac{2}{3}\frac{m_e}{E_e}[\mathrm{Im}(A^*C) - \mathrm{Im}(B^*D)] \\ &- \tfrac{2}{3}[\mathrm{Im}(A^*D) - \mathrm{Im}(B^*C)] + \frac{4}{3}\frac{1}{2m_p}\frac{F_M}{F_A}\Big\{(2E_e - W_0)|A|^2 - W_0|B|^2 + 2\frac{m_e}{E_e}(E_e - W_0)\mathrm{Re}(A^*B) \\ &+ (\tfrac{4}{3}E_e - W_0)|C|^2 + (\tfrac{2}{3}E_e - W_0)|D|^2 - 2\frac{m_e}{E_e}(E_e - W_0)\mathrm{Re}(C^*D) \\ &+ 2\frac{m_e}{E_e}(E_e - W_0)[\mathrm{Im}(A^*C) - \mathrm{Im}(B^*D)] - (\tfrac{8}{3}E_e - 2W_0)[\mathrm{Im}(A^*D) - \mathrm{Im}(B^*C)]\Big\}\Big)\,.\end{aligned} \tag{62}$$

In the estimate of the lowest-order Coulomb correction, it is customary to keep the two large radial functions g_{-1} and f_{+1}, i.e., A and B in Eq. (53). In this approximation Eq. (62) reduces to

$$S(E_e,Z) \cong \frac{1}{F(E_e,Z)}\Big\{\Big[|A|^2 + |B|^2 + 2\frac{m_e}{E_e}\mathrm{Re}(A^*B)\Big] + \frac{4}{3}\frac{1}{2m_p}\frac{F_M}{F_A}\Big[|A|^2(2E_e - W_0) - |B|^2 W_0 + 2\frac{m_e}{E_e}\mathrm{Re}(A^*B)(E_e - W_0)\Big]\Big\} \tag{63}$$

$$\cong \frac{1}{F(E_e,Z)}\Big[|A|^2 + |B|^2 + 2\frac{m_e}{E_e}\mathrm{Re}(A^*B)\Big] + \frac{8}{3}\frac{1}{2m_p}\frac{F_M}{F_A}\Big(E_e - \frac{W_0}{2}\Big)\,, \tag{64}$$

where the second approximate equality, Eq. (64), is due to the replacement of the last square bracket in Eq. (63) by $(2E_e - W_0)F(E_e,Z)$, which is the leading term; this is justified because the (F_M/F_A) term is already of order E_e/m_p. If we use, for example, the A and B which correspond to the shell distribution [Eq. (55)], Eq. (64) becomes, neglecting $(Rp_e)^2$ terms,

$$S(E_e,Z) \cong \frac{\pi}{2E_e p_e}[g_{-1}{}^2(r=R) + f_{+1}{}^2(r=R)] + \frac{8}{3}\frac{1}{2m_p}\frac{F_M}{F_A}\Big(E_e - \frac{W_0}{2}\Big)\,. \tag{65}$$

The first term in Eq. (65) is the well-known Coulomb correction factor for the allowed approximation and is usually denoted by L_0. Substituting g_{-1} and f_{+1} of Eq. (45) into Eq. (65), we finally obtain

$$S(E_e, Z) \cong 1 - \frac{4}{3}\frac{W_0}{2m_p}\frac{F_M}{F_A} + aE_e\,, \tag{66}$$

with

$$a = \frac{8}{3}\frac{1}{2m_p}\frac{F_M}{F_A} - \frac{28}{15}\alpha Z R \quad \text{for shell distribution,}$$

implying that the Coulomb correction modifies the coefficient of E_e by the factor $-\frac{28}{15}\alpha ZR$. For the values of A and B of the uniform distribution in Eq. (56) the radial wave functions g_{-1} and f_{+1} are averaged over the nuclear volume, resulting in the replacement of r^n by $3R^n/(n+3)$. The coefficient a for this case is given by

$$a = \frac{8}{3}\frac{1}{2m_p}\frac{F_M}{F_A} - \frac{22}{15}\alpha Z R \quad \text{for uniform distribution.} \tag{67}$$

The above Coulomb correction agrees with the result of Refs. 2 and 8.

We have also calculated the contribution of the small radial wave functions g_{+1} and f_{-1} through C and D as given in Eq. (62). This modifies Eq. (67) to

$$a \cong \frac{8}{3}\frac{1}{2m_p}\frac{F_M}{F_A} - \frac{24}{15}\alpha Z R\,, \tag{68}$$

implying that the correction due to the small radial wave functions (C and D terms) is about 10% of the usual correction due to the large radial wave functions. Comparing Eq. (68) with the observed value[9] of a, we find

$$F_M/F_A = 4.6 \pm 0.9\,, \tag{69}$$

which yields, with $F_A = 1.03$,[10]

$$F_M = 4.7 \pm 1.0\,. \tag{70}$$

This value is to be compared, as a test of the CVC hypothesis, with the value of F_M obtained from the inelastic electron scattering from C^{12},[11]

$$F_M = 4.1 \pm 0.2\,. \tag{71}$$

In view of the large experimental error in the value of a, evaluation of the Coulomb correction using more-realistic form factors is not warranted at present.

V. SUMMARY AND DISCUSSION

We have formulated an "elementary-particle" treatment of nuclear β decay, which includes the effects of the final-state Coulomb interaction. The completely general transition matrix element which results from this calculation is given in Eq. (25). By restricting our attention to allowed transitions, we are able to accurately approximate this general result with the somewhat more tractable form given in Eq. (40). Simplification results, in this case, primarily because of the similar dependence on q^2 of the nuclear form factors which contribute to the β decay. Equation (40) is, in fact, quite similar in form to the usual impulse-approximation transition matrix element for allowed decay. Differences which exist between the two forms are in the calculation of the average value of the electron wave function, and in the evaluation of the nuclear matrix elements. In the impulse approximation one must have a model-dependent nuclear wave function; in the elementary-particle treatment the nuclear matrix element is given in terms of the nuclear form factors – parameters which can be obtained from a completely independent experiment.

The average over the electron wave function has been carried out for two models of the nuclear-structure function $\varphi(r)$ using the usual Coulomb-distorted wave function for a finite nucleus. Although this calculation can be carried out more accurately if the nuclear form factors are known, this average should be relatively independent of the the exact q^2 dependence of these form factors. Thus, the results of the model calculation should be reasonably reliable. We find that our result in this case agrees with that of the usual Coulomb correction to order Rp_e.

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[5]See the second paper in Kim and Primakoff (Ref. 1).

[6]See, for example, Konopinski (Ref. 3) and Rose (Ref. 4).

[7]We follow the notation given by Rose (Ref. 4). The changes, $g_\kappa \to g_{-\kappa}$ and $f_{-\kappa} \to f_\kappa$, in the notation of Konopinski (Ref. 3) correspond to our notation.

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