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Upper and Lower Bounds on Phase Shifts for Three-Body Systems: n-d Scattering

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A variational-bound formulation of the three-body problem based on the Faddeev equations has been derived previously. This method involves a variational calculation of the exact effective potential between the incident particle and the bound two-body target. Variational upper and lower bounds on this effective potential are obtained by constructing approximate separable three-body Green's operators. The eigenphase shifts determined by using the effective potential in the two-body Lippmann-Schwinger equation are upper and lower bounds on the true eigenphases for energies below the three-body breakup threshold. The method is applied to the neutron-deuteron system, and a set of variationally converged phase shifts is obtained.

I. INTRODUCTION

A considerable effort has been made in recent years to formulate the three-body scattering problem in a manner which will allow practical evaluation of the various scattering parameters.[1] A large part of this effort has been devoted to the investigation of simplified three-body models[2] which take the two-body interaction to be a sum of separable functions in momentum space and thereby reduce the three-body Faddeev equations[3] to a one-dimensional integral equation which is easily solved. However, since the two-body interaction is not separable (at least asymptotically), we must ultimately learn how to solve the three-body problem with local interactions. A direct numerical finite differencing of the Faddeev equations has recently been performed for the S states of a simple three-boson system interacting via local potentials which are dominated by a few angular momentum states.[4] However, this method rapidly becomes orders of magnitude more difficult as the spin and higher angular momenta are introduced. A variational approach to the Faddeev equations also has been formulated and successfully employed to construct the quartet phase shifts for the neutron-deuteron system.[5] This variational approach is applicable to nonzero energies, and differs from previously derived variational-bound formulations[6] in that it does not involve the Feshbach projection operators. This second point is crucial in the nuclear three-body problem, where all the mass ratios are finite and the projection operators are extremely difficult to construct.[7] In this formulation the zero-order variational approximation is the exact solution to a simple separable model. In this way the gross properties of the three-body problem are factored into the variational solution explicitly, and a natural coupling of the finite-difference and variational methods is made. As is well known, the quartet state is well described by most dynamical schemes, and it is the purpose of this work to establish the quality of the variational solution when applied to the more complicated and sensitive doublet state of the neutron-deuteron system.

In Sec. II of this article, we make a specific decomposition of the two-particle scattering amplitude into separable and nonseparable contributions. When this amplitude is inserted into the three-body Faddeev equations, a two-body Lippmann-Schwinger-type equation results for the elastic n-d scattering amplitude. The effective potential that appears in the two-body equation is constructed in Sec. II. In Sec. III formal upper and lower variational bounds on the solution to the elastic two-body equation are derived. In Sec. IV a discussion of the application of this technique to the scattering states of the neutron-deuteron system is presented.

II. EFFECTIVE POTENTIAL FORMALISM

Consider first a two-particle system interacting through a potential v which supports N bound states. The eigenvalue equations are written as

$$(h_0 + v)\,|\chi_i\rangle = -\epsilon_i\,|\chi_i\rangle\,, \quad i = 1, \ldots, N. \tag{1}$$

An exactly solvable separable potential v_B which supports the same N bound states is

$$v_B = \sum_{i,j} v\,|\chi_i\rangle(\underline{v}^{-1})_{ij}\langle\chi_j|\,v\,, \tag{2}$$

where

$$\underline{v}_{ij} = \langle \chi_i | v | \chi_j \rangle . \tag{3}$$

That the two potentials support the same bound states follows from the relation

$$v | \chi_i \rangle = v_B | \chi_i \rangle . \tag{4}$$

The two-body scattering amplitude, $t(\vec{k}', \vec{k}; E)$, possesses bound-state poles at the energies $E = -\epsilon_i$ $(i = 1, \ldots, N)$. We can isolate these poles by writing

$$t = t_A + t_B , \tag{5}$$

where t_A is defined by the Lippmann-Schwinger equation

$$t_A = v_A + v_A g_0 t_A \tag{6}$$

and $v_A = v - v_B$. g_0 is the two-body free Green's operator, $(E - h_0)^{-1}$. It follows from the two-potential theorem[8] that

$$t_B = (1 + t_A g_0) t_{AB} (1 + g_0 t_A) , \tag{7}$$

where t_{AB} is defined by

$$t_{AB} = v_B + v_B g_A t_{AB} , \tag{8}$$

and g_A is given by

$$g_A = g_0 + g_0 t_A g_0 . \tag{9}$$

Since v_B is a separable operator, t_{AB} may be determined algebraically in the usual manner, with the result

$$t_{AB}(E) = \sum_{i,j} v | \chi_i \rangle \frac{\underline{S}_{ij}(E)}{E + \epsilon_i} \langle \chi_j | v . \tag{10}$$

The "self-energy" function $\underline{S}_{ij}(E)$ is defined by

$$[\underline{S}^{-1}(E)]_{ij} = \langle \chi_i | \tilde{\chi}_j(E) \rangle , \tag{11}$$

where the "distorted bound state" $|\tilde{\chi}_j(E)\rangle$ is given by

$$|\tilde{\chi}_j(E)\rangle = g_A(E) v | \chi_j \rangle . \tag{12}$$

It follows from Eqs. (4), (1), and (12) that at $E = -\epsilon_j$ there is no distortion, i.e.,

$$|\tilde{\chi}(E = -\epsilon_j)\rangle = g_A(E = -\epsilon_j) v | \chi_j \rangle = | \chi_j \rangle . \tag{13}$$

This guarantees that $\underline{S}_{ij}(E = -\epsilon_j) = \delta_{ij}$, and that t_B has the correct residue at the bound-state pole.

It is noteworthy that in the context of the three-body problem the two-body amplitude t_B enters through the expression

$$g_0 t_B g_0 = \sum_{i,j} |\tilde{\chi}_i(E)\rangle \frac{\underline{S}_{ij}(E)}{E + \epsilon_i} \langle \tilde{\chi}_j(E) | . \tag{14}$$

In the derivation of the effective potential, it is assumed that the potential v_A does not support any bound states. This is an assumption which is easily checked and in the present application will be shown to be valid.

As is well known, the Faddeev equations are a set of coupled linear integral equations for the nine scattering operators ${}^{(\alpha)}T^{(\beta)}(E)$ $(\alpha, \beta = 1, 2, 3)$ which describe the three-body scattering events of total energy E in which pair α (β) interacts in the final (initial) state. (Upper- and lower-case symbols will represent operators that act in the three- and two-body Hilbert spaces, respectively). The amplitudes $T(\alpha; E)$ $(\alpha = 1, 2, 3)$, which describe the scattering of pair α in the presence of the third spectator particle, are the input to these equations. $T(1; E)$, e.g., is related to the two-body amplitude t by the relation

$$\langle \vec{k}_1' \vec{k}_{23}' | T(1; E) | \vec{k}_1 \vec{k}_{23} \rangle = \delta(\vec{k}_1' - \vec{k}_1) \left\langle \vec{k}_{23}' \middle| t\left(E - \frac{\hbar^2 k_1^2}{2\mu_1} \right) \middle| \vec{k}_{23} \right\rangle , \tag{15}$$

where $\vec{k}_1$ is the momentum of particle 1 relative to the 23 subsystem, and $\vec{k}_{23}$ is the relative momentum of the 23 subsystem.

The amplitude $T_{\alpha\beta}$ for scattering with pair α bound finally and pair β bound initially is given by the residue of the matrix elements of the operator ${}^{(\alpha)}T^{(\beta)}$ at the initial- and final-state poles, located at

$$E = \frac{\hbar^2 k_\alpha'^2}{2\mu_\alpha} - \epsilon_\alpha = \frac{\hbar^2 k_\beta^2}{2\mu_\beta} - \epsilon_\beta .$$

In order to define the effective potential consider the reduced Faddeev operators ${}^{(\alpha)}T_A^{(\beta)}$ defined by the input amplitudes $T_A(\alpha; E)$. When these operators are compared with the full Faddeev operators, ${}^{(\alpha)}T^{(\beta)}$, using the two potential theorem,[8] it may be shown[9, 10] that the amplitude $T_{\alpha\beta}(\vec{k}_\alpha', \vec{k}_\beta; E)$ satisfies the two-body equation

$$T_{\alpha\beta}(\vec{k}_\alpha', \vec{k}_\beta; E) = V_{\alpha\beta}(\vec{k}_\alpha', \vec{k}_\beta; E) + \sum_{\gamma,\lambda} \int d\vec{k}_\gamma'' V_{\alpha\gamma}(\vec{k}_\alpha', \vec{k}_\gamma''; E) \frac{\underline{S}_{\gamma\lambda}\left(E - \frac{\hbar^2 k_\gamma''^2}{2\mu_\gamma} \right)}{E - \frac{\hbar^2 k_\gamma''^2}{2\mu_\gamma} + \epsilon_\gamma + i\eta} T_{\lambda\beta}(\vec{k}_\lambda'', \vec{k}_\beta; E) , \tag{16}$$

where the effective potential is given by

$$V_{\alpha\beta}(\vec{k}_\alpha', \vec{k}_\beta; E) = \left\langle \tilde{\chi}_\alpha\left(E - \frac{\hbar^2 k_\alpha'^2}{2\mu_\alpha} \right); \vec{k}_\alpha' \middle| \left[(E - H_0)(1 - \delta_{\alpha\beta}) + \sum_{\rho \neq \alpha} \sum_{\sigma \neq \beta} {}^{(\rho)}T_A^{(\sigma)}(E) \right] \middle| \tilde{\chi}_\beta\left(E - \frac{\hbar^2 k_\beta^2}{2\mu_\beta} \right); \vec{k}_\beta \right\rangle . \tag{17}$$

H_0 is the three-body kinetic energy operator. We are here using the subscript α to define both the particular two-particle subsystem and the associated bound state. The state $|\tilde{\chi}_1(E-\hbar^2k_1^2/2\mu_1);\vec{k}_1\rangle$ is defined as

$$\left\langle \vec{r}_1\vec{r}_{23}\left|\tilde{\chi}_1\left(E-\frac{\hbar^2k_1^2}{2\mu_1}\right);\vec{k}_1\right.\right\rangle = (2\pi)^{-3/2}e^{i\vec{k}_1\cdot\vec{r}_1}\left\langle \vec{r}_{23}\left|\tilde{\chi}\left(E-\frac{\hbar^2k_1^2}{2\mu_1}\right)\right.\right\rangle . \tag{18}$$

III. MAXIMUM AND MINIMUM PRINCIPLES

It follows from the definition of the Faddeev operators[3] that the sum appearing in $V_{\alpha\beta}(\vec{k}_\alpha,\vec{k}_\beta;E)$ [Eq. (17)] may be expressed in terms of the reduced three-body Green's operator $G_A=(E-H_A)^{-1}$ as

$$\sum_{\sigma\neq\beta}\sum_{\rho\neq\alpha}{}^{(\rho)}T_A^{(\sigma)}=\sum_{\gamma\neq\alpha,\beta}V_A(\gamma)+[V_A-V_A(\alpha)]G_A[V_A-V_A(\beta)]. \tag{19}$$

$V_A(\alpha)$ represents the interaction between pair α, V_A is the sum of all pairwise interactions, and H_A is the reduced three-body Hamiltonian. The starting point in the derivation of the variational principle for G_A is the identity[10]

$$G_A=G_{At}+G_{At}^\dagger+G_{At}^\dagger(H_A-E)G_{At}+\Delta G_A^\dagger(H_A-E)G_A(H_A-E)\Delta G_A, \tag{20}$$

where G_{At} is a trial Green's operator and $G_A=G_{At}+\Delta G_A$. Let G_A^B represent a zeroth order bound on G_A, so that $G_A-G_A^B$ is of definite sign. Then if G_A is replaced by G_A^B in the right-hand side of Eq. (20), the error introduced will be of second order and of definite sign. We then have the following variational bound on G_A:

$$G_A^{VB}=G_{At}+G_{At}^\dagger+G_{At}^\dagger(H_A-E)G_{At}+\Delta G_A^\dagger(H_A-E)G_A^B(H_A-E)\Delta G_A. \tag{21}$$

At this point it is assumed (and will be shown to be the case in the present n-d application) that H_A is a positive operator. If, however, H_A supports a set of three-body bound states, these must be "subtracted out" as were the two-body bound states.[11] The zeroth-order bounds employed in the present calculation are:

$$\text{upper bound: } G_A^B=0\geq G_A; \tag{22}$$

and

$$\text{lower bound: } G_A^B=1/E\leq G_A. \tag{23}$$

These inequalities follow from the fact that H_A is a positive operator, and E is taken to be negative.

For convenience the trial Green's operator appearing in the Eq. (20) is taken to be of the form of a finite sum of separable terms

$$G_{At}=\sum_i a_i\,|\,\psi_i\rangle\langle\psi_i\,|, \tag{24}$$

where the $|\,\psi_i\rangle$ are members of a suitably chosen set of basis functions. If the linear coefficients a_i are evaluated variationally by demanding that $\delta G_A^{VB}=0$, the resulting variational upper and lower bounds are:

$$G_A^{VUB}=\sum_{ij}|\,\psi_i\rangle(\underline{m}_U^{-1})_{ij}\langle\psi_j\,|, \tag{25}$$

where the matrix $\underline{m}_U$ is defined by

$$(\underline{m}_U)_{ij}=\langle\psi_i\,|\,E-H_A\,|\,\psi_j\rangle; \tag{26}$$

and

$$G_A^{VLB}=\frac{1}{E}+\frac{1}{E}\sum_{ij}H_A|\psi_i\rangle(\underline{m}_L^{-1})_{ij}\langle\psi_j|H_A, \tag{27}$$

where the matrix $\underline{m}_L$ is defined by

$$(\underline{m}_L)_{ij}=\langle\psi_i\,|\,H_A(E-H_A)\,|\,\psi_j\rangle. \tag{28}$$

The error introduced into the effective potential $V_{\alpha\beta}$ by replacing G_A with either the variational upper bound, Eq. (25), or the variational lower bound, Eq. (27), may be written

$$\Delta V_{\alpha\beta}(\vec{k}'_\alpha,\vec{k}_\beta)=\langle\Delta\psi_\alpha(\vec{k}'_\alpha)\,|\,(G_A-G_A^B)\,|\,\Delta\psi_\beta(\vec{k}_\beta)\rangle, \tag{29}$$

where

$$|\Delta\psi_\beta(\vec{k}_\beta)\rangle=(H_A-E)\Delta G_A\times[V_A-V_A(\beta)]\,|\tilde{\chi}_\beta\left(E-\frac{\hbar^2k_\beta^2}{2\mu_\beta}\right);\vec{k}_\beta\rangle. \tag{30}$$

Since G_A^B represents either an upper or lower bound on G_A, the diagonal elements of the error operator, $\Delta V_{\alpha\alpha}(\vec{k},\vec{k})$, are of definite sign; positive for $G_A>G_A^B$ and negative for $G_A<G_A^B$. We now recall the monotonicity theorem,[10] which states that if the diagonal elements of the error operator ΔV are positive (negative) the eigenphases determined variationally using Eq. (27) [Eq. (25)] lie above (below) the true eigenphases.

Therefore, by inserting the variational bounds on the Green's operator [Eqs. (25) and (27)] into the effective potential [using Eq. (19)] variational bounds on $V_{\alpha\beta}$ may be constructed. These bounds when inserted into the effective two-body Lippmann-Schwinger equation [Eq. (16)] lead to variational bounds on the eigenphase shifts.

It is noteworthy that there are two possible computational strategies that can be employed: (1) improve the trial Green's operator G_t by adding more separable terms; or (2) iterate the bound by inserting the variationally determined bound into the right-hand side of Eq. (21).

The bounds on $V_{\alpha\beta}$ constructed here are equivalent to the bounds derived by Sugar and Blankenbecler.[12] The advantage of the present formulation is that the Feshbach projection operators do not appear.

IV. NEUTRON-DEUTERON SCATTERING

In order to evaluate the practicality of this formulation we consider the neutron-deuteron $L=0$ scattering states. The two-particle interaction is taken to be central with symmetric exchange mixture having the radial form

$$v(r) = v_0 e^{-r^2/b^2}, \tag{31}$$

where $v_0 = -86.4$ MeV and $b = 1.332$ fm. It is easily shown that the distorted bound state $|\tilde{\chi}(z)\rangle$ satisfies the integral equation

$$|\tilde{\chi}(z)\rangle = g_0(z)v|\chi\rangle + g_0(z)v_A|\tilde{\chi}(z)\rangle . \tag{32}$$

The Christian-Gammel approximate deuteron wave function[13] was used to represent $|\chi\rangle$, and Eq. (32) was solved numerically for $\langle\vec{k}|\tilde{\chi}(z)\rangle$. In order to carry out analytically the integrations occurring in the effective potential, the solution was fitted to a sum of three Gaussians for values of z in the interval $(-\infty, E)$. The self-energy function $S(E)$ was then determined using Eq. (11). The three-body states appearing in G_A^{VUB} and G_A^{VLB} [Eqs. (25) and (27)] were taken to be of the form

$$\psi_i(r_1, r_2, r_3; s) = \exp(-\textstyle\sum_j \alpha_{ij} r_j^2)\sigma_i^s , \tag{33}$$

where r_j is the separation between pair j and σ_i^s is a particular spin state having total spin $s = \frac{1}{2}, \frac{3}{2}$. The advantage of this particular form is that it allows analytic evaluation of the effective potential matrix elements, Eq. (17).

Before the variational principle may be applied, the bound-state spectra of the reduced two- and three-body Hamiltonians h_A and H_A must be established. The two-body Fredholm determinant, $\Delta_A(E) = \mathrm{Det}(1 - v_A g_A)$, has been evaluated numerically and is a smooth, nonnegative function for $E<0$, excluding the presence of any negative-energy bound states in h_A. [The two-body bound-state energies ϵ_i^A are given by the zeros of the Fredholm determinant $\Delta_A(E)$.] In order to establish the presence of any negative-energy states in the spectrum of H_A, a Rayleigh-Ritz eigenvalue calculation was performed. Using a 10-term trial wave function, it was found to be impossible to drive the energy expectation $\langle\psi|H_A|\psi\rangle$ negative, indicating that H_A is a positive operator. It is recognized that this is not a proof of the positivity of H_A; however, since this is a difficult question and of somewhat peripheral interest a more exhaustive study was not performed.

After an angular momentum decomposition and an antisymmetrization to account for the identical neutrons, Eq. (16) reduces to a one-dimensional integral equation for the elastic n-d amplitude which was solved numerically. [The equation actually solved was that for the reaction matrix which amounts to replacing the $+i\eta$ in Eq. (16) by the principal-value prescription.]

In the quartet state, as is well known, the neutrons are spatially well separated as a result of the Pauli principle, and most dynamical schemes give essentially the same results. This spatial separation has the consequence, in the present formulation, that the variational terms arising from G_A^{VUB} and G_A^{VLB} are only a small contribution to the effective potential, and for a simple trial Green's operator G_{At} the upper and lower bounds on $V_{\alpha\beta}$ are separated by only a few percent. The variational bounds on $k\cot^4\delta$, where $^4\delta$ is the quartet $L=0$ phase shift, at $E=-1.2$ MeV are:

$$k\cot^4\delta_{\mathrm{U}} = -0.136 \geq k\cot^4\delta \geq -0.153 = k\cot^4\delta_{\mathrm{L}} . \tag{34}$$

The upper bound is in reasonable agreement with previous calculations[14] and with experiment,[15] while the lower bound may be improved by the addition of more terms in the variational Green's operator, Eq. (27).

In the doublet state the situation is different and the variational contributions to the effective potential are of the same order as $V_{\alpha\beta}$. As in bound-state calculations[16] the variational upper bound is,

TABLE I. Doublet scattering lengths.

	Present variational upper bound	Experiment: Van Oers and Seagrave [a]	Experiment: Koester and Ungerer [b]	Separable model [c]	Kohn variational upper bound: Yukawa [d]	Kohn variational upper bound: Gaussian [e]
Doublet scattering length (fm)	0.118	0.15 ± 0.05	0.46 ± 0.06	-1.0	-2.0	0.19

[a] Reference 15.
[b] Reference 17.
[c] Aaron, Amado, and Yam in Reference 14.
[d] Reference 18.
[e] Reference 19.

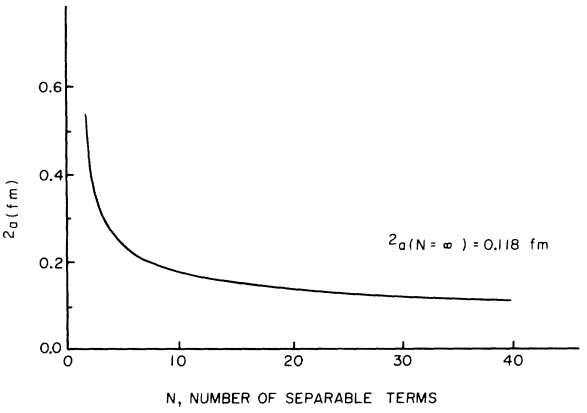

FIG. 1. The doublet scattering length as a function of N, the number of separable terms in the expansion of G_A^{VUB}.

in general, of better quality than the comparable lower bound and we therefore consider the upper-bound scattering-length calculation first.

Inserting G_A^{VUB} into $V_{\alpha\beta}$ we determine a lower bound on the doublet phase shift $^2\delta$, and, at zero energy, an upper bound on the doublet scattering length ^2a. The nonlinear parameters α_{ij} appearing in Eq. (33) were determined by numerically minimizing the scattering length. For convenience, the same values of these nonlinear parameters were also used for nonzero energies. In Fig. 1, the doublet scattering length is presented as a function of the number of separable terms in G_A^{VUB}, and it is observed that the variational bound has essentially converged for $N \approx 30$. In Table I, the experimental value of Van Oers and Seagrave[15] and a new value based on the recent experiment of

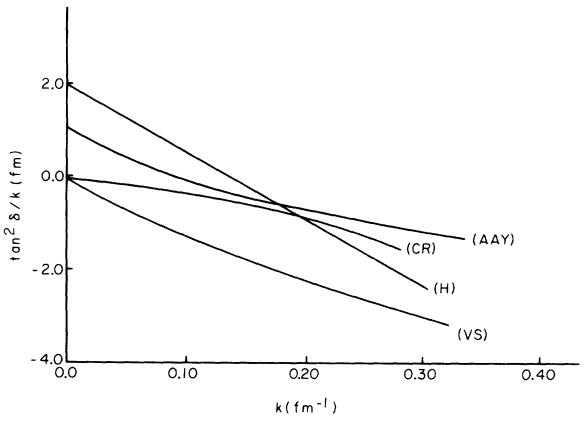

FIG. 2. Variational lower bound on $(\tan^2\delta)/k$ (CR). For comparison the experimental results of Van Oers and Seagrave (VS) (Ref. 15), the separable results of Aaron, Amado, and Yam (AAY) (Ref. 14), and the Kohn variational calculation of Humberston (H) (Ref. 18) are also presented.

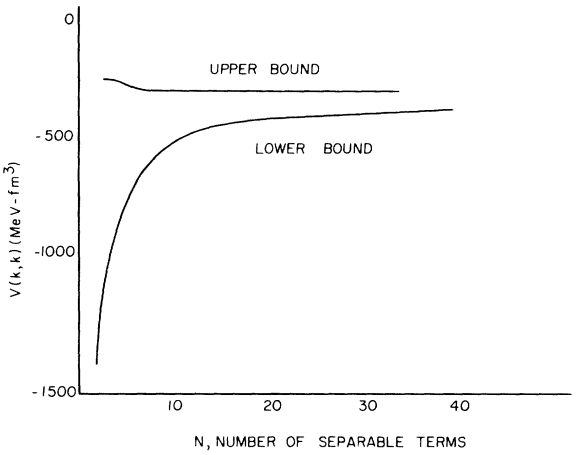

FIG. 3. Variational upper and lower bounds on a typical diagonal momentum-space matrix element of the effective potential vs N – the number of separable terms in the expansion of G_A^{VUB} and G_A^{VLB} – at zero energy ($s = \frac{1}{2}$).

Koester and Ungerer,[17] together with the result of the present calculation are given for the doublet scattering length. For comparison the calculations of Aaron, Amado, and Yam,[14] who used a Yamaguchi separable interaction, of Humberston,[18] who used the Kohn variational principle with a Yukawa local interaction, and of Pett,[19] who used the Kohn variational principle with a Gaussian local interaction (Serber exchange mixture) having the same radial form as our potential, are also presented.[20] When comparing our results with the separable approximation, it should be noted that the neglect of the variational terms arising from G_A^{VUB} and G_A^{VLB} is equivalent to making the separable approximation $V_A = V - V_B \equiv 0$. That is, the contribution to the three-body amplitude of the separable component of the two-body interaction v_B is exactly

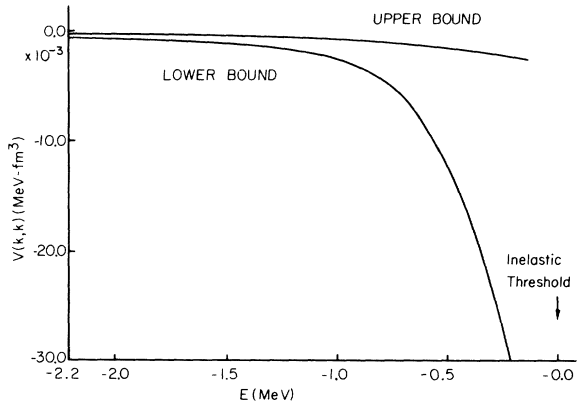

FIG. 4. Variational upper and lower bounds on a typical diagonal momentum-space matrix element of the effective potential vs the three-body energy ($s = \frac{1}{2}$). The separable expansion of $G_A^{\mathrm{VUB}}(G_A^{\mathrm{VLB}})$ contains 30 (36) terms.

represented by the Born-exchange term (the leading term) in Eq. (17). From Fig. 1 it is clear that in the present application the variational terms, which represent the nonseparable component of the potential, are indeed important.

In Fig. 2 the variational estimate of $(\tan^2\delta)/k$ is presented along with the experimental curve of Van Oers and Seagrave.[15] Since the variational result is also a bound, the exact value for the Gaussian potential must lie above our estimate of $(\tan^2\delta)/k$. [An upper bound on $k\cot^2\delta$ yields a lower bound on $(\tan^2\delta)/k$]. In these calculations G_A^{VUB} [Eq. (25)] contained 30 separable terms.

The lower-bound calculation is defined by inserting G_A^{VLB} [Eq. (27)] into $V_{\alpha\beta}$ and then solving Eq. (16). The nonlinear parameters α_{ij} contained in G_A^{VLB} were evaluated by numerically maximizing the effective potential at zero energy. The difficulty in the doublet lower-bound calculation is twofold. In the first place, the operator G_A^{VLB} requires matrix elements of the square of the Hamiltonian (G_A^{VUB} only requires matrix elements of H_A) and consequently is considerably more difficult to evaluate than G_A^{VUB} (a factor of $\approx$100 in computer time is involved). Secondly, the convergence of the lower bound is much slower than the upper-bound calculation. To illustrate this, in Fig. 3 a typical matrix element of the effective potential, $V(k, k)$, for the lower-bound calculation (at zero energy) is plotted versus the number of terms in the separable expansion of G_A^{VLB}; with 36 terms in the expansion the calculation has yet to converge.[21] For comparison, the corresponding upper-bound results are also presented in Fig. 3. It is important to note that as the number of separable terms in the expansion of G_A^{VLB} is increased the machine time required to evaluate $V_{\alpha\beta}$ increases rapidly.

In Fig. 4 the variational upper and lower bounds on $V(k, k)$ are presented as a function of E. The divergence of the lower bound from the variationally converged upper bound as E approaches the inelastic threshold is due to the $1/E$ behavior in G_A^{VLB}.

An interesting comparison may be drawn between these n-d lower bounds and the corresponding bounds on the e^--H scattering phase shifts recently reported by Madan.[22] By neglecting the variational contributions to G_A^{VLB}, i.e., by taking

$$G_A^{\mathrm{VLB}} = 1/E\,, \tag{35}$$

he has shown that away from the inelastic threshold the upper and lower bounds on the phase shifts are separated by less than 20%. In our problem such a crude approximation is grossly inaccurate, as shown in Fig. 3 for zero-energy scattering. The divergence of the upper and lower bounds at the inelastic threshold is also present in Madan's calculations.

The variational upper-bound approach presented here has been shown to be a practical one for three-body systems interacting via local two-body potentials. While an evaluation of the utility of this formulation will be incomplete until realistic two-body forces are employed, we believe that the labor involved in such extended calculations will not be prohibitive. The crucial test will be the convergence properties of the upper-bound variational calculation.

In the quartet state the lower-bound variational solution is sufficiently close to the upper bound to yield a meaningful bound on the error in the variational calculation. In the doublet state, where there is considerable polarization of the deuteron, the lower-bound solution does not yield a useful estimate of the error, and in this case the lower-bound calculation can only serve as a guide to the upper-bound calculation. We emphasize, however, that the difficulty in the doublet lower-bound calculation is not one of principle. In a more systematic study, involving realistic forces, the considerable investment in computer time required by such a calculation might very well be considered worthwhile.

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Isospin-Forbidden Particle Decay of the First $T=\frac{3}{2}$ States in ^{9}Be and ^{9}B

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The particle decays of the first $T=\frac{3}{2}$ state in ^{9}Be (^{9}B) have been studied by the ^{7}Li(^{3}He,pn) reaction at $E(^3\text{He})=10.0$ MeV (8.7 and 7.95 MeV). Protons (neutrons) emitted at 0° were detected in coincidence with decay neutrons (protons). Neutron energy was determined by the associated-particle time-of-flight technique. The ratios of partial neutron (proton) widths to the ground and first excited states of ^{8}Be to the ground-state radiative width Γ_{γ_0} are found to be, respectively, 6.4 ± 2.0 and 20.4 ± 4.6 for the ^{9}Be analog state (<1.5 and 13.9 ± 2.1 for ^{9}B). Taking $\Gamma_{\gamma_0}=10.5\pm1.5$ eV, these results give $\Gamma_{n_0}=67\pm26$ eV, $\Gamma_{n_1}=213\pm56$ eV, $\Gamma_{p_0}<18$ eV, and $\Gamma_{p_1}=147\pm30$ eV for the mirror decays.

1. INTRODUCTION

In spite of the isospin selection rule, it has recently been found that the $T=\frac{3}{2}$ states in several $A=4n+1$ nuclei[1-4] decay to $T=0$ low-lying states of the corresponding $A=4n$ nuclei with $Z=N$. The branching ratios of these decays are very charge asymmetric. Since the isospin-forbidden width is a measure of isospin impurity in the analog state, one hopes to obtain information on the charge-dependent effects inside the nucleus from such measurements. Theoretical calculations for $A=13$ by Arima and Yoshida[5] show that the observed partial widths[1] can be qualitatively explained by consideration of isospin mixing due to residual Coulomb interaction within the $1p$-shell configuration states and mixing with the continuum states due to single-particle Coulomb interactions. Presence of a charge-dependent nuclear force is, however, not essential for theoretical-experimental agreement. Recent calculations for the isospin mixing in the first $T=\frac{3}{2}$ states of ^{9}Be and ^{9}B by one of the authors[6] have shown that the residual Coulomb interaction within $1p$-shell configuration states is too small to account for the observed total widths[7] of the analog states. Measurements of the isospin-forbidden particle decays in mass-9 nuclei may be able to give better evidence concerning the existence of a charge-dependent nuclear force.

In the present experiment, the reactions ^{7}Li(^{3}He,p)^{9}Be and ^{7}Li(^{3}He,n)^{9}B were used to populate analog states. The decays were observed by detecting the protons (neutrons) in coincidence with the decay neutrons (protons). The processes in-