On the Pion-Nucleus Interaction in the (3, 3)-Resonance Region*

Göran Fäldt†
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 11 May 1971)

Current theoretical models for the pion-nucleus interaction in the (3,3)-resonance region are discussed. We conclude that nonlocal potentials of the Kisslinger type are inadequate. The most promising potential is local and closely related to the Glauber model. Some properties of the Glauber model are also studied. Thus we show that the multiple-scattering series can be summed and charge-exchange effects can easily be accounted for.

I. INTRODUCTION

The elastic scattering of pions by carbon nuclei in the (3, 3)-resonance region has recently attracted much interest both experimentally[1] and theoretically.[2] The theoretical approach has been based on the multiple-scattering model of Glauber[3] and on the optical model of Kisslinger.[4] Confusion about the applicability of the two models has resulted, and they have been found to give different results. Here we want to point out that part of this confusion is due to an incorrect application of both the Glauber model[2(b),2(c)] and of the Kisslinger model.[2(d)]

In the Kisslinger model, one solves a Klein-Gordon equation for the pion. The potential is derived from the free pion-nucleon amplitude. It contains a term proportional to the nuclear density and a nonlocal velocity-dependent term. The latter is due to the p-wave term in the pion-nucleon interaction. This model was originally intended to describe pion-nucleus scattering at somewhat lower energies. Extending the model to scattering in the (3, 3)-resonance region creates new problems. In fact, we point out that the standard form of the potential has a serious deficiency. It cannot simultaneously reproduce the forward pion-nucleon amplitude and the angular dependence near the forward direction. In the (3, 3)-resonance region the pion-nucleus cross sections depend critically on these two properties.

In the Glauber model the scattering amplitude is a sum of multiple-scattering terms up to the 12th order. This explicit binomial form has been used in most of the theoretical calculations. Needless to say, such an approach is not very transparent but was used in order to account for the large charge-exchange cross section. We show that despite this complication the multiple-scattering series can be summed into a nice and compact formula which takes charge-exchange contributions into account.

We discuss three different potential models. Two of them are nonlocal and of the Kisslinger type, while the third one is local. We conclude that the Kisslinger potentials are inadequate in the (3, 3)-resonance region. The local potential, which seems more promising, is just the potential one would naïvely associate with the Glauber model. We also find that the cross sections obtained with the Glauber model approximate rather closely those obtained with the potential model, and this up to surprisingly large angles. A local potential has independently been suggested by Lee and McManus[2(f)] and by Wilkin.[2(k)]

We are mainly interested in the theoretical aspects of the Glauber and Kisslinger models. We do not compare our results with experiments, because our wave function for the carbon nucleus is unrealistic. Moreover, it is well known that Fermi-motion effects are important in the (3, 3)-resonance region,[5,6] at least in the forward direction. On the other hand recoil corrections and corrections due to the nonvanishing longitudinal momentum transfer[6,7] can be neglected.

II. GLAUBER MULTIPLE-SCATTERING SERIES

We shall use the independent-particle model for C^{12} and the wave function derived from the harmonic-well potential. Explicit expressions for densities, form factors, and other quantities of interest are given in Appendix A.

In the multiple-scattering theory of Glauber[3] the amplitude for pion-nucleus scattering is expressed in terms of pion-nucleon amplitudes. For the latter he introduces a profile function $\Gamma(\vec{b})$ defined by

$$f(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\,\Gamma(\vec{b}), \tag{2.1}$$

with all quantities being taken in the lab system. This relation is easily inverted to give the profile function $\Gamma(\vec{b})$ in terms of the scattering amplitude $f(\vec{q})$. The pion-nucleus elastic scattering ampli-

tude is then constructed in the following way:

$$F(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle\Psi|\Gamma(\vec{b}, \vec{s}_1, \ldots, \vec{s}_A)|\Psi\rangle, \tag{2.2}$$

where Ψ is the nuclear wave function

$$\Psi(\vec{r}_1, \ldots, \vec{r}_A) = \prod_{k=1}^{A} \varphi(\vec{r}_k), \tag{2.3}$$

and $\vec{s}_k$ is the component of $\vec{r}_k$ orthogonal to the direction of motion of the projectile meson. The nuclear profile function is defined by

$$\Gamma(\vec{b}, \vec{s}_1, \ldots, \vec{s}_A) = 1 - \prod_{k=1}^{A} [1 - \Gamma_k(\vec{b} - \vec{s}_k)]. \tag{2.4}$$

This definition is valid only when the operators $\Gamma_k(\vec{b} - \vec{s}_k)$ commute. When this is not the case it is necessary to account for the temporal order of the individual collisions. This is achieved by replacing the right-hand side of Eq. (2.4) by its space-ordered product. For two profile functions this product is defined as

$$s[\Gamma_1(\vec{b} - \vec{s}_1)\Gamma_2(\vec{b} - \vec{s}_2)] = \begin{cases} \Gamma_1(\vec{b} - \vec{s}_1)\Gamma_2(\vec{b} - \vec{s}_2), & \text{when } z_1 > z_2 \\ \Gamma_2(\vec{b} - \vec{s}_2)\Gamma_1(\vec{b} - \vec{s}_1), & \text{when } z_2 > z_1 \end{cases}, \tag{2.5}$$

z_j being the component of $\vec{r}_j$ parallel to the direction of motion of the projectile meson. The generalization to an arbitrary number of profile functions is obvious.

The approach above does not account for the c.m. correlation within the nucleus. For harmonic-well wave functions this effect is easily incorporated.[8] It amounts to multiplying the scattering amplitude of Eq. (2.2) by a factor

$$R(\vec{q}) = e^{a^2\vec{q}^2/4A}, \tag{2.6}$$

with a as defined in Appendix A. The numerical effect of the c.m. correlation is shown in Fig. 1. It is quite small, given the experimental uncertainties, and in the following we shall neglect it.

III. CONVERGENCE OF MULTIPLE-SCATTERING SERIES

In this section we shall assume $f_{\pi p}(\vec{q}) = f_{\pi n}(\vec{q}) = f(\vec{q})$ for the pion-nucleon amplitudes. This means that there is no charge-exchange scattering and that the Γ_k's of Eq. (2.4) will commute. This restriction will be relaxed in the following section.

We introduce the nucleon density $\rho(\vec{x})$ normalized as

$$\int d^3x\, \rho(\vec{x}) = A, \tag{3.1}$$

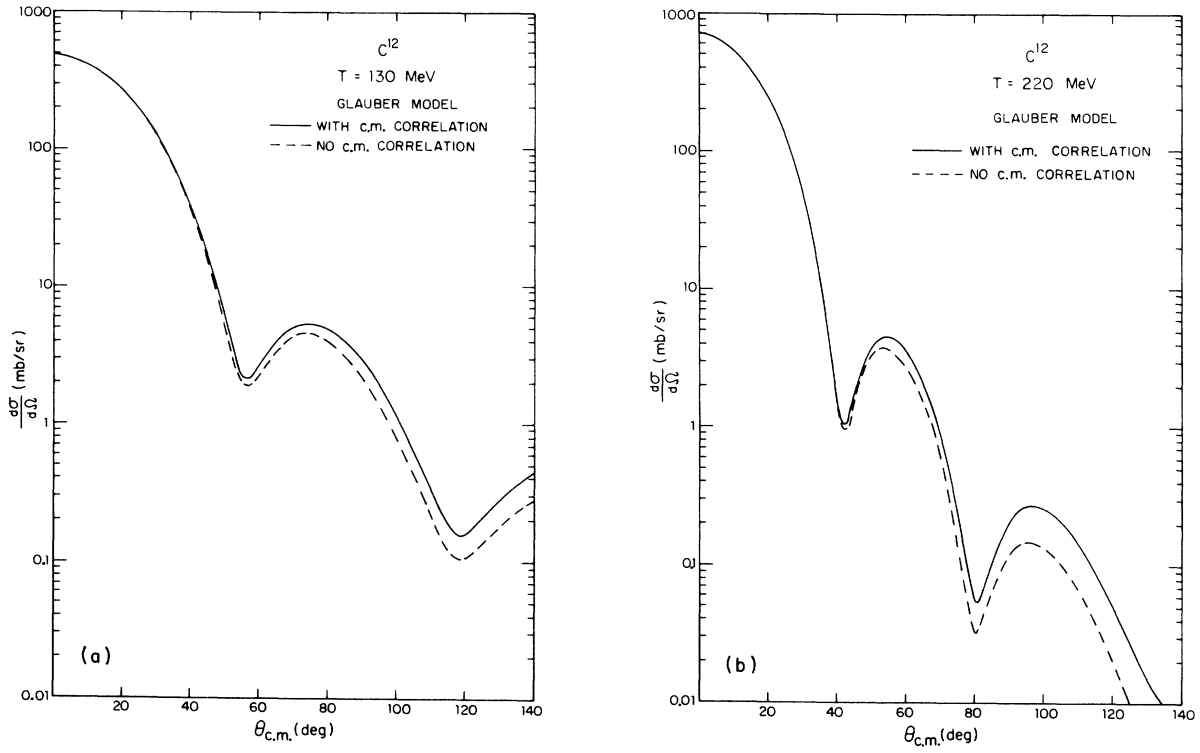

FIG. 1. (a), (b) Elastic differential cross section. Predictions of Glauber model (dashed line) and Glauber model corrected for the c.m. correlation (solid line).

and define

$$\chi(\vec{b}) = i\int d^3x\,\rho(\vec{x})\Gamma(\vec{b}-\vec{s})$$
$$= \frac{1}{2\pi k}\int d^2q\, e^{-i\vec{q}\cdot\vec{b}} f(\vec{q})S(\vec{q}). \quad (3.2)$$

where $S(\vec{q})$ is the nuclear form factor and $\vec{s}$ the component of $\vec{x}$ in the impact-parameter plane. The elastic scattering amplitude, Eq. (2.2), can then be written as

$$F(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left[1-\left(1+\frac{i}{A}\chi(\vec{b})\right)^A\right]. \quad (3.3)$$

We shall call this expression the binomial form of the multiple-scattering series. It is complicated to handle in the general case and we want to simplify it. To this end we study

$$X = (1+i\chi/A)^A$$
$$= \sum_{k=0}^{A}\frac{1}{k!}(i\chi)^k a_k. \quad (3.4)$$

The "binomial" coefficients a_k are given by

$$a_k = \frac{1}{A^k}A(A-1)\cdots(A-k+1)$$
$$\cong 1-\frac{1}{2A}k(k-1)+\frac{1}{24A^2}k(k-1)(k-2)(3k-1), \quad (3.5)$$

where the last line gives the coefficient as an expansion in $1/A$.

For heavy nuclei, i.e., large A, expression (3.4) can be replaced by its "optical limit,"

$$X_{\text{opt}} = e^{i\chi(\vec{b})}. \quad (3.6)$$

In a nucleus as light as C^{12} it is not immediately clear that the optical limit is a good approximation. It approximates the binomial coefficients a_k of Eq. (3.5) by $a_k(\infty)=1$. Thus, taken term by term, the optical limit misinterprets the higher-order multiple-scattering terms quite strongly. Close to the forward direction this is not too serious, because the multiple-scattering series converges quite rapidly, but when we move away from the forward direction higher-order multiple-scattering terms become more and more important and one would expect the optical approximation to break down. But this argument ignores the delicate cancellation between different terms in the multiple-scattering series. In fact, as demonstrated in Fig. 2 the optical limit is a very accurate approximation of the binomial form. For comparison with experiment the small difference found is unimportant.

It is quite simple to improve upon the optical approximation. By retaining the $1/A$ term when exponentiating (3.4) we get

$$X \cong \exp\left[i\chi(\vec{b})+\frac{1}{2A}\chi^2(\vec{b})\right]. \quad (3.7)$$

This approximation correctly reproduces also the $1/A$ term in the expansion (3.5). When the differential cross section is calculated in approximation (3.7) the result turns out to be practically indistinguishable from the exact results.

In order to avoid confusion we remark that the expansion in Eq. (3.7) is only formally an expansion in $1/A$, because the phase-shift function χ is expected to increase as $A^{1/3}$. This is due to the fact that χ is roughly proportional to the path length within the nucleus. As a result the actual A dependence becomes $1/A^{2/3}$ rather than $1/A$.

IV. INCLUSION OF CHARGE EXCHANGE

In the (3, 3)-resonance region the charge-exchange cross section is quite large and our formalism must be extended to account for charge-exchange contributions to the pion-nucleus amplitude. We shall then assume that protons and neutrons in C^{12} are bound into pairs of isospin zero.

We decompose $\chi(\vec{b})$ of Eq. (3.2) as

$$\chi(\vec{b}) = \chi_0(\vec{b}) + \vec{t}\cdot\vec{\tau}\,\chi_1(\vec{b}), \quad (4.1)$$

$\vec{t}$ being the isospin operator of the pion and $\vec{\tau}$ the Pauli isospin operator of the nucleon. Thus

$$\chi_p(\vec{b}) = \chi_0(\vec{b}) + t_3\chi_1(\vec{b}), \quad (4.2a)$$
$$\chi_n(\vec{b}) = \chi_0(\vec{b}) - t_3\chi_1(\vec{b}). \quad (4.2b)$$

For C^{12} with an equal number of protons and neutrons the approximation (3.7) reads

$$X = \left(1+\frac{i\chi_p}{A}\right)^{A/2}\left(1+\frac{i\chi_n}{A}\right)^{A/2}$$
$$\cong \exp\left[i\chi_0+\frac{1}{2A}(\chi_0^2+t_3^2\chi_1^2)\right]. \quad (4.3)$$

This result is not isospin-invariant, but not much imagination is required to realize that the proper generalization must be obtained through the replacement $t_3^2 \to \vec{t}^2$. In order to prove it we shall have to analyze the meaning of approximations (3.6) and (3.7).

We first consider the optical limit. According to Eq. (4.3) the result depends only on $\chi_0 = \frac{1}{2}(\chi_p+\chi_n)$, and therefore certain multiple-scattering terms must have been left out. In fact, we shall show that the optical approximation is equivalent to counting only those multiple-scattering terms for which scattering takes place off at most one nucleon for each proton-neutron pair. To this end

we decompose the profile functions as in Eq. (4.1):

$$\Gamma(\vec{b}) = \Gamma_0(\vec{b}) + \vec{t}\cdot\vec{\tau}\Gamma_1(\vec{b}), \tag{4.4}$$

and consider pair No. 1, p_1 and n_1. As the profile functions are noncommutative, we must be careful with the ordering. A typical scattering proceeding through p_1 is

$$\Gamma(1)\cdots\Gamma(k-1)\Gamma(p_1)\Gamma(k+1)\cdots\Gamma(l), \tag{4.5}$$

with all Γ's belonging to different pairs. Another possibility is to scatter off n_1 instead:

$$\Gamma(1)\cdots\Gamma(k-1)\Gamma(n_1)\Gamma(k+1)\cdots\Gamma(l), \tag{4.6}$$

with all other factors remaining the same. As all nucleons have the same density (4.5) and (4.6) will be additive. We get

$$\Gamma(1)\cdots\Gamma(k-1)[\Gamma(p_1)+\Gamma(n_1)]\Gamma(k+1)\cdots\Gamma(l). \tag{4.7}$$

But $\Gamma(p_1)+\Gamma(n_1)=2\Gamma_0$ and is isospin-independent. Repeating this argument for all pairs we conclude that each nucleon contributes effectively only with the isospin-independent part Γ_0. Hence the space-ordering prescription can be ignored. The total contribution from lth-order multiple scattering becomes

$$\binom{\frac{1}{2}A}{l}\left(\frac{i\chi_0(\vec{b})}{A}\right)^l = \frac{1}{l!}(i\chi_0)^l\left[1-\frac{1}{A}l(l-1)+\cdots\right]. \tag{4.8}$$

The first term in this expansion agrees with the first term in the expansion (3.5). In the optical limit they are therefore equivalent. The $1/A$ terms do not agree. To achieve agreement we must include terms for which the pion scatters off both nucleons in exactly one pair.

Assume the pion scatters off both nucleons in pair No. 1, p_1 and n_1, but only off at most one nucleon in all other pairs. The contributions are then of the form (4.5) but contain both $\Gamma(p_1)$ and $\Gamma(n_1)$. For nucleons not in pair No. 1 we conclude as above that they will effectively contribute with the isospin-independent part Γ_0 only. By summing over all permutations between these nucleons we arrive at an expression of the form

$$\prod_k \Gamma_0(j_k)[\Gamma(p_1)\Gamma(n_1)\theta(z)+\Gamma(n_1)\Gamma(p_1)\theta(-z)], \tag{4.9}$$

with $z = z_{p_1} - z_{n_1}$. But as p_1 and n_1 form an isospin

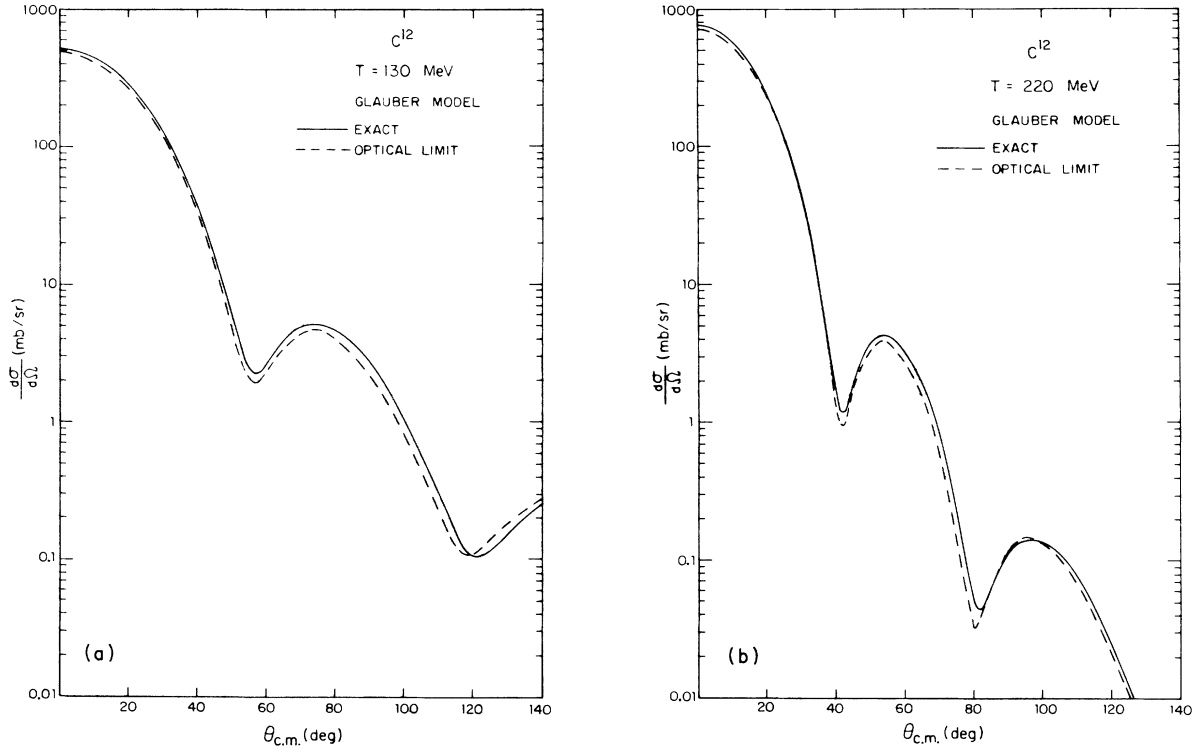

FIG. 2. (a), (b) Elastic differential cross section. Predictions of Glauber's multiple-scattering series (solid line) and its optical limit (dashed line).

$T=0$ pair we have

$$\Gamma(p_1)\Gamma(n_1)\theta(z)+\Gamma(n_1)\Gamma(p_1)\theta(-z)=\Gamma_0(p_1)\Gamma_0(n_1)-\vec{t}^2\Gamma_1(p_1)\Gamma_1(n_1)+\tfrac{1}{2}\epsilon(z)\Gamma_1(p_1)\Gamma_1(n_1)i(\vec{\tau}_p\times\vec{\tau}_n)\cdot\vec{t}. \tag{4.10}$$

Here the last term does not contribute to elastic scattering as it will induce transition to the $T=1$ state only. The total contribution to lth-order scattering proceeding through any one of the $\frac{1}{2}A$ pairs is then

$$\frac{A}{2}\,\frac{1}{A^2}(-\chi_0{}^2+\vec{t}^2\chi_1{}^2)\frac{1}{(l-2)!}(i\chi_0)^{l-2}\left(\frac{A}{2}-1\right)\cdots\left(\frac{A}{2}-1-l+3\right)$$
$$=\frac{1}{l!}(-)^l(i\chi_0)^{l-2}(-\chi_0{}^2+\vec{t}^2\chi_1{}^2)\frac{l(l-1)}{2A}\left[1-\frac{1}{A}(l-1)(l-2)+\cdots\right]. \tag{4.11}$$

This contribution is of the same magnitude as the $1/A$ term in Eq. (4.8). Combining Eqs. (4.8) and (4.11) we conclude that taken together they correctly reproduce the first two terms in the expansion (3.5) of the binomial coefficients. To this accuracy the result is equivalent to

$$X\cong\exp\left[i\chi_0+\frac{1}{2A}(\chi_0{}^2+\vec{t}^2\chi_1{}^2)\right]. \tag{4.12}$$

Being isospin-invariant, this is the desired generalization of Eq. (4.3).

We conclude that charge-exchange scattering within the nucleus, being a $1/A$ correction to the optical approximation, plays only a minor role in elastic pion-nucleus scattering.

We note that the $1/A$ term in (4.12) is proportional to the square of the nuclear density. Therefore when this correction is taken into account we should distinguish between s- and p-state densities. We also remark that (4.12) is easily extended to account for nuclear correlations.

V. RESULT OF GLAUBER'S MODEL

In previous sections it was shown that pion-carbon scattering can be accurately described by the optical-limit approximation and that charge-exchange scattering and other $1/A$ effects can be ignored. As a result the scattering amplitude for elastic scattering becomes

$$F(\vec{q})=\frac{ik}{2\pi}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}[1-e^{i\chi(\vec{b})}], \tag{5.1}$$

with

$$\chi(\vec{b})=\frac{1}{2\pi k}\int d^2q\,e^{-i\vec{q}\cdot\vec{b}}f(\vec{q})S(\vec{q}), \tag{5.2}$$

where $S(\vec{q})$ is the nuclear form factor and

$$f(\vec{q})=\tfrac{1}{2}[f_{\pi p}(\vec{q})+f_{\pi n}(\vec{q})]=\tfrac{1}{3}[2f_{3/2}(\vec{q})+f_{1/2}(\vec{q})]. \tag{5.3}$$

The pion-nucleon amplitudes are to be evaluated in the lab system and according to Glauber's prescription at $t=-\vec{q}^2$. Equations (5.1)–(5.3) are valid for any form of $f(\vec{q})$, but in the (3, 3)-resonance region it is sufficient to keep the s- and p-wave contributions. When this is done

$$f(t)=u_0+v_0t, \tag{5.4}$$

with u_0 and v_0 as defined in Appendix B. Going back to Eq. (5.2) we get

$$\chi(\vec{b})=\frac{2\pi}{k}[u_0T(\vec{b})+v_0\Delta_2T(\vec{b})]. \tag{5.5}$$

Here $T(\vec{b})=\int_{-\infty}^{\infty}dz\,\rho(\vec{b},z)$ is the transmission factor and Δ_2 is the two-dimensional Laplace operator. For analytic expressions for $T(\vec{b})$ and $\Delta T(\vec{b})$ we refer to Appendix A.

The term proportional to v_0 in Eq. (5.5) is entirely due to the angular dependence of the pion-nucleon p wave. The cross section is quite sensitive to this term. This is illustrated in Fig. 3, where we compare the result of a calculation using Eq. (5.5) with one in which the v_0 term is neglected. It is clearly seen that the latter approximation is unsatisfactory. It shifts the second minimum by about 20° and reduces the forward amplitude by a substantial amount. The reason for the failure of this approximation, so successful at high energies, is intimately connected with the relative sizes of the nucleus and the pion-nucleon interaction. At high energies the nuclear radius is much larger than the range of the pion-nucleon forces, and it follows that the nuclear form factor $S(\vec{q})$ is much steeper than the pion-nucleon amplitude $f(\vec{q})$. In the (3, 3)-resonance region, however, the pion-nucleon cross section is several times larger than at high energies, and the range of the pion-nucleon forces becomes comparable to the nuclear radius. Thus, for this case the slope of $f(\vec{q})$ is comparable to that of $S(\vec{q})$.

In many of the applications of the Glauber model the important v_0 term was neglected. This was done in analogy with high-energy scattering. In the (3, 3)-resonance region a proper application of Glauber's multiple-scattering theory must use the complete expression (5.5).

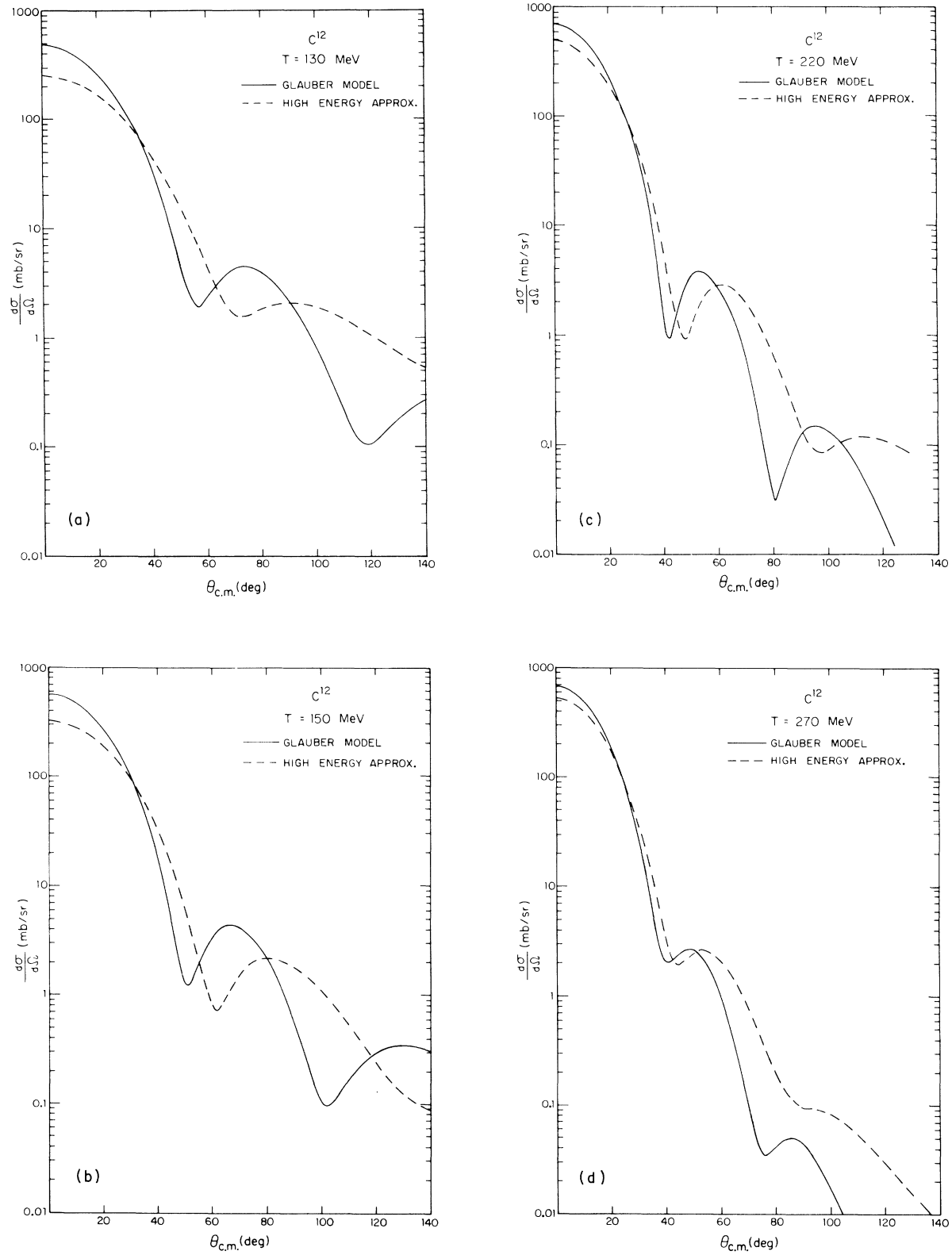

FIG. 3. (a)–(d) Elastic differential cross section. Predictions of Glauber's model (solid line) and the high-energy approximation of the same model (dashed line). Both calculations in the optical-limit approximation.

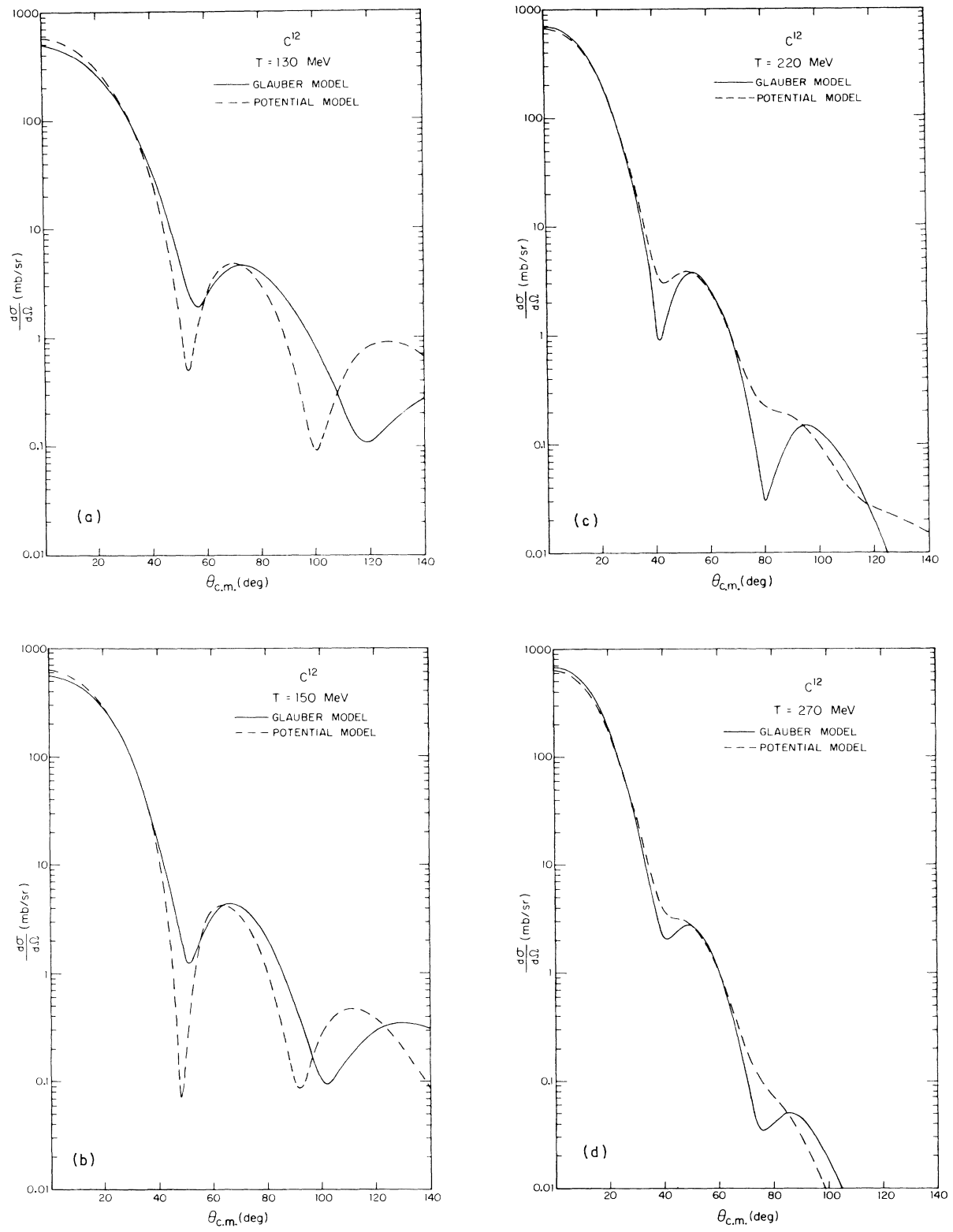

FIG. 4. (a)–(d) Elastic differential cross section. Predictions of Glauber's model (solid line) and the corresponding local optical potential (dashed line).

VI. DERIVATION OF AN OPTICAL POTENTIAL

In high-energy potential scattering the phase function $\chi(\vec{b})$ of Eq. (5.1) is related to the potential through

$$\chi(\vec{b}) = -\frac{1}{v}\int_{-\infty}^{\infty} dz\, V(\vec{b}^2 + z^2)^{1/2}), \tag{6.1}$$

for spherically symmetric potentials. When instead the phase function is assumed to be known, this equation defines the potential through an Abel integral equation. In the previous section we found

$$\chi(\vec{b}) = \frac{2\pi}{k}\left[u_0\int_{-\infty}^{\infty} dz\,\rho(\vec{b}, z) + v_0\int_{-\infty}^{\infty} dz\,\Delta_b\rho(\vec{b}, z)\right]. \tag{6.2}$$

In the last integrand we can add a term $\partial^2(\rho(\vec{b}, z))/\partial z^2$ because it integrates to zero. It follows that the multiple-scattering theory of Glauber is equivalent to an optical potential

$$V_G(\vec{r}) = -\frac{2\pi}{\omega_l}[u_0\rho(\vec{r}) + v_0\Delta_3\rho(\vec{r})]. \tag{6.3}$$

This method of arriving at an optical potential is very attractive because it can easily be extended to include $1/A$ effects, correlations, and absorption on nucleon pairs. A more conventional approach is given in Sec. VII.

In high-energy scattering one expects the potential V_G to give a result similar to the Glauber model. We have investigated this correspondance in the (3, 3)-resonance region. The pion is then described by a Klein-Gordon equation,

$$(\Delta + \omega_l^2 - \mu^2)\psi = 2\omega_l V_G\psi, \tag{6.4}$$

with V_G as in Eq. (6.3). This equation was solved numerically by a computer program developed by M. Krell.[9] In Fig. 4 we compare the result with that of the Glauber model. The general agreement between the two methods and especially in the forward direction is quite surprising. We must remember that the Glauber approximation is intended for quite a different situation. It assumes high-energy and small-angle scattering and also that the pion-nucleon scattering itself is of a diffractive nature. Despite the fact that these conditions are not at all fulfilled in the (3, 3)-resonance region, the general agreement extends to very large angles. Of course there are differences and they are large enough to be significant also with present experimental accuracy. On the low-energy slope of the (3, 3) resonance the Glauber approximation smears out the first minimum and displaces the second one quite considerably. On the high-energy slope the Glauber approximation gives deeper minima but otherwise follows the potential model quite closely. We also conclude that the agreement between the potential model and the Glauber model improves with increasing energy. This is of course not a very astonishing result. In general the potential model and the Glauber model differ appreciably for very large scattering angles and experiments there favor the potential model.

VII. COMPARISON WITH OTHER OPTICAL MODELS

The potential V_G discussed in the previous section is local in contrast to the Kisslinger potential[4] which is given by

$$V_K(\vec{r}) = -\frac{2\pi}{\omega_l}[a_0\rho(\vec{r}) + b_0\vec{\nabla}\cdot(\rho(\vec{r})\vec{\nabla})], \tag{7.1}$$

and

$$a_0 = u_0 - 2k_c^2 v_0, \tag{7.2a}$$

$$b_0 = -2v_0k_c^2/k_l^2, \tag{7.2b}$$

where k_l is the appropriate pion-nucleon lab momentum and k_c the corresponding pion-nucleon c.m. momentum. The solution to the Klein-Gordon equation, Eq. (6.4), with the Kisslinger potential is displayed in Fig. 5. The cross sections obtained differ very much from those obtained with the local potential V_G or the Glauber model. Krell and Barmo[2(a)] and also Sternheim and Auerback[2(f)] used b_0 as a free parameter and looked for the best fit to the experimental data. This was obtained for an imaginary part of b_0 about 50% higher than the predicted one. This is quite a significant difference and indicates that the Kisslinger potential is in fact not a suitable starting point for a description of pion-nucleus scattering in the (3, 3)-resonance region. Theoretically, this inability can also be understood as follows.

The pion-nucleon lab amplitude which determines the optical potential is given by Eq. (B2):

$$f_l = u_0 + v_0 t. \tag{7.3}$$

For free pion-nucleon scattering $-t = (\vec{k}_l - \vec{k}_l')^2 - (\omega_l - \omega_l')^2$. For scattering by a nucleon bound in a nucleus the collision will take place off the energy shell. Because of the large nuclear mass the recoil energy ω_l' will differ only very slightly from ω_l and we can therefore put $-t \simeq (\vec{k}_l - \vec{k}_l')^2$. If we assume that the correct extrapolation of the pion-nucleon amplitude is obtained by choosing the correct t, we get

$$f_l = u_0 - v_0(\vec{k}_l - \vec{k}_l')^2. \tag{7.4}$$

From this amplitude all phenomenological potentials can be derived.

We first discuss Kisslinger-type potentials. As

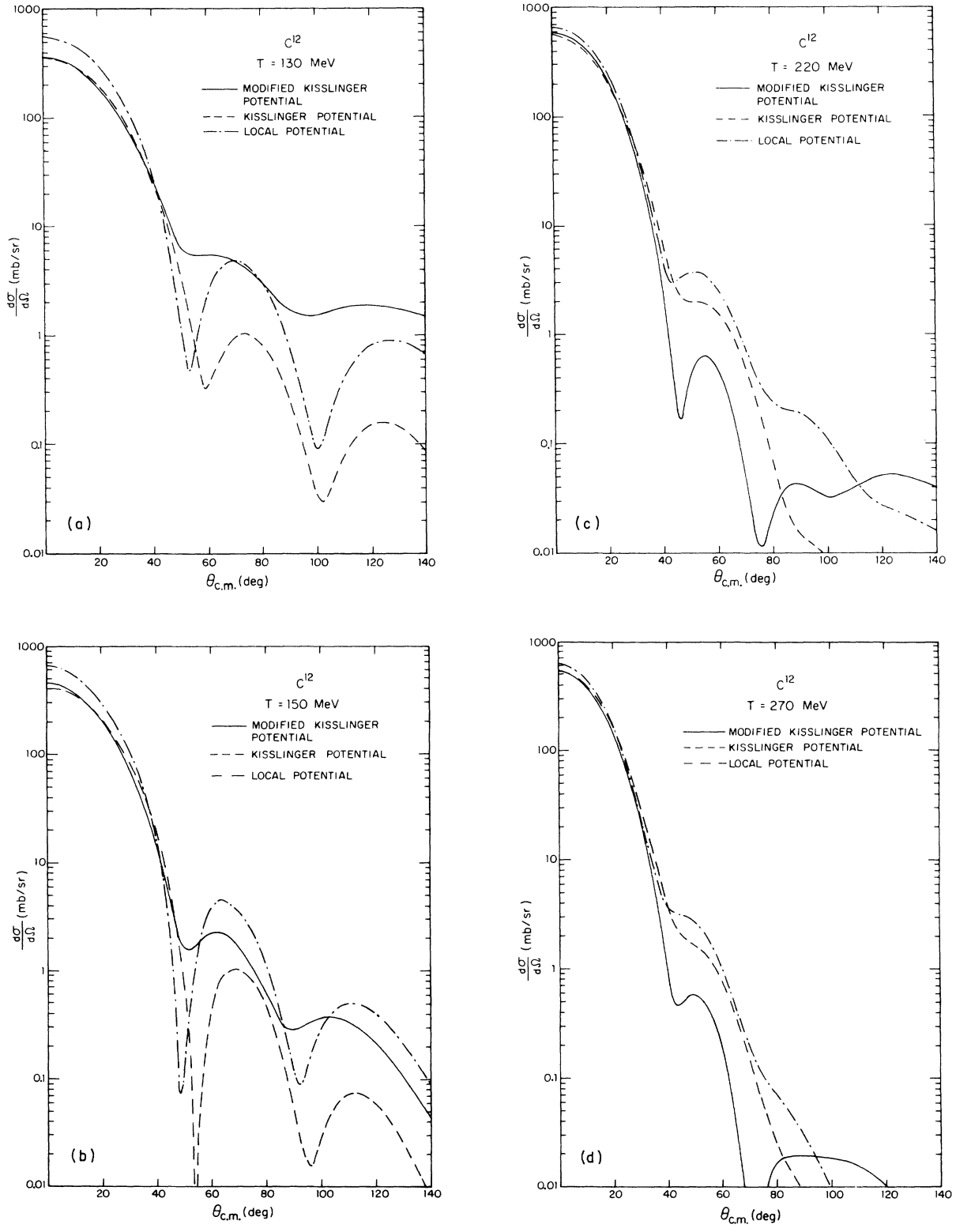

FIG. 5. (a)–(d) Elastic differential cross section. Predictions of the different optical models: modified Kisslinger potential (solid line), Kisslinger potential (dashed line), and local potential (dash-dotted line).

the energy loss has been neglected, we can rewrite Eq. (7.4) as

$$f_N = (u_0 - 2k_l^2 v_0) + 2v_0 \vec{k}_l \cdot \vec{k}_l' . \tag{7.5}$$

Treating all quantities in this f_l except $\vec{k}_l \cdot \vec{k}_l'$ as phenomenological constants we arrive at a Kisslinger potential (7.1) with

$$a_0 = u_0 - 2k_l^2 v_0 , \tag{7.6a}$$

$$b_0 = -2v_0 . \tag{7.6b}$$

Comparing with the standard Kisslinger potential we see that b_0 is enhanced by a factor $(k_l/k_c)^2 = 1 + 2\omega_l/m + \mu^2/m^2$, where m is the nucleon mass. At threshold this factor is only $(1+\mu/m)^2$, but in the (3, 3)-resonance region it is considerable and explains why the best fit of Sternheim and Auerbach requires a significantly larger b_0 than the Kisslinger prescription. However, in order to also reproduce $f_l(0)$ correctly this larger b_0 requires a smaller a_0. In fact in the (3, 3)-resonance region a_0 gets a large contribution from the pion-nucleon p wave. As a result the imaginary part of a_0 becomes negative. When the corresponding potential is used in the first Born approximation, physically inadmissable results are obtained. Such a potential yields production of pions instead of absorption. However, when the corresponding Klein-Gordon equation is solved, no production is observed but the cross sections have a somewhat strange behavior as shown in Fig. 5. However, when the energy is lowered, the p-wave contribution to a_0 will diminish and we finally reach an energy where the imaginary part of a_0 is positive. This is particularly true at threshold, where the phenomenological potential obtained from (7.6a) and (7.6b) is more appropriate than the standard form (7.1). On the other hand we have not included the true absorption by nucleon pairs which will cut down the unphysical production and hence reduce its associated problems.

The Kisslinger potential on the other hand is based on a pion-nucleon lab amplitude of the form

$$f_K = (u_0 - 2k_c^2 v_0) + 2(k_c/k_l)^2 v_0 \vec{k}_l \cdot \vec{k}_l' . \tag{7.7}$$

When deriving the Kisslinger potential (7.1) from Eq. (7.7) all quantities except $\vec{k}_l \cdot \vec{k}_l'$ are treated as phenomenological constants. In the forward direction the amplitudes f_K and f_N are the same. The a_0 of the Kisslinger potential, Eq. (7.2a), has no admixture of the pion-nucleon p wave and therefore its imaginary part is always positive. The price one has to pay for this convenience is an incorrect description of the angular dependence, the factor $(k_c/k_l)^2$ discussed above.

The upshot of this discussion is that a Kisslinger-type potential is totally inadequate for a quantitative theoretical description of pion-nucleus scattering in the (3, 3)-resonance region. The phenomenological potential based on approximation N which correctly approximates the basic pion-nucleon interaction develops unphysical properties in the (3, 3)-resonance region. The phenomenological potential based on approximation K avoids this difficulty but on the other hand strongly misrepresents the basic pion-nucleon interaction.

The only surviving potential is therefore our local potential V_G of Eq. (6.3). This potential can be derived in a manner similar to the Kisslinger-type potentials discussed above. We then start from the original Eq. (7.4)

$$f_l = u_0 - v_0(\vec{k}_l - \vec{k}_l')^2 ,$$

and treat u_0 and v_0 as phenomenological constants. The operator $\vec{q} = \vec{k}_l - \vec{k}_l'$ acts on the nuclear density function $\rho(\vec{r})$ and yields the term $\Delta\rho(\vec{r})$ in Eq. (6.3). It is therefore not at all necessary to go through the Glauber model and relation (6.1) to motivate our local potential, but one can directly use the original method of Kisslinger.

VIII. GENERALIZATIONS

In this section we generalize some of our results to nuclei with $T \neq 0$. This will include charge-exchange and double-charge-exchange reactions. We limit the discussion to transitions within the same nuclear isospin multiplet. Such transitions will be called coherent.

Consider a nucleus with $T = \frac{1}{2}$ and assume protons and neutrons bound into P pairs of isospin zero except for the valence nucleon. It is not difficult to show that for coherent scattering the nuclear profile function factorizes; i.e.,

$$\langle f|\prod(1-\Gamma_k)|i\rangle = \langle f|(1-\Gamma_v)|i\rangle\langle 0|\prod(1-\Gamma_k)|0\rangle . \tag{8.1}$$

The scattering by the core is treated as in Sec. IV. We get the results

$$F_{\mathrm{el}}(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} \times\left\{1 - \exp\left[i\frac{1}{A}(Z\chi_p + N\chi_n) + \frac{P}{A^2}(\chi_0^2 + \vec{t}^2\chi_1^2)\right]\right\} . \tag{8.2}$$

and

$$F_{\mathrm{ce}}(\vec{q}) = i\frac{\sqrt{2}}{A}\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\chi_1(\vec{b}) \times \exp\left[i\frac{2P}{A}\chi_0 + \frac{P}{A^2}(\chi_0^2 + \vec{t}^2\chi_1^2)\right] . \tag{8.3}$$

The extension to isospin $T = 1$ nuclei is straightforward.

Then consider a heavy nucleus. We assume that all M valence nucleons are neutrons and that the core consists of P pairs of isospin zero. We also assume M and P are so large that $1/A$ effects can be neglected. For elastic scattering this gives the well-known result

$$F_{\mathrm{el}}(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left\{1-\exp\left[i\frac{1}{A}(Z\chi_p+N\chi_n)\right]\right\}. \tag{8.4}$$

In a coherent charge-exchange reaction the initial π^+ turns into a π^0. Nevertheless, the methods of Sec. IV show that the scattering by the core remains the same. For the valence nucleons, however, we must distinguish between scattering before and after the charge exchange. Introducing

$$\chi_i(\vec{b}, z) = i\int d^2s\,\rho(\vec{s}, z)\Gamma_i(\vec{b}-\vec{s}), \qquad i=0, 1 \tag{8.5}$$

we get

$$\langle\psi_f|\prod(1-\Gamma_k)|\psi_i\rangle = 2i\sqrt{M}\exp\left[i\frac{2P}{A}\chi_0(\vec{b})\right]\int_{-\infty}^{\infty}dz\frac{1}{A}\chi_1(\vec{b}, z)\exp\left[i\frac{M-1}{A}\int_{-\infty}^{z}\chi_+(\vec{b}, z')dz' + i\frac{M-1}{A}\int_{z}^{\infty}\chi_0(\vec{b}, z')dz'\right]. \tag{8.6}$$

Here the subscripts + and 0 stand for π^+n and π^0n scattering, respectively. The integration of (8.6) is trivial and we get

$$F_{\mathrm{ce}}(\vec{q}) = i\frac{2\sqrt{M}}{M-1}\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\sin\left(\frac{M-1}{A}\frac{\chi_1}{2}\right)\exp\left\{i\frac{1}{A}\left[2P\chi_0+(M-1)\frac{\chi_+ + \chi_0}{2}\right]\right\}. \tag{8.7}$$

Finally, we consider coherent double charge exchange where the projectile π^+ leaves the nucleus as a π^-. Equation (8.6) is then generalized in an obvious way and we get

$$F_{\mathrm{dce}}(\vec{q}) = \frac{-8M}{(M-2)^2}\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left[\sin\left(\frac{M-2}{A}\frac{\chi_1}{2}\right)\right]^2\exp\left[i\frac{1}{A}\left(2P\chi_0+(M-2)\frac{\chi_+ + \chi_0 + \chi_-}{3}\right)\right]. \tag{8.8}$$

Cross sections for other charge combinations are easily derived using the Wigner-Eckart theorem.

IX. CONCLUSIONS

The Glauber and Kisslinger models have recently been used to describe the pion-carbon interaction in the (3, 3)-resonance region. None of the models were originally intended to apply in this energy region, the Kisslinger model being a threshold approximation and the Glauber model a high-energy approximation. We have therefore investigated some properties of these models relevant in this particular energy region.

We conclude that the nonlocal Kisslinger potential in its standard form is inadequate because it misrepresents the basic pion-nucleon interaction quite strongly. If one tries to correct for this defect a potential with unphysical properties is obtained. The s-wave part of the modified potential becomes productive instead of absorptive. This complication does not occur at threshold and there we suggest the use of the new modified potential rather than the original one.

We have also shown that the methods of Kisslinger can be used to derive a phenomenological local potential. In the (3, 3)-resonance region this potential seems to be more promising than the nonlocal potentials of the Kisslinger type. Its detailed properties and implications will be studied in forthcoming publications.

Our new local potential is very closely related to the Glauber model. In fact we have found that the Glauber model reproduces the cross sections of our local potential quite accurately and up to very large angles. This is rather surprising because the Glauber approximation is a small-angle approximation. In addition it assumes a diffractive pion-nucleon interaction, whereas in the (3, 3)-resonance region this interaction is dominated by one particular partial wave. We have also investigated other aspects of the Glauber multiple-scattering series. Thus we have shown that in spite of the large charge-exchange cross sections the series can be conveniently summed and charge-exchange effects can be accounted for.

ACKNOWLEDGMENTS

It is a great pleasure to thank Professor M. Krell for many valuable discussions and for providing us with the numerical solutions to the Klein-Gordon equation. We would also like to thank Pro-

fessor T. E. O. Ericson and Professor R. J. Glauber for helpful discussions and suggestions.

APPENDIX A

The following analytic expressions are valid for nuclei $4 \leq A \leq 16$ that can be described by the harmonic-oscillator potential. We assume that the s-states are always filled (four nucleons) and that the remaining $A-4$ nucleons are in the p states.

Single-Particle Densities

$$s \text{ state: } \rho_s(\vec{r}) = \frac{1}{\pi^{3/2}a^3} e^{-\vec{r}^2/a^2}, \tag{A1}$$

$$p \text{ state: } \rho_p(\vec{r}) = \frac{2}{3\pi^{3/2}a^3}\left(\frac{r}{a}\right)^2 e^{-\vec{r}^2/a^2}. \tag{A2}$$

The nuclear rms radius determines a;

$$\langle \vec{r}^2 \rangle = a^2\left(\frac{5}{2} - \frac{4}{A}\right). \tag{A3}$$

Form Factors

$$\text{Defined by: } S(\vec{q}) = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\rho(\vec{r}); \tag{A4}$$

$$s \text{ state: } S_s(\vec{q}) = e^{-a^2\vec{q}^2/4}, \tag{A5}$$

$$p \text{ state: } S_p(\vec{q}) = (1 - \tfrac{1}{6}a^2\vec{q}^2)e^{-a^2\vec{q}^2/4}. \tag{A6}$$

Nuclear Density and Form Factor

$$\rho(\vec{r}) = 4\rho_s(\vec{r}) + (A-4)\rho_p(\vec{r})$$

$$= \frac{4}{\pi^{3/2}a^3}\left[1 + \frac{A-4}{6}\left(\frac{r}{a}\right)^2\right]e^{-\vec{r}^2/a^2}, \tag{A7}$$

$$S(\vec{q}) = A\left(1 - \frac{A-4}{6A}a^2\vec{q}^2\right)e^{-a^2\vec{q}^2/4}, \tag{A8}$$

$$\Delta_3\rho(\vec{r}) = \frac{4}{\pi^{3/2}a^5}\left[A - 10 - \frac{7A-40}{3}\left(\frac{r}{a}\right)^2 + \frac{2(A-4)}{3}\left(\frac{r}{a}\right)^4\right]e^{-\vec{r}^2/a^2}, \tag{A9}$$

where Δ_3 is the three-dimensional Laplace operator.

Transmission Factors

$$\text{Defined by: } T(\vec{b}) = \int_{-\infty}^{\infty} dz\, \rho(\vec{b}, z), \tag{A10}$$

$$T_s(\vec{b}) = \frac{1}{\pi a^2} e^{-\vec{b}^2/a^2}, \tag{A11}$$

$$T_p(\vec{b}) = \frac{1}{3\pi a^2}\left[1 + 2\left(\frac{b}{a}\right)^2\right]e^{-\vec{b}^2/a^2}, \tag{A12}$$

$$T(\vec{b}) = \frac{A+8}{3}\,\frac{1}{\pi a^2}\left[1 + 2\frac{A-4}{A+8}\left(\frac{b}{a}\right)^2\right]e^{-\vec{b}^2/a^2}, \tag{A13}$$

$$\Delta_2 T(\vec{b}) = \frac{-4}{3\pi a^4}\left[16 - A + (5A-32)\left(\frac{b}{a}\right)^2 - 2(A-4)\left(\frac{b}{a}\right)^4\right]e^{-\vec{b}^2/a^2}, \tag{A14}$$

where Δ_2 is the two-dimensional Laplace operator.

APPENDIX B

Here we discuss the form of the pion-nucleus amplitudes to be used in our calculations. We denote quantities in the c.m. system by a subscript c and those in the lab system by a subscript l. $h(\vec{q})$ will be used as a common notation for $f_{3/2}(\vec{q})$ and $f_{1/2}(\vec{q})$.

In the region around and below the (3, 3)-resonance only s and p waves are important in $h(\vec{q})$. Thus

$$h_c(t) = \frac{1}{k_c}[a_0 + (2a_{1+} + a_{1-})\cos\theta_c], \tag{B1}$$

$$\cos\theta_c = 1 + t/2k_c^2, \tag{B2}$$

and

$$a_{l\pm} = e^{i\delta_{l\pm}}\sin\delta_{l\pm}, \tag{B3}$$

where $l^{\pm}$ denotes angular momentum states with $j = l \pm \frac{1}{2}$. Going over to the lab system we get an additional kinematic transformation factor:

$$h_l(\vec{k}_l, \vec{k}_l') = Jh_c(\vec{k}_c, \vec{k}_c'). \tag{B4}$$

The exact form for J is easily calculated but uninteresting. The leading terms in an expansion are

$$J = \frac{k_l}{k_c}\left(1 + \frac{t}{4k_c^2}\,\frac{\omega_c}{E_c + \omega_c} + \cdots\right). \tag{B5}$$

We have $\omega_c/(E_c + \omega_c) \simeq \frac{1}{5}$ and can therefore put $J \simeq k_l/k_c$. At this point it is also important to remember that a large-angle pion-nucleus collision will be predominantly made up by high-order multiple-scattering collisions and that the momentum transfer will be roughly equally divided between them. As a result the momentum transfer in the individual pion-nucleon collisions will be much smaller than k_c and the leading term in

Eq. (B5) becomes entirely sufficient. Thus

$$h_l(t) = u + vt\,, \tag{B6}$$

$$u = \frac{k_l}{k_c^2}(a_0 + 2a_{1+} + a_{1-})\,, \tag{B7}$$

$$v = \frac{k_l}{2k_c^4}(2a_{1+} + a_{1-})\,. \tag{B8}$$

By taking the appropriate isospin average, Eq. (5.3), we get

$$f(l) = u_0 + v_0 t\,, \tag{B9}$$

$$u_0 = \tfrac{1}{3}[2u(\tfrac{3}{2}) + u(\tfrac{1}{2})]\,, \tag{B10}$$

$$v_0 = \tfrac{1}{3}[2v(\tfrac{3}{2}) + v(\tfrac{1}{2})]\,, \tag{B11}$$

with $u(T)$ and $v(T)$ as defined in Eqs. (B7) and (B8).

---

*Supported in part by the Air Force Office of Scientific Research, Contract No. F44620-70-C-0030.

†On leave from the Institute of Theoretical Physics, University of Lund, Lund, Sweden.

[1]F. Binon, P. Duteil, J. P. Garron, J. Görres, L. Hugon, J. P. Peigneux, C. Schmit, M. Spighel, and J. P. Stroot, Nucl. Phys. B17, 168 (1970).

[2](a) M. Krell and S. Barmo, Nucl. Phys. B20, 461 (1970); (b) K. Bjørnenak, J. Finjord, P. Osland, and A. Reitan, Nucl. Phys. B22, 179 (1970); (c) C. Schmit, Lettere Nuovo Cimento 4, 454 (1970); (d) J. P. Maillet, C. Schmit, and J. P. Dedonder, Lettere Nuovo Cimento 1, 191 (1971); (e) C. Wilkin, Lettere Nuovo Cimento 4, 491 (1970); (f) M. M. Sternheim and E. H. Auerbach, Phys. Rev. Letters 25, 1500 (1970); (g) T. E. O. Ericson and J. Hüfner, Phys. Letters B33, 601 (1970); (h) M. Locher, O. Steinmann, and N. Straumann, Nucl. Phys. B27, 598 (1971); (i) H. K. Lee and H. McManus, Nucl. Phys. A167, 257 (1971); (j) G. W. Edwards and E. Rost, Phys. Rev. Letters 26, 786 (1971); *ibid.* 26, 1354(E) (1971); (k) C. Wilkin, CERN Report No. 71-14, 1971 (unpublished).

[3]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience, New York, 1959), Vol. 1, p. 315.

[4]L. S. Kisslinger, Phys. Rev. 98, 761 (1955).

[5]G. Fäldt, Nucl. Phys. B10, 597 (1969).

[6]G. Fäldt, Pion-Deuteron Scattering in the Resonance Region (to be published).

[7]G. Fäldt, Nucl. Phys. B29, 16 (1971).

[8]L. J. Tassie and C. Baker, Phys. Rev. 111, 940 (1958).

[9]M. Krell, private communication.