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Three-Body Clusters in Nuclear Matter*

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It is shown that starting with a correlated wave function for nuclear matter of a general form one may make direct correspondence to the conventional diagrammatic theory of Brueckner, Bethe, and Goldstone. In particular, in this work we show that the expression for the three-body-cluster energy is the same as that obtained by Bethe. Also, it is shown that the current approximation which neglects the potential for particle states in the diagrammatic approach corresponds to a neglect of a *coupling* between the two-body (Bethe-Goldstone) equation and the equation for the three-body-cluster wave function (Bethe-Faddeev). The theory presented here provides a clarification of the relation of the n-body-cluster wave functions and the structure of the wave function of the entire system.

In two previous works[1,2] we have discussed the theory of correlated Fermi systems which obtains if one begins with a wave function of general form,

$$|\Psi\rangle = \frac{e^S|\Phi\rangle}{\langle\Phi|e^{S\dagger}e^S|\Phi\rangle^{1/2}}, \tag{1}$$

where $|\Phi\rangle$ is an uncorrelated state.[3] In the case of nuclear matter $|\Phi\rangle$ represents a Fermi gas, while for finite systems $|\Phi\rangle$ represents a Slater determinant of appropriately chosen orbitals. In the first paper of this series we studied the structure of the theory for the *special case* in which e^S induced correlations having Jastrow structure.[1] In the second paper[2] we showed how the theory could be extended to include occupation-factor corrections and, in addition, the equations were written so as to be appropriate to the treatment of finite systems. In that second work we limited ourselves to the case, $S=S^{(2)}$, where $S^{(2)}$ is a two-body operator. It was shown there that that choice, along with a cluster-expansion method, enabled us to obtain the Bethe-Goldstone equation for the two-body cluster wave function. Also, it was clear that the neglect of three-body-cluster effects resulted in a theory in which there was no potential in the *particle* states of the Bethe-Goldstone equation.

In this work we are interested in demonstrating, through three-body-cluster terms, the complete correspondence of the Brueckner-Bethe-Goldstone approach and that obtained starting with the wave function of Eq. (1). We are able to make this correspondence if we neglect a coupling term between the two- and three-body-cluster wave functions. If this coupling is neglected we obtain correspondence with the conventional diagrammatic approach if, in the latter theory, the particle potential is placed equal to zero.

To carry out this program it is necessary to write

$$S = S^{(2)} + S^{(3)}, \tag{2}$$

where $S^{(n)}$ is an n-body operator.[4] The operator e^S may be expanded as

$$e^S = 1 + F^{(2)} + F^{(3)} + \cdots$$

and the $F^{(n)}$'s may also be written

$$F^{(2)} = \frac{1}{2!}\sum_{mnij} a_m^\dagger a_n^\dagger \langle mn|f_{12}|ij\rangle a_j a_i, \tag{3}$$

$$F^{(3)}=\frac{1}{3!}\sum_{mnpijk} a_m^\dagger a_n^\dagger a_p^\dagger \langle mnp|\, f_{123}\,|ijk\rangle a_k a_j a_i\,, \tag{4}$$

etc. Note that $F^{(2)}=S^{(2)}$, $F^{(3)}=S^{(3)}$, $F^{(4)}=(S^{(2)})^2/2!$, etc., if S is given as in Eq. (2). [As in previous works we use $i,j,k,\ldots$ to refer to occupied (hole) states, $m,n,p,\ldots$ to refer to unoccupied (particle) states, and $\alpha,\beta,\gamma,\ldots$ to refer to either.]

Using the cluster-expansion procedure outlined in Ref. 2, we may write the energy of nuclear matter (see Appendix A)

$$\mathcal{E}=\langle\Psi|\,H\,|\Psi\rangle=\sum_i t_{i,i}\gamma_i+\frac{1}{2!}\sum_{ij}h_{ij,ij}\gamma_i\gamma_j+\frac{1}{3!}\sum_{ijk}h_{ijk,ijk}\gamma_i\gamma_j\gamma_k+\cdots \tag{5}$$

with $\gamma_i=\langle\Psi|\,a_i^\dagger a_i\,|\Psi\rangle$, and where the h's are cluster integrals defined below. We introduce the operator

$$\beta_{12}=(1+f_{12}^\dagger)(t_1+t_2+v_{12})(1+f_{12})-(t_1+t_2)\,, \tag{6}$$

and in terms of this operator we have

$$h_{ij,i'j'}\equiv\langle ij\,|\beta_{12}|\,i'j'\rangle_A\,, \tag{7}$$

where the subscript A refers to an antisymmetrization prescription for kets

$$|ij\rangle_A=|ij\rangle-|ji\rangle\,,$$

$$|ijk\rangle_A=|ijk\rangle-|jik\rangle-|ikj\rangle+|jki\rangle+|kij\rangle-|kji\rangle\,,$$

etc. The three-body cluster integral is given by

$$\begin{aligned}h_{ijk,i'j'k'}=\langle ijk|\{&(1+f_{12}^\dagger+f_{23}^\dagger+f_{13}^\dagger+f_{123}^\dagger)(t_1+t_2+t_3+v_{12}+v_{23}+v_{13})(1+f_{12}+f_{23}+f_{13}+f_{123})-(\beta_{12}+\beta_{13}+\beta_{23})\\&-[t_1(1+f_{23}^\dagger)(1+f_{23})+t_2(1+f_{13}^\dagger)(1+f_{13})+t_3(1+f_{12}^\dagger)(1+f_{12})]\}\,|i'j'k'\rangle_A\,.\end{aligned} \tag{8}$$

It is also useful to define the two- and three-body-cluster wave functions:

$$|\Psi_{ij}\rangle=(1+f_{12})|ij\rangle_A\,, \tag{9}$$

$$|\Psi_{ijk}\rangle=(1+f_{12}+f_{23}+f_{13}+f_{123})|ijk\rangle_A\,. \tag{10}$$

These are normalized as follows:

$$\langle ij\,|\Psi_{i'j'}\rangle=\langle ij\,|i'j'\rangle_A=\langle ij\,|i'j'\rangle-\langle ij\,|j'i'\rangle\,, \tag{11}$$

$$\langle ijk\,|\Psi_{i'j'k'}\rangle=\langle ijk|i'j'k'\rangle_A\,. \tag{12}$$

We will also need the following cluster integrals:

$$\kappa_{ij,i'j'}=\langle ij\,|\,[(1+f_{12}^\dagger)(1+f_{12})-1]\,|\,i'j'\rangle_A\,, \tag{13}$$

and

$$\begin{aligned}\kappa_{ijk,i'j'k'}=\langle ijk|\{&(1+f_{12}^\dagger+f_{13}^\dagger+f_{23}^\dagger+f_{123}^\dagger)(1+f_{12}+f_{13}+f_{23}+f_{123})-[(1+f_{12}^\dagger)(1+f_{12})-1]-[(1+f_{13}^\dagger)(1+f_{13})-1]\\&-[(1+f_{23})(1+f_{23})-1]-1\}\,|i'j'k'\rangle_A\,.\end{aligned} \tag{14}$$

Finally, we write the equation for the occupation factors which are obtained by a selective summation of cluster diagrams as discussed in Ref. 2 (see Appendix A),

$$\gamma_i=1-\sum_k\kappa_{ik,ik}\gamma_i\gamma_k-\tfrac{1}{2}\sum_{kl}\kappa_{ikl,ikl}\gamma_i\gamma_k\gamma_l\,. \tag{15}$$

At this point we may proceed with various algebraic manipulations; however, it is useful to derive equations which allow us to specify the best correlation operators f_{12} and f_{123}.[5] We may use variational techniques if we introduce Lagrange multipliers. We multiply Eq. (15) by ϵ_i and sum over i to obtain

$$\sum_i\epsilon_i=\sum_i\gamma_i\epsilon_i+\sum_{ik}\epsilon_i\kappa_{ik,ik}\gamma_i\gamma_k+\tfrac{1}{2}\sum_{ikl}\epsilon_i\kappa_{ikl,ikl}\gamma_i\gamma_k\gamma_l\,. \tag{16}$$

Thus we have for the energy

$$\mathcal{E}=\sum_i\epsilon_i+\sum_i(t_{i,i}-\epsilon_i)\gamma_i+\frac{1}{2!}\sum_{ij}K_{ij,ij}\gamma_i\gamma_j+\frac{1}{3!}\sum_{ijk}K_{ijk,ijk}\gamma_i\gamma_j\gamma_k\,, \tag{17}$$

where we have defined

$$K_{ij,ij} = h_{ij,ij} - (\epsilon_i + \epsilon_j)\kappa_{ij,ij} \tag{18}$$

and

$$K_{ijk,ijk} = h_{ijk,ijk} - (\epsilon_i + \epsilon_j + \epsilon_k)\kappa_{ijk,ijk}\,. \tag{19}$$

We may now freely vary $\mathcal{E}$ with respect to the γ and the correlation operators f. Variation with respect to γ_i yields the expression for ϵ_i,[6]

$$\epsilon_i = t_{i,i} + \sum_k K_{ik,ik}\gamma_k + \tfrac{1}{2}\sum_{jk} K_{ijk,ijk}\gamma_j\gamma_k\,. \tag{20}$$

Variation with respect to $\langle ij|\, f_{12}^{\dagger}|\, mn\rangle$ yields

$$\langle mn|\,(t_1+t_2+v_{12})|\,\Psi_{ij}\rangle - \langle mn|\,\Psi_{ij}\rangle(\epsilon_i+\epsilon_j) + \sum_k [\langle mnk|\,(t_1+t_2+t_3+v_{12}+v_{23}+v_{13})|\,\Psi_{ijk}\rangle$$

$$- \langle mnk|\,(t_1+t_2+v_{12})(1+f_{12})|\,ijk\rangle_A - \langle mnk|\,t_3(1+f_{12})|\,ijk\rangle_A]\gamma_k = 0\,. \tag{21}$$

We note that in the square bracket of Eq. (21), the second two terms which we have derived serve to cancel the "unlinked" terms in the first term of this bracket – thus the bracketed expression is of "linked" character.

We also note at this point that if we drop the last term in Eq. (21), which represents a coupling between the amplitudes $|\Psi_{ij}\rangle$ and $|\Psi_{ijk}\rangle$, we find that Eq. (21) leads to the equation

$$|\Psi_{ij}\rangle = |ij\rangle - \frac{1}{t_1+t_2-\epsilon_i-\epsilon_j} Q v_{12}|\Psi_{ij}\rangle\,, \quad Q = \sum_{mn} |mn\rangle\langle mn|\,, \tag{22}$$

which is the Bethe-Goldstone equation with zero potential for the particle states. Also, as shown previously,[1,2] we have the relations (see Appendix B)

$$K_{12} = v_{12}(1+f_{12})\,, \tag{23}$$

and

$$f_{12} = -\frac{Q}{e}K_{12}\,, \tag{24}$$

where the K_{12} of Eqs. (23) and (24) is the same as that defined in Eq. (18), and $e = t_1 + t_2 - \epsilon_i - \epsilon_j$.

We will return to the discussion of the neglected term in Eq. (21) at a later stage. At this point we vary the expression for the energy with respect to $\langle ijk|\, f_{123}^{\dagger}|\, mnp\rangle$ and obtain

$$\langle mnp|\,t_1+t_2+t_3+v_{12}+v_{23}+v_{13})|\,\Psi_{ijk}\rangle - (\epsilon_i+\epsilon_j+\epsilon_k)\langle mnp|\,\Psi_{ijk}\rangle = 0\,. \tag{25}$$

Equations (21) and (25) provide a set of equations which enable us to calculate the correlation operators f_{12} and f_{123}. We proceed now to establish the connection between the theory developed above and the conventional diagrammatic methods of Brueckner, Bethe, and Goldstone. The relation between these theories through the consideration of two-body clusters is apparent from our previous work.[1,2]

If we note that

$$\kappa_{ijk,ijk} = \sum_{mnp}\langle\Psi_{ijk}|\, mnp\rangle\langle mnp\,|\,\Psi_{ijk}\rangle\,, \tag{26}$$

we may use Eq. (25) to simplify Eqs. (8) and (19). We find

$$K_{ijk,ijk} = \langle ijk|\,(1+f_{12}^{\dagger}+f_{13}^{\dagger}+f_{23}^{\dagger})(t_1+t_2+t_3+v_{12}+v_{13}+v_{23})|\,\Psi_{ijk}\rangle$$

$$- \langle ijk|\,[(\beta_{12}+\beta_{13}+\beta_{23}) + t_1(1+f_{23}^{\dagger})(1+f_{23}) + t_2(1+f_{13}^{\dagger})(1+f_{13}) + t_3(1+f_{12}^{\dagger})(1+f_{12})]\,|\,ijk\rangle_A\,. \tag{27}$$

Next we notice that the kinetic energy operators do not contribute to Eq. (27). Therefore, we have

$$K_{ijk,ijk} = \langle ijk|\,(1+f_{12}^{\dagger}+f_{13}^{\dagger}+f_{23}^{\dagger})(v_{12}+v_{23}+v_{13})|\,\Psi_{ijk}\rangle - \langle ijk|\,[(1+f_{12}^{\dagger})K_{12} + (1+f_{13}^{\dagger})K_{13} + (1+f_{23}^{\dagger})K_{23}]|\,ijk\rangle_A\,, \tag{28}$$

where we have used Eq. (23). To further simplify Eq. (28) we introduce the wave matrix Ω and the T ma-

trix for the three-body-cluster amplitude $|\Psi_{ijk}\rangle$:

$$|\Psi_{ijk}\rangle = \Omega |ijk\rangle_A , \tag{29}$$

and

$$(v_{12}+v_{23}+v_{13})|\Psi_{ijk}\rangle = T|ijk\rangle_A . \tag{30}$$

At this point we follow Bethe and Rajaraman[7] and write (see Appendix C)

$$T = T^{(1)} + T^{(2)} + T^{(3)} , \tag{31}$$

where

$$T^{(1)} = K_{23} - K_{23}\frac{Q}{e}[T^{(2)} + T^{(3)}] , \tag{32}$$

etc. We may also write

$$T = K_{12} + K_{13} + K_{23} + \Delta T , \tag{33}$$

with

$$\Delta T \equiv [T^{(1)} - K_{23}] + [T^{(2)} - K_{13}] + [T^{(3)} - K_{12}] .$$

Using Eqs. (30) and (33) in Eq. (28) we find

$$K_{ijk,ijk} = \langle ijk|\,(f_{12}^{\dagger} + f_{13}^{\dagger} + f_{23}^{\dagger})\Delta T\,|ijk\rangle_A , \tag{34}$$

taking note of the fact that, for example, $\langle ijk|\, f_{12}^{\dagger}K_{23}|ijk\rangle_A = 0$, etc., and $\langle ijk|\,\Delta T|ijk\rangle_A = 0$. Also note that $\langle ijk|\, f_{12}^{\dagger}[T^{(3)} - K_{12}]|ijk\rangle_A = 0$, etc., so that we may write

$$\begin{aligned} K_{ijk,ijk} = {} & \langle ijk|\, f_{12}^{\dagger}[T^{(1)} - K_{23}] + f_{12}^{\dagger}[T^{(2)} - K_{13}]|ijk\rangle_A + \langle ijk|\, f_{13}^{\dagger}[T^{(1)} - K_{23}] + f_{13}^{\dagger}[T^{(3)} - K_{12}]|ijk\rangle_A \\ & + \langle ijk|\, f_{23}^{\dagger}[T^{(2)} - K_{13}] + f_{23}^{\dagger}[T^{(3)} - K_{12}]|ijk\rangle_A . \end{aligned} \tag{35}$$

Again to make contact with Bethe's work we may examine the first term of Eq. (35), which corresponds to particles 1 and 2 interacting last. Noting that [see Appendix,. Eq. (C7)]

$$T^{(1)} - K_{23} = K_{23}[\Omega^{(1)} - 1] , \quad T^{(2)} - K_{13} = K_{13}[\Omega^{(2)} - 1] , \tag{36}$$

the first term of Eq. (35) becomes

$$\langle ijk|\, f_{12}^{\dagger}\{K_{23}[\Omega^{(1)} - 1] + K_{13}[\Omega^{(2)} - 1]\}\,|ijk\rangle_A = \langle ijk|\, f_{12}^{\dagger}K_{23}\Omega^{(1)} + f_{12}^{\dagger}K_{13}\Omega^{(2)}|ijk\rangle_A = \langle ijk|\,[T^{(3)} - K_{12}]|ijk\rangle_A , \tag{37}$$

with similar expressions for the other terms of Eq. (35). Comparison may be made, for example, with Eq. (4.12) of Bethe and Rajaraman,[7] noting that their $\eta(\vec{r}_{12}) = -f_{12}$, and their $Z^{(1)} = [1 - \Omega^{(1)}]|ijk\rangle$.

Using Eqs. (33), (35), and (37) our expression for the energy, Eq. (17), becomes

$$\mathcal{E} = \sum_i \epsilon_i + \sum_i (t_{i,i} - \epsilon_i)\gamma_i + \frac{1}{2!}\sum_{ij} K_{ij,ij}\gamma_i\gamma_j + \frac{1}{3!}\sum_{ijk}\langle ijk|\,\Delta T\,|ijk\rangle_A \gamma_i\gamma_j\gamma_k . \tag{38}$$

This result agrees with that of Ref. 7 if we set the occupation factors equal to unity.

Finally, we return to the question of the coupling between the two- and three-particle wave functions exhibited in Eq. (21). We may ask under what circumstances can this coupling term be replaced or approximated by a potential in the particle states of the Bethe-Goldstone equation. To derive such an approximation we may assume that particles 1 and 2 are close together and 3 is far from either. More precisely, we assume that $f_{13} = f_{23} = 0$.[8] Then note that in that case $|\Psi_{ijk}\rangle \simeq (1 + f_{12})|ijk\rangle_A$. Keeping two terms of $|ijk\rangle_A$, we may write

$$|\Psi_{ijk}\rangle \simeq (1 + f_{12})|ij\rangle_A |k\rangle = |\Psi_{ij}\rangle|k\rangle . \tag{39}$$

Using these approximations, Eq. (21) becomes

$$\langle mn|\,(t_1 + W_1 + t_2 + W_2 + v_{12})|\Psi_{ij}\rangle - \langle mn|\Psi_{ij}\rangle(\epsilon_i + \epsilon_j) = 0 , \tag{40}$$

which is a Bethe-Goldstone equation with *particle* potentials given by

$$\langle m|W_1|n\rangle = \sum_k \langle mk|v_{13}^{l}|nk\rangle . \tag{41}$$

The superscript l reminds us that this approximation only holds for the "long-range" part of v_{13} (or v_{23}),

since Eqs. (38) and (39) were derived in the limit $f_{13}=f_{23}=0$. The result in which the particle potential is given by the antisymmetrized matrix element

$$\langle m|W_1|n\rangle=\sum_k\langle mk|v_{13}^t|nk\rangle_A\,, \tag{42}$$

may be obtained by replacing the coupling term in Eq. (21) by

$$\langle m(1)n(2)k(3)|\,[(v_{23}^t+v_{13}^t)(1+f_{12})|\,i(1)j(2)\rangle_A|\,k(3)\rangle-v_{13}^t(1+f_{23})|\,i(3)j(2)\rangle_A|\,k(1)\rangle-v_{23}^t(1+f_{13})|\,i(1)j(3)\rangle_A|\,k(2)\rangle]\,. \tag{43}$$

Further work is called for in order to determine the particle potential most suitable for uncoupling the two- and three-body equations. The "uncoupling" of the two- and three-body equations in nuclear-matter calculations has been discussed previously. However, the *explicit form* of the coupling term as given in Eq. (21) will allow one to check which particle-state potential best approximates the effect of such coupling.

APPENDIX A

In order to draw the diagrams describing the cluster expansion of the expectation value of the energy, up to three-body clusters, we need the elements shown in Fig. 1, in addition to the elements which have been considered in Ref. 2. With the help of these elements the expectation value of the Hamiltonian may be represented by the diagrams of Fig. 2(a) in an approximation which corresponds to the partial summation of a set of primary diagrams (containing no thick lines). The thick lines represent the occupation factors γ, and Fig. 2(b) provides a diagrammatic representation of Eq. (15), which relates the occupation factors to the two- and three-body-cluster integrals $\kappa_{ik,ik}$ and $\kappa_{ikl,ikl}$.

APPENDIX B

In this appendix we derive Eqs. (23) and (24) for completeness. If one neglects three-body clusters one has from Eq. (21) or (22)

$$\langle mn|\,(t_1+t_2+v_{12})|\,\Psi_{ij}\rangle=(\epsilon_i+\epsilon_j)\langle mn|\,\Psi_{ij}\rangle\,, \tag{B1}$$

or

$$Q(t_1+t_2+v_{12})|\,\Psi_{ij}\rangle=(\epsilon_i+\epsilon_j)Q|\,\Psi_{ij}\rangle\,. \tag{B2}$$

We may write Eq. (18) as

$$\begin{aligned}K_{ij,ij}&=\langle ij|\,[(1+f_{12}^\dagger)(t_1+t_2+v_{12})(1+f_{12})-(t_1+t_2)]|\,ij\rangle_A-(\epsilon_i+\epsilon_j)\kappa_{ij,ij}\\&=\langle ij|\,[(1+f_{12}^\dagger)Q(t_1+t_2+v_{12})(1+f_{12})-(t_1+t_2)]|\,ij\rangle_A\\&\quad+\langle ij|\,[(1+f_{12}^\dagger)(1-Q)(t_1+t_2+v_{12})(1+f_{12})-(t_1+t_2)]|\,ij\rangle_A-(\epsilon_i+\epsilon_j)\kappa_{ij,ij}\,.\end{aligned} \tag{B3}$$

Making use of Eqs. (9) and (B2) and the relations

$$\langle ij|\,(1+f_{12}^\dagger)(1-Q)=\langle ij|\,, \tag{B4}$$

$$\kappa_{ij,ij}=\langle\Psi_{ij}|\,\Psi_{ij}\rangle-\langle ij|\,ij\rangle_A=\langle\Psi_{ij}|\,Q|\,\Psi_{ij}\rangle\,, \tag{B5}$$

we find

$$\begin{aligned}K_{ij,ij}&=\langle ij|\,v_{12}(1+f_{12})|\,ij\rangle_A\\&=\langle ij|\,v_{12}|\,\Psi_{ij}\rangle=\langle ij|\,K_{12}|\,ij\rangle_A\,,\end{aligned} \tag{B6}$$

where

$$K_{12}=v_{12}(1+f_{12})\,, \tag{B7}$$

and [see Eqs. (9) and (22)]

$$f_{12}=-\frac{1}{t_1+t_2-(\epsilon_i+\epsilon_j)}\,QK_{12}=-\frac{Q}{e}\,K_{12}\,. \tag{B8}$$

APPENDIX C

From Eq. (30) onward we have tacitly assumed that we can use the Bethe-Faddeev solution[7] to define our three-body-cluster amplitude $|\Psi_{ijk}\rangle$. In this appendix we show that this is indeed the case.

We recall that we had

$$\begin{aligned}|\,\Psi_{ijk}\rangle&=(1+f_{12}+f_{13}+f_{23}+f_{123})|\,ijk\rangle_A\\&=\left(1-\frac{Q}{e}\,K_{12}-\frac{Q}{e}\,K_{13}-\frac{Q}{e}\,K_{23}+f_{123}\right)|\,ijk\rangle_A\end{aligned} \tag{C1}$$

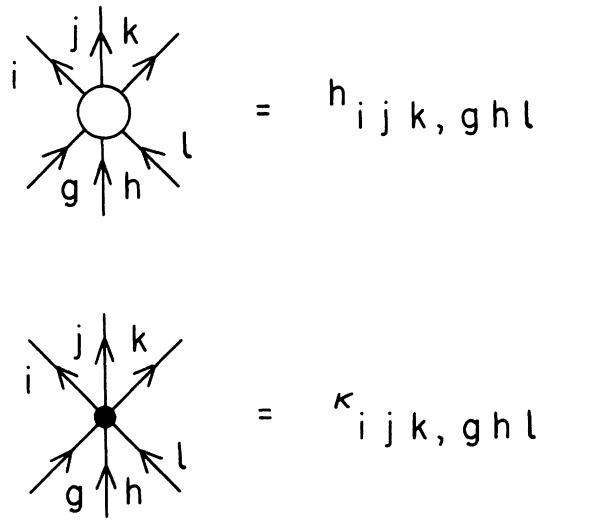

FIG. 1. Diagrammatic representations of three-body-cluster integrals defined in Eqs. (8) and (14). (Diagrammatic elements which have only one or two lines entering and leaving were defined previously in Ref. 2.)

using Eq. (24). As defined, f_{123} excited all three particles so that

$$\langle i'j'k'|\, f_{123}|\, ijk\rangle_A = \langle i'j'm|\, f_{123}|\, ijk\rangle_A = \langle i'mn|\, f_{123}|\, ijk\rangle_A = 0\,. \tag{C2}$$

Also we must have

$$\langle mnp|\,\Psi_{ijk}\rangle = \langle mnp|\, f_{123}|\, ijk\rangle_A = -\frac{1}{t_m + t_n + t_p - (\epsilon_i + \epsilon_j + \epsilon_k)}\langle mnp|\, V|\,\Psi_{ijk}\rangle\,, \tag{C3}$$

where $V = v_{12} + v_{13} + v_{23}$.

Now we study the Bethe-Faddeev solution to see that all these properties hold true. The wave matrix for the Bethe-Faddeev solution may be written

$$\Omega = 1 - \frac{Q}{e}T = 1 - \frac{Q}{e}[T^{(1)} + T^{(2)} + T^{(3)}] = 1 - \frac{Q}{e}(K_{12} + K_{23} + K_{13}) - \frac{Q}{e}(\Delta T)\,. \tag{C4}$$

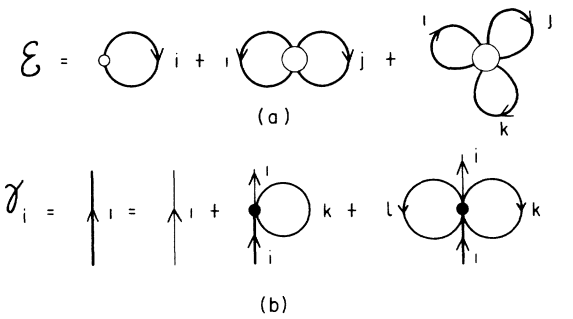

FIG. 2. (a) and (b) Diagrammatic representation of the cluster expansions for the energy, Eq. (5), and for the occupation factors, Eq. (15). (See Ref. 2 for a definition of the diagrammatic elements other than those defined in Fig. 1.)

Comparing Eqs. (C4) and (C1) we see that we should define

$$f_{123} = -\frac{Q}{e}(\Delta T)\,. \tag{C5}$$

With this definition we see that the requirements given by Eq. (C2) are satisfied.

Also note that for the Bethe-Faddeev solution,

$$\Omega = 1 + \sum_{i=1}^{3}[\Omega^{(i)} - 1]\,, \tag{C6}$$

with

$$\Omega^{(i)} \equiv 1 - \frac{Q}{e}T^{(i)}\,. \tag{C7}$$

Thus

$$\langle mnp|\,\Omega|\, ijk\rangle = \sum_{i=1}^{3}\langle mnp|\,\Omega^{(i)}|\, ijk\rangle_A = -\frac{1}{(t_m + t_n + t_p) - (\epsilon_i + \epsilon_j + \epsilon_k)} \times \langle mnp|\,\Delta T|\, ijk\rangle_A\,, \tag{C8}$$

so that we see that the requirement of Eq. (C3) is also met, since [recall Eqs. (30) and (33)]

$$\langle mnp|\,\Delta T|\, ijk\rangle_A = \langle mnp|\, V|\,\Psi_{ijk}\rangle\,. \tag{C9}$$

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1969), Vol. XI B.

[4]Of course, we can limit consideration only to $S = S^{(2)}$ if we go no further than the study of two-body clusters. The specification of $S^{(n)}$ is necessary if one wishes to study the role of the n-body cluster.

[5]We realize that we may use variational techniques only in a limited sense. We proceed by stopping the cluster expansion at some point and apply variational methods to the resulting expression for the energy. Ideally one might check that the neglected terms in the expansion are indeed small.

[6]Single-particle energies defined to include occupation factors were first introduced by B. H. Brandow, Phys. Rev. 152, 873 (1966); Rev. Mod. Phys. 39, 771 (1967).

[7]H. A. Bethe, Phys. Rev. 138, B804 (1965); R. Rajaraman and H. A. Bethe, Rev. Mod. Phys. 39, 745 (1967).

[8]This is a reasonable assumption for the simplest form of the Jastrow theory in which f_{12} is a short-range function of $r_{12} = |\vec{r}_1 - \vec{r}_2|$. For a more general theory we would have to work harder to understand the exact nature of the approximation obtained by setting $f_{13} = f_{23} = 0$.

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Method to Investigate the Off-Shell Effects of the High-Energy Part of the Two-Nucleon Interaction*

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Two classes of two-body Hamiltonians are constructed such that the energy eigenfunctions of the members of each class are identical below some cutoff energy E_c. In one class the phase shifts for energies greater than E_c remain fixed for all Hamiltonians in the class, although the corresponding eigenfunctions are different for small distances of separation. In the other class the phase shifts for the various members of the class also differ for energies greater than E_c.

It has become conventional in nuclear physics to invent a two-body potential V_{12} that reproduces two-nucleon scattering data reasonably well below a laboratory energy of ~300 MeV and to use this potential to calculate properties of systems that contain more than two nucleons. Clearly, such a potential is not unique. Several groups[1] have recently initiated systematic studies of the ambiguities that possible choices of V_{12} may introduce into the many-nucleon calculations. One approach has been to generate a family of equivalent potentials, each member of which produced eigenfunctions that differ for small values of the relative coordinate r_{12}, but are identical for large values of this coordinate. These potentials thus give rise to T matrices that are identical on the energy shell, but differ off shell. Such a family of equivalent potentials allows one to investigate the sensitivity of various many-body calculations to off-shell differences in the corresponding family of two-body T matrices.

Of equal importance is the question of the sensitivity of many-nucleon calculations to ambiguities in the high-energy behavior of the two-nucleon interaction. Without recourse to a fundamental theory, the high-energy part of the two-nucleon interaction cannot be fixed and even a careful analysis of future nucleon-nucleon scattering data will not resolve this difficulty. It is of interest, therefore, to be able to isolate the off-shell effects of the high-energy tail of any given two-body interaction. To this end we propose a method to generate a family of two-body Hamiltonians, each member of which has identical eigenfunctions for energies below some specified cutoff energy E_c. For energies greater than E_c the eigenfunctions differ. We consider two classes of such Hamiltonians: (i) the class for which the high-energy ($E > E_c$) eigenfunctions differ at small values of the separation distance r_{12}, but for which the high-energy phase shifts remain unaltered, and (ii) the class for which each Hamiltonian generates different high-energy phase shifts.

The Hamiltonian operators in class (i) are of the form[2]

$$\hat{H} = UHU^{\dagger} = H_0 + \hat{V}\,, \tag{1}$$

where H_0 is the kinetic energy operator for the rel-