Unified Model in Intermediate Coupling Applied to the Odd-Mass Isotopes of Indium

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(Received 21 May 1971)

The unified model in intermediate coupling has been applied to the odd-mass isotopes of indium $In^{109-117}$. Using three free parameters: the dimensionless interaction parameter, ξ; the effective 2^+ phonon energy of the core, $\hbar\omega$; and the single-particle energy separations, E_j, theoretical energy spectra were fitted to the experimental spectra. The wave functions obtained during the diagonalization of the Hamiltonian were used in calculating magnetic dipole moments for the ground and first excited states, the quadrupole moments of the ground states, and $M1$ and $E2$ reduced transition probabilities for transitions between the second and first excited states. In reproducing the observed values of the magnetic dipole and electric quadrupole moments, effective values of the spin g factor and interaction strength parameter were employed. These were found to vary remarkably smoothly with the mass number A. The following values for the hitherto unmeasured properties of the indium isotopes are predicted: magnetic moment of the ground state of In^{117}, $5.516\mu_N$; the quadrupole moment of the ground state of In^{117}; 1.057×10^{-24} cm^2; and the $B(M1)$ and $B(E2)$ values for In^{113} for the decay of the second excited state to the first excited, $0.708e^2\times10^{-24}$ cm^2 and $0.149e^2\times(10^{-24}$ cm$^2)^2$, respectively. The results compare favorably with those obtained by other investigators using other models.

I. INTRODUCTION

In nuclei in which the coupling of single-particle motion to the vibrations of the core is weak, the energy levels as well as the wave functions can be calculated by perturbation theory. The groundwork for such a treatment was laid out by Bohr and Mottelson,[1] and later expanded by Choudhury.[2] The treatment, called the unified model in intermediate coupling, is essentially a phenomenological model, in which a set of parameters is employed to reproduce the experimental energy levels together with the correct sequence of spins and parities. These parameters are: ξ, the interaction strength between single particle and core; $\hbar\omega$, the phonon energy for the core; and E_j, the effective single-particle energies. The interaction parameter is obtained from the best fit of the theory to experiment, the phonon energy $\hbar\omega$ from the first excited 2^+ state of a neighboring even-mass isotone, and the effective single-particle energies E_j from the fitting procedure. The E_j can be treated as either adjustable parameters or calculated by many-body techniques. They cannot be deduced directly from the experimental values of the low-lying spectra (single-particle excitations). Because of the interactions between the odd nucleons and the nucleons of the core, the motions of the nucleons cannot be described by uncorrelated wave functions of the single-particle potential, and the higher-energy levels may be occupied before some of the lower ones are completely filled. However, the entire procedure is not simply an *ad hoc* exercise in fitting parameters, as the resulting parameters are expected to exhibit a systematic dependence on the neutron number rather than to vary randomly, and the wave functions obtained should yield reasonable values of the various physical quantities.

The unified model has been applied with considerable success to nuclei for which $50 \leq (N \text{ or } Z) \leq 82$, more specifically to the odd-mass nuclei: to $Te^{123, 125}$ and $Xe^{127, 129}$ by Glendenning[3]; to $I^{127, 129}$ by Banerjee and Gupta[4]; to Pr^{143} by Choudhury and Kujawski[5]; to $Pm^{145, 147, 149}$ by Choudhury and O'Dwyer[6]; to $Mo^{93, 95, 97}$ by Choudhury and Clemens[7]; to $Sb^{121, 125}$, $I^{129, 131}$, Pr^{143}, and $Pm^{147, 149}$ by Heyde and Brussard[8]; and to Ce^{139} by Heyde and Vanden Berghe.[9] It has also been applied to the even-mass nucleus La^{140} by Heyde and Brussard.[10] Further, the unified model has been tested in the lighter nuclei Si^{30} by Thankappan and Pandya[11] and in Ca^{40} by Raz.[12] Finally, in the case of heavy nuclei, the model has very recently been applied to the odd-mass bismuth isotopes $Bi^{203, 205, 207, 211}$ by Bradley and Meder.[13]

In a previous theoretical study of the odd-mass indium isotopes, Silverberg[14] estimated the position of the energy levels by quasiparticle methods similar to those of Kisslinger and Sorrensen,[15] as well as the levels formed by the coupling of the $1g_{9/2}$ and $2p_{1/2}$ levels to a 2^+ phonon of the core. In his single-particle core-coupled treatment the other two single-particle orbitals present in the 50–82 shell, viz. $2p_{3/2}$ and $1f_{5/2}$, were not considered as being coupled to the core. The sequence of energy levels obtained was in agreement with the experimental results available when

Silverberg made his calculations. With the advent of solid-state detectors, many new levels have been discovered in the indium isotopes,[16-31] and the spins and parities of many of the previously known levels have been revalued. Some investigators[21-24] have suggested that some of the levels are due to core-particle coupling, and others[19,23,28,29] have interpreted the same or other levels as belonging to rotational bands. In the case of In^{117}, Pandharipande *et al.*[19] have suggested one set of levels as made up by the rotational band $\frac{1}{2}^+[431]$ and some of the other levels, as being due to the coupling of the $1g_{9/2}$ hole with the quadrupole vibrations of the Sn core.

In this paper the odd-mass isotopes of indium are studied using the unified model in intermediate coupling. The calculations presented herein offer rather strong evidence that at least for In^{115} and In^{117}, the energy levels can be identified quite clearly as to their vibrational or rotational origins.

II. MATHEMATICAL FORMULATION

A. Hamiltonian and Basis Functions

In the particle-core coupled model the total Hamiltonian is given by

$$H_{\text{tot}} = H_{\text{s.p.}} + H_{\text{coll}} + H_{\text{int}} . \tag{1}$$

$H_{\text{s.p.}}$ and H_{coll} are diagonal in the representation $|nlj; NR; IM\rangle$. Here nlj defines the single-particle orbitals and for convenience will be abbreviated to j; N is the number and R the angular momentum of the core phonons; I is the total angular momentum of the coupled system; and M is the z component of I. For a given I the matrix elements for $H_{\text{coll}} + H_{\text{s.p.}}$ between states $|j, NR; IM\rangle$ and $|j', N'R'; IM\rangle$ are

$$\langle j', N'R'; IM | H_{\text{coll}} + H_{\text{s.p.}} | j, NR; IM\rangle = [(N + \tfrac{5}{2})\hbar\omega + E_j]\,\delta_{NN'}\delta_{RR'}\delta_{jj'} . \tag{2}$$

Here E_j stands for the effective single-particle energy.

Under the assumption that the equipotentials for the independent nucleon motion follow adiabatically the surface motion as they vary in time about their equilibrium, spherical shape, H_{int} can then be written as

$$H_{\text{int}} = -k(r)\sum_{\mu}\alpha_{2\mu}Y_{2\mu}(\theta, \phi), \tag{3}$$

where $k(r)$ represents the strength of the interaction. Following Choudhury,[2] a dimensionless interaction parameter ξ is introduced:

$$\xi = k\left(\frac{5}{2\pi\hbar\omega C_2}\right)^{1/2} . \tag{4}$$

In the representation $|j, NR; IM\rangle$ the matrix elements of H_{int} are given by

$$\begin{aligned}\langle j', N'R'; IM | H_{\text{int}} | j, NR; IM\rangle &= \tfrac{1}{2}\xi(\hbar\omega)(-1)^{I+j+j'+1/2} \\ &\times [(2j+1)(2j'+1)]^{1/2} \begin{Bmatrix} j & R & I \\ R' & j' & 2 \end{Bmatrix} \begin{pmatrix} j' & 2 & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\ &\times [(-1)^{R'}\langle N'R' \| b_2 \| NR\rangle + (-)^{R}\langle NR \| b_2 \| N'R'\rangle]\delta^{l'+l}_{\text{even}} .\end{aligned} \tag{5}$$

The reduced matrices are governed by the selection rules $\Delta N = 1$ and $R \leq 2$. The 3-j and 6-j symbols and the Kroenecker δ give the further selection rules $\Delta l = 0$, or 2, and $\Delta j \leq 2$.

B. Diagonalization Procedure

In order to obtain the energy eigenvalues for a given I^π the Schrödinger equation

$$H_{\text{tot}} | E^{(\alpha)}; IM\rangle = E^{(\alpha)} | E^{(\alpha)}; IM\rangle \tag{6}$$

is solved with the state vectors $|E^{(\alpha)}; IM\rangle$ expanded in the basis $|j, NR; IM\rangle$.

In the present calculations matrices have been diagonalized for all values of $\frac{1}{2}^\pm \leq I^\pi \leq \frac{11}{2}^\pm$ and for $\frac{13}{2}^+$. In the initial calculation core coupling up to three phonons was calculated. However, the third phonon had rather negligible effect on the low-lying energy levels, at least in the region ξ of interest, and only succeeded in making the diagonalization more difficult. Subsequent calculations involved single-particle coupling to only two phonons of the core. Since the separation between energy levels rather than the absolute values of energy are of interest here, we define

$$\Delta_j = \lim_{\xi \to 0} \frac{E_j - E_{9/2}}{\hbar\omega} . \tag{7}$$

The eigenvalues are calculated in units of $\hbar\omega$. The diagonalization was done by the Jacobi method which is applicable to real and symmetric matrices.

For the first run on the computer the starting values of Δ_j were obtained from the experimental values for the $\frac{9}{2}^+$, $\frac{1}{2}^-$, and $\frac{3}{2}^-$ states of In^{115}. The interaction parameter was varied between 0 and 6.0 in steps of 0.5. Subsequent runs with different combinations of $\Delta_{1/2}$ and $\Delta_{3/2}$ were made in order to provide a wide range with which to compare the experimental and theoretical values of the energy levels for all of the odd-mass indium isotopes. A typical run is plotted in Fig. 1.

The effective phonon energy can be treated as a scaling factor to compare the experimental and theoretical spectra. In Fig. 2 are plotted the ratios of the experimental energy levels of In^{115} to values of $\hbar\omega$, varying $\hbar\omega$ from 0.5 to 1.4 MeV. The resulting curves are compared with the theoretical curves of Fig. 1 to select the best combination of $\hbar\omega$, ξ, and Δ_j. The resulting spectra for In^{115}, as well as for the other odd indium isotopes will be discussed in Sec. III.

III. ENERGY LEVELS

In every instance, the ground state of the odd-mass indium isotopes has been established to have spin and parity $\frac{9}{2}^+$. The first excited state, in each isotope for which it has been measured, has spin and parity $\frac{1}{2}^-$. For In^{113}, In^{115}, and In^{117} the second excited state is $\frac{3}{2}^-$. For In^{109} and In^{111} the spins of the second excited states have not been established. It was assumed in the present calculations that the 0.97-MeV level shown by Lederer, Hollander, and Perlman[27] for In^{111} was $\frac{3}{2}^-$. Recently a level at 0.801 MeV was found by Kim, Robinson, and Johnson[30] with strong evidence that it is a $\frac{3}{2}^-$ state. In In^{109} a great profusion of levels has been uncovered very recently by Shastry, Bakhru, and Ladenbauer-Bellis (SBL-B)[31]; however, the second excited state is suspected to be $\frac{5}{2}^-$.

The $\frac{1}{2}^-$ first excited and $\frac{3}{2}^-$ second excited states are presumably the $2p_{1/2}$ and $2p_{3/2}$ single-particle states. The separation between the $\frac{1}{2}^-$ and $\frac{9}{2}^+$ levels decreases with increasing mass number. From $A = 109$ to 113 the decrease is sharp and

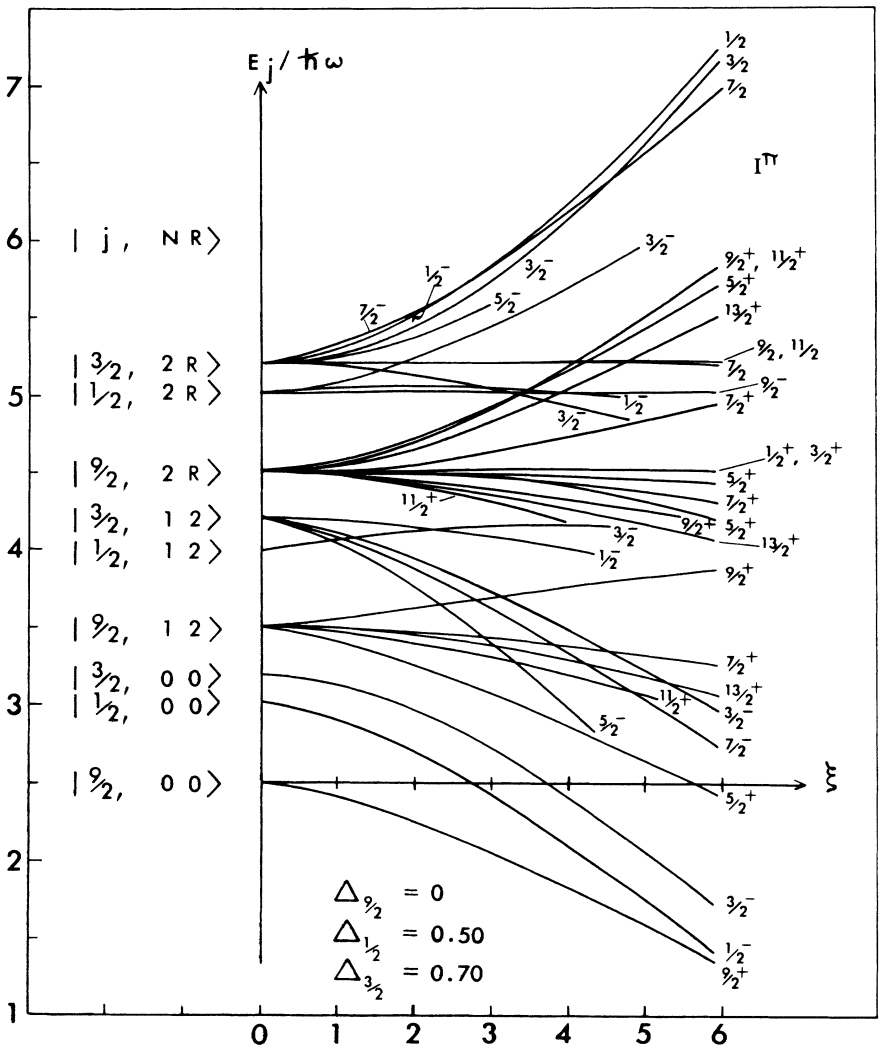

FIG. 1. The energy levels of the Hamiltonian consisting of a single proton hole coupled to quadrupole oscillations of the core versus the interaction parameter.

from 113 to 121 there is a leveling off. It has been suggested[14] that the reason for these systematics is a residual proton-neutron interaction. The $1g_{7/2}$ and $2d_{5/2}$ neutron subshells are filled up at 64 neutrons (In^{113}) where the tendency to depress the $2p_{1/2}$ proton level decreases, suggesting that additional neutrons in $1h_{11/2}$, $2d_{3/2}$, and $3s_{1/2}$ subshells do not affect the $p_{1/2}$ protons as strongly. The same argument should be valid for the $p_{3/2}$ protons, since the separation between the $\frac{3}{2}^-$ and the $\frac{1}{2}^-$ levels remains almost constant at approximately 0.260 MeV for all isotopes.

Because the energy spectra of In^{113}, In^{115}, and In^{117} are known most accurately, these isotopes will be discussed first. In^{119} and In^{121}, for which very little information is available, have not been treated in this paper.

A. In^{113}

The parameters obtained for In^{113} are $\Delta_{9/2}=0$, $\Delta_{1/2}=0.45$, $\Delta_{3/2}=0.65$, $\xi=2.1$, and $\hbar\omega=0.952$ MeV. The principal reference for the experimental energy spectrum for this nucleus is the work by Stewart, Baron, and Leonard.[22] The experimental and theoretical energy levels are seen in Fig. 3(a). The two positive-parity states, with spins $\frac{3}{2}$ and $\frac{1}{2}$ which were evident in In^{115} and In^{117} immediately above the second excited state and suggested to be members of rotational bands, do not appear in In^{113}. Over-all collective effects may be somewhat less developed in In^{113} than they are in In^{115} and in In^{117}.

B. In^{115}

The parameters obtained for In^{115} are $\Delta_{9/2}=0$, $\Delta_{1/2}=0.46$, $\Delta_{3/2}=0.65$, $\xi=2.5$, and $\hbar\omega=0.9575$ MeV. The experimental and theoretical energy levels are presented in Fig. 4(a). There is satisfactory agreement between theoretical and experimental levels. However, one can account for the experimental levels more completely by taking into consideration a rotational band.[23] According to the rotational model for odd-A nuclei the rotational energy is given by

$$E_{\rm rot}=\frac{\hbar^2}{2\mathcal{I}}\left[I(I+1)+(-1)^{I+1/2}a(I+\tfrac{1}{2})\delta_{K,1/2}\right], \qquad (8)$$

where a is the decoupling parameter and $\mathcal{I}$ is the inertial parameter of the core. In constructing the rotational band the $\frac{3}{2}^+$ state at 0.829 MeV was taken to be the lowest level of a $K=\frac{1}{2}^+$ rotational band, and the 0.864-MeV ($\frac{1}{2}^+$) and the 0.934-MeV ($\frac{7}{2}^+$) levels to be the first and second excited levels of the band, respectively. With $a=-1.9$ and $\hbar^2/2\mathcal{I}=13$ keV, the higher levels of the band were calculated[23] as 1.018 MeV ($\frac{5}{2}^+$), 1.144 MeV ($\frac{11}{2}^+$), 1.275 MeV ($\frac{9}{2}^+$), 1.461 MeV ($\frac{15}{2}^+$), 1.692 MeV ($\frac{13}{2}^+$), 1.884 MeV ($\frac{19}{2}^+$), and 2.151 MeV ($\frac{17}{2}^+$). This band would presumably be characterized by the Nilsson state $K^\pi[Nn_z\Lambda]$: $\frac{1}{2}^+[431]$.

The presence of deformation in this region is also shown by some recent investigations on the first excited (2^+) state of Cd^{114}. Measurement of the static quadrupole moment of the 2^+ state by de Boer *et al.*[32] has yielded $(-0.70\pm0.21)\times(10^{-24}$ $cm^2)$. More-recent measurements of the magnetic dipole moment of the 2^+ state by Bhattacherjee, Bowman, and Kaufmann[33] are also compatible with a deformed picture for the excited levels.

C. In^{117}

The parameters obtained for In^{117} are $\Delta_{9/2}=0$, $\Delta_{1/2}=0.51$, $\Delta_{3/2}=0.81$, $\xi=2.92$, and $\hbar\omega=0.792$ MeV. In Fig. 4(b) are presented the experimental and theoretical spectra. The primary reference for the measured values of energies, spins, and parities is the paper by Pandharipande *et al.*[19] Many of the measured levels are accounted for by the unified model. However, a number of positive-parity states – 0.660 MeV ($\frac{3}{2}^+$), 0.748 MeV ($\frac{1}{2}^+$, $\frac{3}{2}^+$), 0.881 MeV ($\frac{5}{2}^+$, $\frac{7}{2}^+$), 1.249 MeV ($\frac{11}{2}^+$, $\frac{9}{2}^+$), and 1.715 MeV ($\frac{9}{2}^+$, $\frac{11}{2}^+$) – do not find correspondence in the theoretical levels. These levels have been suggested[19] to constitute a rotational band characterized by the Nilsson state $K^\pi[Nn_z\Lambda]$: $\frac{1}{2}^+[431]$. With the values $a=-2.2$ and $\hbar^2/2\mathcal{I}=28$ keV, Pandharipande has constructed a rotational band with the members 0.660 MeV ($\frac{3}{2}^+$), 0.760 MeV ($\frac{1}{2}^+$), 0.872 MeV ($\frac{7}{2}^+$), 1.108 MeV ($\frac{5}{2}^+$), 1.310 MeV ($\frac{11}{2}^+$), and 1.670 MeV ($\frac{9}{2}^+$). Taking the same value of the decoupling parameter, but considering the factor $\hbar^2/2\mathcal{I}$ to be a free parameter, an alternate band can be proposed in which the two lowest experimental values cited (0.660 and 0.748 MeV)

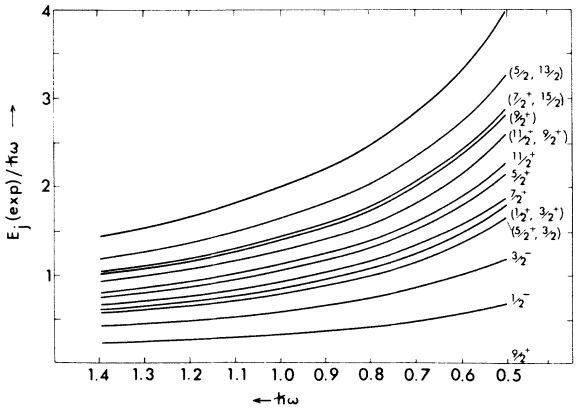

FIG. 2. $E_j(\exp)/\hbar\omega$ versus the effective phonon energy, $\hbar\omega$, for $0.5\leq\hbar\omega\leq1.4$.

are treated as the values of the two lowest members of the $K=\frac{1}{2}$ band. The subsequent levels can be determined very easily and the entire scheme would appear as in Fig. 4(b). The sequence of levels in this band are $\frac{3}{2}^+$ (0.660 MeV), $\frac{1}{2}^+$ (0.784 MeV), $\frac{7}{2}^+$ (0.838 MeV), $\frac{5}{2}^+$ (1.053 MeV), $\frac{11}{2}^+$ (1.229 MeV), $\frac{9}{2}^+$ (1.551 MeV), $\frac{15}{2}^+$ (1.778 MeV), and $\frac{13}{2}^+$ (2.272 MeV). This consideration yields the value of 24.4 keV for $\hbar^2/2\mathcal{I}$.

Further evidence for the rotational band here is the success with which the measured value of the magnetic moment for the 0.660-MeV state has been explained[29] using Nilsson's formulation.

The clear separation of the rotational and vibrational bands implies that the interaction between the two modes of excitation must be quite small for In^{115} and In^{117}.

D. In^{109}

The parameters obtained for In^{109} are $\Delta_{9/2}=0$, $\Delta_{1/2}=0.50$, $\Delta_{3/2}=0.73$, $\xi=1.08$, and $\hbar\omega=1.418$ MeV.

In Fig. 3(b) are presented the experimental and theoretical spectra for In^{109}. Until recently very few levels had been detected. Recently SBL-B[31] have discovered a large number of levels, but the spins and parities have not been identified. At 0.980 MeV Graeffe and Gordon[21] show a state with a tentative spin assignment of $\frac{5}{2}^+$; the lowest $\frac{5}{2}^+$ level in the theoretical spectrum occurs at 1.434 MeV. At 1.4 MeV of the experimental spectrum there has been a tentative spin assignment by Graeffe and Gordon of $\frac{13}{2}^+$; in the theoretical spectrum a $\frac{13}{2}^+$ state occurs at 1.5 MeV which is the result of coupling the $1g_{9/2}$ single-particle state to a single 2^+ phonon excitation of the core. The only other state whose spin has been determined is the $I=\frac{19}{2}$ (2.1-MeV) level. In the framework of the unified model, the lowest $\frac{19}{2}$ core-coupled excitation would be a state resulting from the coupling of the $1g_{9/2}$ single-particle orbit to the three-phonon state of the core. This would be expected to occur at energies around 4 MeV for a core phonon of $\hbar\omega=1.4$ MeV.

Moreover, there appears to be no evidence in the work of SBL-B[31] for the low-lying $\frac{3}{2}^-$ and $\frac{1}{2}^-$ levels, which in In^{115} and In^{117} were designated as members of a rotational band. It might be noted that for In^{109} the particle-core interaction is the smallest. Thus, absence of rotational bands would not be surprising.

E. In^{111}

The parameters obtained for In^{111} are $\Delta_{9/2}=0$,

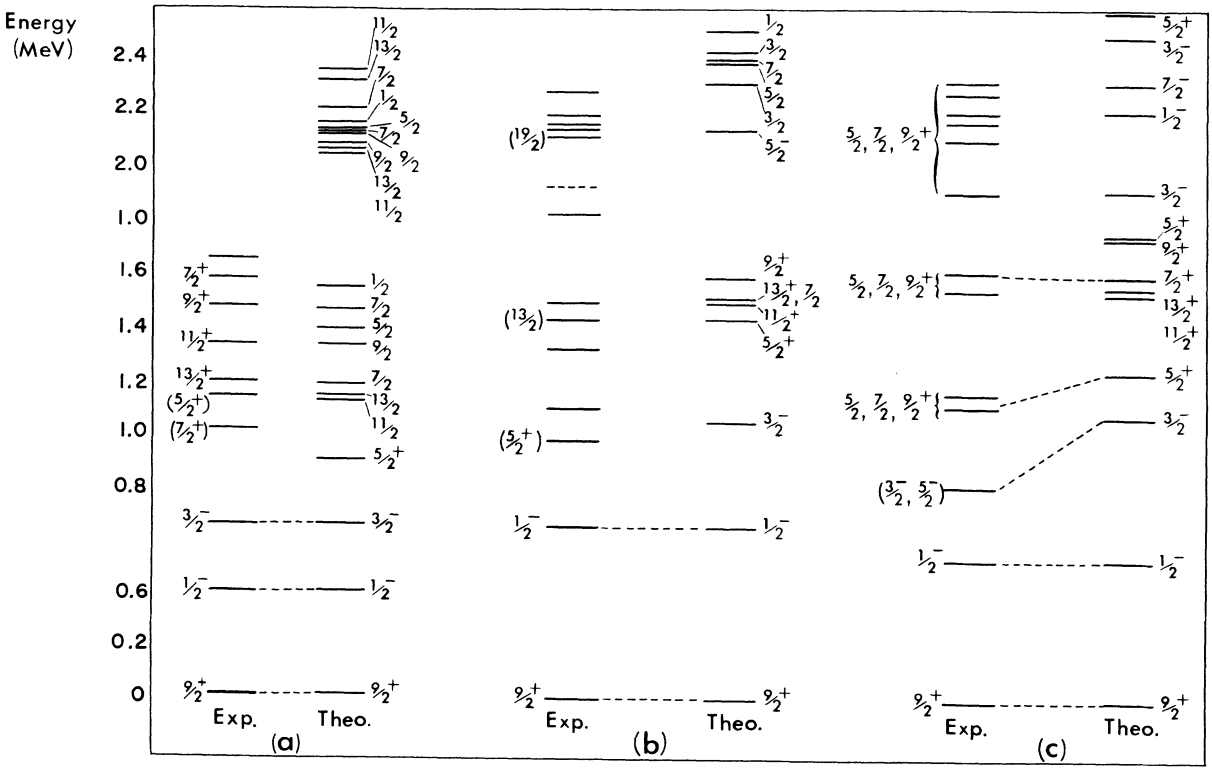

FIG. 3. Experimental and theoretical energy spectra for: (a) In^{113}, (b) In^{109}, and (c) In^{111}.

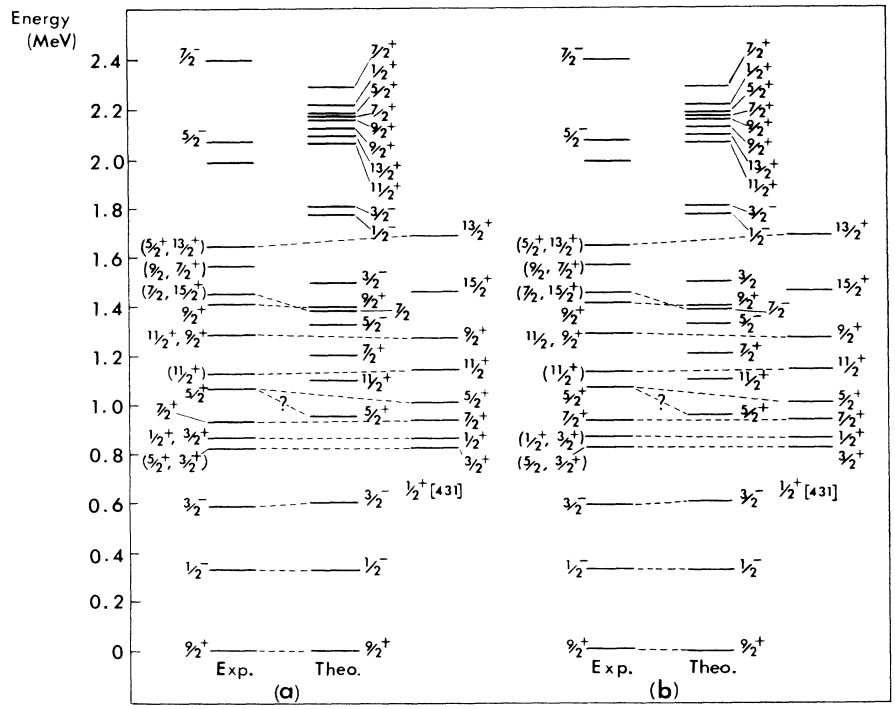

FIG. 4. Experimental and theoretical energy spectra for: (a) In^{115} and (b) In^{117}.

$\Delta_{1/2} = 0.45$, $\Delta_{3/2} = 0.76$, $\xi = 1.62$, and $\hbar\omega = 1.2$ MeV. The experimental spectrum has been taken mainly from Lederer, Hollander, and Perlman[27] and Shilin *et al.*[34] and appears in Fig. 3(c). The new level at 0.801 MeV has been discovered recently by Kim, Robinson, and Johnson,[30] who have indicated that it has a spin and parity of either $\frac{3}{2}^-$ or $\frac{5}{2}^-$, but more probably the former. This would support our theoretical scheme. As in In^{109} and In^{113}, the $\frac{3}{2}^-$ and $\frac{1}{2}^-$ levels have not been identified in In^{111}. Again this can be explained in terms of the relatively small interaction between the single-particle motion and the nuclear surface.

F. Energy-Center-of-Gravity Theorem

There have been some attempts[22,23] to analyze the energy spectra of some of the odd-mass indium isotopes in the framework of the energy-center-of-gravity theorem.[35-38] However, application of the theorem to complicated schemes involves some guess work. As an example, consider In^{115}. Applying the theorem to the levels $\frac{5}{2}^+$ (1.078 MeV), $\frac{13}{2}^+$ (1.133 MeV), $\frac{11}{2}^+$ (1.291 MeV), and $\frac{9}{2}^+$ (1.45 MeV) one calculates an energy center of gravity of 1.27 MeV, which is the same as the energy of the first 2^+ state of Sn^{116}. Alternately one may take as the "core multiplet" the levels of $\frac{13}{2}^+$ (1.645 MeV), $\frac{9}{2}^+$ (1.291 MeV), $\frac{11}{2}^+$ (1.133 MeV), $\frac{5}{2}^+$ (1.078 MeV), and $\frac{7}{2}^+$ (0.934 MeV), which also has a cen-

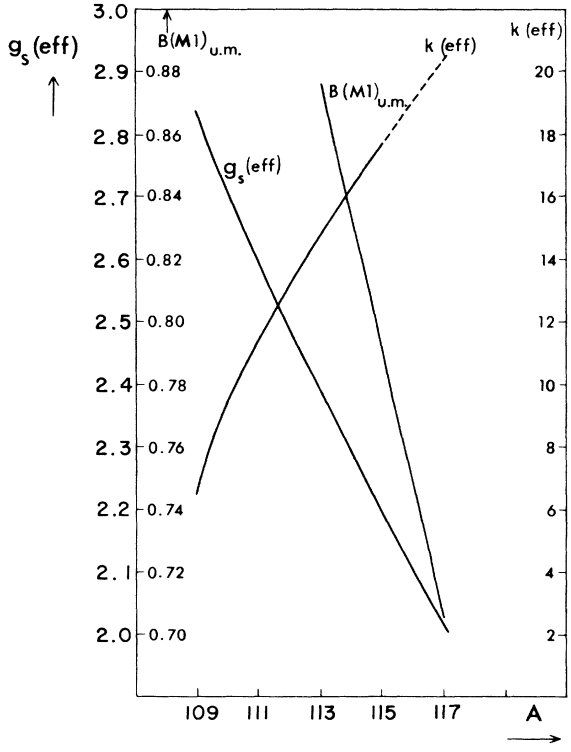

FIG. 5. g_s(eff) for the ground state in the unified model, magnified scale. $B(M1)$ in the unified model (u.m.), with $g_s = g_s$(free). k_{eff} obtained from fitting the theoretical values of Q to the observed value.

ter of gravity of 1.27. In both cases, the multiplet is assumed to be the result of coupling the core phonon to the $\frac{9}{2}^+$ state.

There are several arguments against this analysis:

(1) No transitions are expected between members of a core multiplet. Nevertheless, a strong intermultiplet transition is observed in both schemes, i.e., in $\frac{11}{2}^+$ (1.291 MeV)→ $\frac{13}{2}^+$ (1.133 MeV) and in $\frac{9}{2}^+$ (1.291 MeV)→ $\frac{11}{2}^+$ (1.133 MeV). There is evidence for other such intermultiplet transitions. This transition in our scheme occurs between two levels within a rotational band and is allowed.

(2) According to the unified-model calculations carried out here, the levels with spins $\frac{5}{2}^+$, $\frac{11}{2}^+$, and $\frac{13}{2}^+$ should be below the $\frac{7}{2}^+$ and $\frac{9}{2}^+$ levels.

(3) In the presence of particle-core interaction one would not expect the core excitation of the odd-even nucleus to have the same value as that of a neighboring even-even isotone. In the present calculations it was found that the effective $\hbar\omega$ is 0.957 MeV, whereas the 2^+ states of Cd^{114} and Sn^{116} are 0.555 and 1.27 MeV, respectively. In the center-of-gravity calculation for the odd-mass Tl isotopes de-Shalit[36] also found that the center of gravity for a core multiplet was not the same as the energy of the 2^+ state of a neighboring isotone.

IV. MAGNETIC DIPOLE MOMENTS AND TRANSITION RATES

The magnetic dipole tensor is defined as

$$\mathfrak{M}_\mu(M1)=\left(\frac{3}{4\pi}\right)^{1/2}\frac{e\hbar}{2M_p c}(\vec{\mu})_\mu, \tag{9}$$

where $(\vec{\mu})_\mu$ denotes the spherical components $\mu = -1, 0, 1$ of the dipole vector $\vec{\mu}$. The diagonal elements of $\mathfrak{M}_\mu(M1)$ for $\mu = 0$ give the magnetic dipole moment, and the nondiagonal elements determine the amplitudes of $M1$-transition processes.

TABLE I. Magnetic dipole moments μ (μ_N) for the ground state of odd-mass indium isotopes calculated by the single-particle model (s.p.) and the unified model (u.m.). g_s(free) = 5.586.

Isotope	I^π	$\mu_{s.p.}$ g_s(free)	$\mu_{u.m.}$ g_s(free)	g_s(eff)	$\mu_{u.m.}$	μ_{obs}
In^{109}	$\frac{9}{2}^+$	6.79	6.901	2.837	5.54	5.54 (6) [a]
In^{111}	$\frac{9}{2}^+$		6.997	2.598	5.53	5.53 (6) [a]
In^{113}	$\frac{9}{2}^+$		7.077	2.395	5.523	5.5233 [a]
In^{115}	$\frac{9}{2}^+$		7.175	2.201	5.535	5.5351 [a]
In^{117}	$\frac{9}{2}^+$		7.234	2.019	5.52	

[a] I. Lindgren, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy* (see Ref. 39), p. 1621.

In the unified model the magnetic dipole moment is given by

$$\mu=\langle E^{(\alpha)}; I, M=I\,|\,(\tfrac{4}{3}\pi)^{1/2}\mathfrak{M}(M1,\ \mu=0)\,|\,E^{(\alpha)}; I, M=I\rangle, \tag{10}$$

and the reduced $M1$ transition probability is obtained by evaluating

$$B(M1;\ \alpha I\to\beta I')=\frac{1}{2I+1}\times\sum_{M'\mu M}|\langle E^{(\beta)}; I'M'\,|\,\mathfrak{M}_\mu(M1)\,|E^{(\alpha)}; I, M\rangle|^2. \tag{11}$$

These expressions have been worked out in the $|j, NR; IM\rangle$ representation by Heyde and Brussard.[8] In the calculations the core g factor g_R is taken to be Z/A.

V. ELECTRIC QUADRUPOLE MOMENTS AND TRANSITION RATES

In the collective model, the electric quadrupole operator has been given as[1]

$$\mathfrak{M}(E2,\ \mu)=\frac{3}{4\pi}ZeR_0^2\alpha^\dagger_{2\mu}+\sum_i e_i r_i^2 Y_{2\mu}(\theta,\phi). \tag{12}$$

In the unified model the quadrupole moment and the reduced $E2$ transition probability are obtained by evaluating

$$Q=\langle E^{(\alpha)}; I, M=I\,|\sqrt{\tfrac{16\pi}{5}}\mathfrak{M}(E2,\ \mu=0)\,|\,E^{(\alpha)}; I, M=I\rangle \tag{13}$$

and

$$B(E2)=\frac{1}{2I+1}|\langle E^{(\beta)}; I'\,\|\,\mathfrak{M}(E2)\,\|\,E^{(\alpha)}; I\rangle|^2. \tag{14}$$

Again, these expressions have been given in the $|j, NR; IM\rangle$ representation.[8]

For the matrix $\langle l'j'\,\|\,r^2\,\|\,lj\rangle$, the shell-model assumption $\langle l'j'\,\|\,r^\lambda\,\|\,lj\rangle = 3R_0^\lambda/(3+\lambda)$ was made. The sign of $\pm e_{\rm eff}$ is positive for particle states and negative for hole states. In the present calculations of indium the 49th proton has been considered twofold: first as a particle state coupled to a Cd core, and then as a hole state coupled to an Sn core. Finally, $e_{\rm eff}$ is taken to be the effective proton charge in order to account for the polarization effects of the core. Following Heyde and Brussard[8] and Kisslinger and Sorensen[15] $e_{\rm eff}$ has been calculated with both $e_{\rm eff} = 2e_p$ and $e_{\rm eff} = e_p$.

VI. RESULTS OF THE INTERMEDIATE-COUPLING CALCULATIONS

Calculations of the magnetic dipole moment, the electric quadrupole moment, and the reduced tran-

TABLE II. Magnetic dipole moments μ (μ_N) for the first excited state of the odd-mass indium isotopes calculated by the single-particle model and the unified model. g_s(free) = 5.586.

Isotope	I^π	$\mu_{s.p.}$ g_s(free)	$\mu_{u.m.}$ g_s(free)	g_s(eff)	$\mu_{u.m.}$	μ_{obs}
In^{113}	$\frac{1}{2}^-$	−0.2643	−0.3549	4.7198	−0.21051	−0.21051(2) [a]
In^{115}	$\frac{1}{2}^-$		−0.3743	4.8026	−0.24375	−0.24375(5) [a]
In^{117}	$\frac{1}{2}^-$		−0.3817	4.8045	−0.25146	−0.25146(3) [b]

[a] I. Lindgreen, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy* (see Ref. 39).
[b] A. R. Mufti, J. A. Cameron, J. C. Waddington, and R. G. Summers-Gill, Can. J. Phys. 46, 177 (1968).

sition probabilities for $M1$ and $E2$ radiation, have been restricted mainly to cases in which comparison with experimental results was possible.

A. Magnetic Dipole Moments

The results of computations of the magnetic moments of the ground states are summarized in Table I. Using the free-proton value of g_s, both the single-particle model and the unified model predict exeessively high values. However, using effective g_s values, one can reproduce the observed values of μ precisely. These values for the unified model appear in the fifth column. g_s(eff) in the single-particle model is nearly constant at ~$0.55g_s$(free). g_s(eff) in the unified model has been plotted in Fig. 5 and is seen to be a smoothly varying function of A. The magnetic moment for In^{117} calculated from the extrapolated g_s curve is $\mu = 5.516\mu_N$. The decrease of the ratio g_s(eff)/g_s(free) with increasing A is reconcilable in terms of the isospin-dependent spin-spin interaction suggested by Nathan and Nilsson[39]:

$$V_{\sigma\tau}(r_{12})(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2) .$$

This force gives rise to strong interactions in the singlet state (^1S) between like nucleons, and to a predominance of the triplet state (^3S) between unlike nucleons. Since the magnetic moments of protons and neutrons have opposite signs, the spin-spin interaction will cause the alignment of the moments of both protons and neutrons of the core antiparallel to the moment of the extracore particle. The magnitude of the reduction in the spin g factor is proportional to the number of polarizable spins within the core. For the indium isotopes the addition of pairs of neutrons increases the number of polarizable spins; e.g., the paired $2d_{5/2}$ neutrons can be lifted into states of $2d_{3/2}$ because of their spin-spin interaction with the $(g_{9/2})^{-1}$ proton. The magnetic moments calculated for the first excited states are presented in Table II.

TABLE III. g_s(eff)/g_s(free) for the $\frac{1}{2}^-$ state.

Isotope	s.p.	u.m.
In^{113}	0.9422	0.8449
In^{115}	0.9779	0.8598
In^{117}	0.9861	0.8601

In Table III are compared the ratios g_s(eff)/g_s(free) required in the single-particle and unified models. The corresponding values for the two models are only 12–13% apart. g_s(eff)/g_s(free) for the $(p_{1/2})^{-1}$ state is close to unity, whereas it varied between 0.3 and 0.2 for the ground state, $(g_{9/2})^{-1}$. The different amounts of magnetic polarization can be explained by the spin quadrupole tensor force also suggested by Nathan and Nilsson[39]

$$V_{\sigma Y_2}[\vec{\sigma}(1)Y_2(1)] \cdot [\vec{\sigma}(2)Y_2(2)] .$$

This interaction produces a spin polarization whose magnitude depends on the orbital and on the relative orientation of the spin and orbital angular momentum of the polarizing particle.

The smoothness in the variation of g_s as a function of the mass, that was prevalent for the ground states, is not present in the $\frac{1}{2}^-$ state. The g_s(eff) values for In^{115} and In^{117} are very close to each other in both models, whereas the value for In^{113} in comparison is considerably depressed. This is very probably related to the fact that at 64 neutrons (corresponding to In^{113}) the neutron subshells $1g_{7/2}$ and $2d_{5/2}$ are filled up. In the systematics of the energy levels, this condition has

TABLE IV. $B(M1)$ ($e^2 \times 10^{-24}$ cm^2) for transitions from $\frac{3}{2}^-$ to $\frac{1}{2}^-$ states.

Isotope	g_s	$B(M1)_{u.m.}$	$B(M1)_{obs}$	Retardation factor $F_{u.m.}$	$F_{s.p.}$	F_{Dav}
In^{113}	5.586	0.8755				
	4.720	0.5134				
In^{115}	5.586	0.7952	1.11	1.40	200	5.3
	4.803	0.4925	2.25			
In^{117}	5.586	0.7082	0.954	1.35	200	4.5
	4.802	0.4381		2.18		

TABLE V. Quadrupole moments Q ($e\times 10^{-24}$ cm^2) for the ground state of the odd-mass indium isotopes. Column 3 gives the results of Kisslinger and Sorensen's model; columns 4 through 7 are the results of the unified-model calculations.

Isotope	$Q_{s.p.}$	$Q_{K.S.}$ $e_{eff}=2e$	$Q_{u.m.}$ $k=40$ MeV $e_{eff}=e$	$Q_{u.m.}$ $k=40$ MeV $e_{eff}=2e$	$Q_{u.m.}$ k_{eff} (Mev)	$Q_{u.m.}$ $e_{eff}=2e$	Q_{obs}
In^{109}	0.1434		0.2868	0.4263	6.38	1.20	1.20 [a]
In^{111}	0.1451	0.97	0.3985	0.5360	11.52	1.18	1.18 [a]
In^{113}	0.1469	0.98	0.4522	0.5880	14.56	1.140	1.14(5) [b]
In^{115}	0.1486	1.01	0.5261	0.6615	17.56	1.160	1.16(5) [b]
In^{117}	0.1503	1.01	0.5387	0.6732	20.4	1.057	

[a] I. Lindgren, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy* (see Ref. 39).
[b] G. Eder, *Nuclear Forces* (Massachusetts Institute of Technology Press, Cambridge, Mass., 1968).

manifested itself as a sudden leveling off of the energy levels. In any event, a more rigorous analysis of the behavior of g_s(eff) would have to wait until the magnetic moment of the newly discovered 0.801-MeV ($\frac{3}{2}^-$) level in In^{111} is measured.

B. $B(M1)$ Transition Rates

The half-lives for the $\frac{3}{2}^- \rightarrow \frac{1}{2}^-$ transitions have been measured for In^{115} and In^{117} by Begzhanov and Khodzhaev.[17] They have calculated $B(E2)$ and $B(M1)$ in the Davydov model, and compared these results with those of the single-particle model. In Table IV the results of the present calculations are presented along with those of Begzhanov and Khodzhaev.[17] The F factor, defined as $B(M1)_{obs}/B(M1)_{theo}$, has been given for the single-particle model, the Davydov model, and the unified model. The values of g_s(eff) used in the unified model are the free-proton g_s value (5.586) and the g_s(eff) obtained from the magnetic moments of the first excited states. Since the decay being investigated involves the second excited state ($\frac{3}{2}^-$) as well, one would expect a slightly different effective g_s value than the ones used. For both In^{115} and In^{117}, the closest agreement of all comes in the unified-model treatment using the free-proton g_s value. The core g factor has been taken as the usual value, $g_R = Z/A$. For In^{113}, the reduced transition probabilities are not available. From the similarities in the other systematics, one would expect the $\frac{3}{2}^-$ to $\frac{1}{2}^-$ transition in this isotope to be very close to those of In^{115} and In^{117}.

The unified-model results for $B(M1)$ have been plotted in Fig. 5. The g_s values used are g_s(free). The curve for these three isotopes appears to be very nearly linear. This is noteworthy, since in these calculations, as well as in those for μ, entirely different wave functions are used for each of the three isotopes.

C. Electric Quadrupole Moments

Calculations of Q and $B(E2)$ have been carried out with the effective proton charge $e_{eff}=2e_p$ and $e_{eff}=e_p$. Noninteger values of e_{eff} were not considered, so that it is not taken as a free parameter in the sense of Nathan and Nilsson.[39]

Following the suggestion of Brink[40] the surface coupling constant was taken to be a free parameter and fitted by comparison with experiment. In most unified-model calculations until now, the value of $k=40$ MeV had been used. This value

TABLE VI. $B(E2)$ [$e^2\times(10^{-24}$ cm$^2)^2$] for transitions from $\frac{3}{2}^-$ to $\frac{1}{2}^-$ states in In^{115}.

k (MeV)	C (MeV/cm^2)	e_{eff}	Core nucleus	$B(E2)_{u.m.}$	$B(E2)_{obs}$	$F_{u.m.}$	$F_{s.p.}$	F_{Dav}
17.56	41.00	$2e$	Sn^{116}	0.159	0.231 [a]	1.45	50	19.4
			Cd^{114}	1.22		0.190		
		e	Sn^{116}	0.129		1.79		
			Cd^{114}	0.654		0.353		
40	212.76	$2e$	Sn^{116}	4.83×10^{-2}		4.78		
			Cd^{114}	0.121		1.91		
		e	Sn^{116}	3.24×10^{-2}		7.13		
			Cd^{114}	3.00×10^{-2}		7.69		

[a] R. B. Begzhanov and M. Kh. Khodzhaev (see Ref. 17).

TABLE VII. $B(E2)$ $[e^2\times(10^{-24}\ \mathrm{cm}^2)^2]$ for transitions from $\frac{3}{2}^-$ to $\frac{1}{2}^-$ states in In^{117}.

k (MeV)	C (MeV cm^2)	e_{eff}	Core nucleus	$B(E2)_{\mathrm{u.m.}}$	$B(E2)_{\mathrm{obs}}$	$F_{\mathrm{u.m.}}$	$F_{\mathrm{s.p.}}$	F_{Dav}
20.4	49.05	$2e$	Sn^{118}	0.127	0.189 [a]	1.49	50	16.3
			Cd^{116}	0.949		0.199		
		e	Sn^{118}	0.103		1.84		
			Cd^{116}	0.505		0.374		
40	188.55	$2e$	Sn^{118}	4.71×10^{-2}		4.01		
			Cd^{116}	0.135		1.40		
		e	Sn^{118}	3.28×10^{-2}		5.76		
			Cd^{116}	3.90×10^{-2}		4.85		

[a] R. B. Begzhanov and M. Kh. Khodzhaev (see Ref. 17).

comes from a Taylor-series expansion of $V(r)$

$$k(r)=r\left(\frac{\partial V}{\partial r}\right)_{r=r_0},$$

which for a simple-harmonic oscillator is $2V(r)$ and requires k to be ~40 MeV. However, as pointed out by Brink,[40] following the work of Feenberg,[41] the amount of admixed state in the ground state of the odd nucleus due to particle-core coupling is given by

$$k^2/16\pi E^2,$$

and for perturbation theory to hold this expression must be very small compared with unity. However, with $k=40$ MeV and typical values of E, this condition is not satisfied. Brink suggests values of $k=10$ MeV as reasonable predictions. Heyde and Brussard[8] have carried out calculations in the 50–82 shell in which they have taken the values of 20 and 40 MeV. For their region of interest they found 40 MeV suitable. Thankappan and Pandya[11] found 20 MeV to be the correct value for Si^{30}; Choudhury and Clemens[7] used the values of $k=32$, 12, and 40 MeV, respectively, for the three odd-mass Mo isotopes, $Mo^{97,95,93}$.

In Table V are presented the results of the unified-model calculations on the quadrupole moments. In addition, the predictions of Kisslinger and Sorensen and of the single-particle model have been listed.

TABLE VIII. $B(E2)$ $[e^2\times(10^{-24}\ \mathrm{cm}^2)^2]$ for transitions from $\frac{3}{2}^-$ to $\frac{1}{2}^-$ states in In^{113}.

k (MeV)	e_{eff}	Core nucleus	$B(E2)_{\mathrm{u.m.}}$	$B(E2)_{\mathrm{s.p.}}$
14.56	$2e$	Sn^{114}	0.149	0.0033
		Cd^{112}	4.64×10^{-2}	
	e	Sn^{114}	0.118	
		Cd^{112}	6.66×10^{-2}	
40	$2e$	Sn^{114}	3.81×10^{-2}	
		Cd^{112}	5.77×10^{-2}	
	e	Sn^{114}	2.32×10^{-2}	
		Cd^{112}	4.45×10^{-2}	

The values of k_{eff} vs the mass number have been plotted in Fig. 5. The linearity of the points corresponding to the isotopes In^{111}, In^{113}, and In^{115} is immediately apparent. This supports the hypothesis that k should be taken as a free parameter. Had k been ~8.4 rather than 6.4 for In^{109}, the curve would be very nearly linear. (For a value of $k_{\mathrm{eff}}=8.4$ the quadrupole-moment calculation would have yielded a value of $Q=1.0$ rather than 1.20).

D. $B(E2)$ Transition Rates

As in the case of the $B(M1)$ values, the results of the unified-model calculations are compared with those of the single-particle model and the Davydov model. In Tables VI and VII are present-

TABLE IX. The $\hbar\omega$ values for the odd-mass indium isotopes and the energies of the first 2^+ states of the neighboring e-e isotones.

Isotope	Neighboring e-e isotone	2^+ State (MeV)	$\hbar\omega$ (MeV)
In^{109}	Cd^{108}		1.418
	Sn^{110}		
In^{111}	Cd^{110}	0.6576 [a]	1.20
	Sn^{112}	1.257 [a]	
In^{113}	Cd^{112}	0.610 [b]	0.952
	Sn^{114}	0.6174 [a]	
		1.299 [a]	
In^{115}	Cd^{114}	0.5581 [a]	0.9575
	Sn^{116}	1.274 [c]	
		1.287 [b]	
In^{117}	Cd^{116}	0.5131 [a]	0.792
	Sn^{118}	1.230 [a]	

[a] See Ref. 27. [b] See Ref. 22. [c] See Ref. 23.

ed the results for In^{115} and In^{117}. In these tables the two values of k employed are $k = 40$ MeV and the values of k_{eff} obtained from fitting the quadrupole moment.

In the calculations using the effective values of k, the picture of the 49th proton as a hole coupled to the Sn core is definitely superior to the one in which it is considered as a particle coupled to the Cd core. Further, taking $e_{\text{eff}} = 2e_p$ gives only a slight improvement over $e_{\text{eff}} = e_p$. However, in all cases the unified-model results are distinctly superior to those of the Davydov model ($F = 19.4$ for In^{115} and $F = 16.3$ for In^{117}) and to those of the single-particle model ($F = 50$ for both isotopes).

$B(E2)$ values for In^{113} computed in the unified model and in the single-particle model appear in Table VIII. It would be of considerable interest to compare these with experimental values of $B(E2)$, if the latter become available.

VII. CONCLUSIONS

It has been shown that the unified model with intermediate coupling provides a very satisfactory description of many of the properties of the odd-mass indium isotopes. The properties tested and found in good agreement with observed values were mainly properties of the low-lying states: magnetic dipole moments of the ground and first excited states, electric quadrupole moments of the ground state, and $M1$ and $E2$ transition rates for the decay of the second excited state. Aside from the essential fact that the vibrational picture

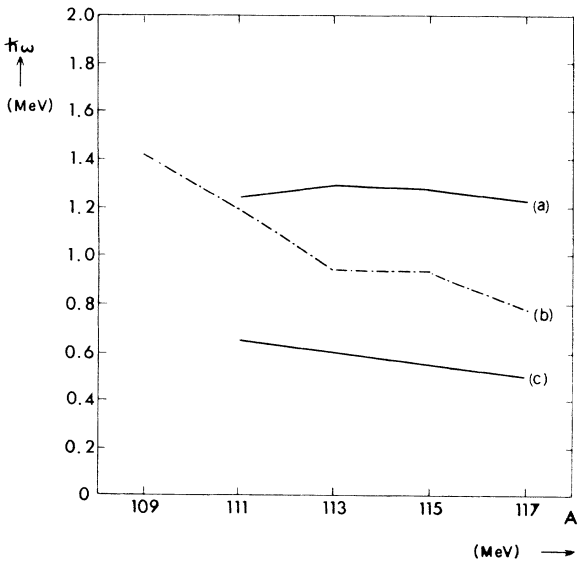

FIG. 6. Values of $\hbar\omega$ for the odd-mass indium isotopes together with the excitation energies of the first 2^+ levels in the neighboring even-even isotones, versus the mass number A: (a) Sn, (b) In, (c) Cd.

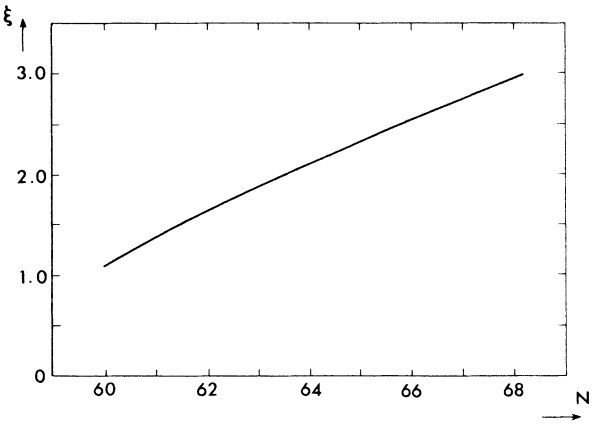

FIG. 7. The dimensionless interaction parameter ξ versus the number of neutrons N.

appears to be correct for the odd-mass indium isotopes, the success of the present application can be attributed primarily to a judicious selection of the set of adjustable parameters ξ, $\hbar\omega$, and Δ_j. In the calculations of the magnetic moments and the $B(M1)$ values, effective values of the spin g factor were used; and in the calculation of the quadrupole moments and $B(E2)$ values, effective values of the interaction strength k were used. It was shown that these quantities exhibited very stable systematics and made extrapolation meaningful. The smooth variation of $g_s(\text{eff})$ and $k(\text{eff})$ with the mass number can again be attributed to the initial selection of ξ, $\hbar\omega$, and Δ_j.

In Table IX are shown the values of the effective phonon energy $\hbar\omega$ used for each of the indium isotopes, along with the energies of the first 2^+ states of the neighboring even-even isotones. These values are plotted in Fig. 6. The 2^+ states of the even-mass Sn isotopes are very nearly constant for $Sn^{112-118}$. For the even-mass Cd isotopes the 2^+ states show a slow and smooth decrease as the

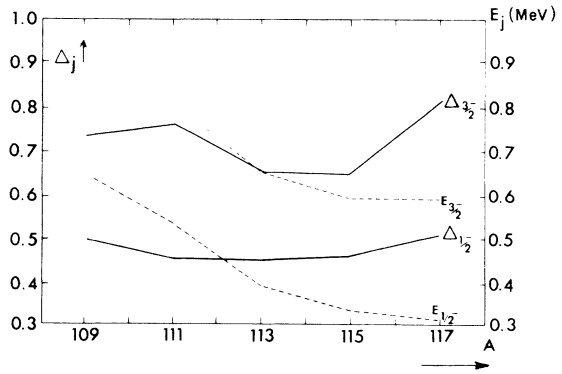

FIG. 8. The effective single-particle energies $\Delta_{1/2}$ and $\Delta_{3/2}$ versus A. The systematics of the $\frac{1}{2}^-$ and $\frac{3}{2}^-$ levels for the odd-mass indium isotopes.

mass increases. The values of $\hbar\omega$ are roughly midway between the 2^+ states of Cd and Sn for the isotopes $In^{113, 115, 117}$. For In^{111} $\hbar\omega$ is much closer to the isotone with the magic number, namely Sn^{112}; and with decreasing mass, $\hbar\omega$ continues to climb. The 2^+ states for Cd^{108} and Sn^{110} are not presently known. Heyde and Brussard[8] also found that $\hbar\omega$ is always very close to one of, if not in between, the values of the neighboring even-even isotones.

The systematics in the interaction parameter are seen in Fig. 7. ξ increases smoothly with the addition of pairs of neutrons starting from the neutron magic number at 50. This is the same as the behavior found by Heyde and Brussard[8] from a least-squares-fitting procedure in obtaining ξ. One would expect the interaction between the single particle and surface to increase proceeding away from the magic numbers to produce nuclear deformation. This can explain the formation of the rotational bands in In^{115} and In^{117}, which were not evident in In^{113} for which the interaction parameter is smaller.

In Fig. 8 are plotted $\Delta_{1/2}$ and $\Delta_{3/2}$, as well as $E_{1/2}$ and $E_{3/2}$ for the isotopes $In^{109-117}$. The general systematics of the Δ_j drops steeply between $A = 109$ and 113, and levels off between 113 and 117. Here when it is considered that an interaction which is increasing with A is applied to the $\frac{1}{2}^-$ state, the $\Delta_{1/2}$ curve will be drawn down, and will appear as in the curves of $E_{1/2}$. The same argument would apply to the systematics of $\Delta_{3/2}$ and $E_{3/2}$. However, there appears to be a discrepancy in that the value of $\Delta_{3/2}$ for In^{111} is too high. This value was deduced from the assumption that the level at 0.970 MeV in In^{111} was a $\frac{3}{2}^-$ state. The recently discovered level at 0.801 MeV, which has tentatively been identified by Kim, Robinson, and Johnson[30] as $I = \frac{3}{2}^-$ would lead to better accord with the rest of the $\Delta_{3/2}$ curve.

The expansion coefficients for the low-lying states show that the admixture of the two-phonon excitations is quite small. However, for energies above 2 MeV it would be recommended to take into account contributions of three-phonon excitations. In order to explain, within the framework of the unified model, the wealth of new levels recently discovered in In^{109}, one would have to consider three- and possibly four-phonon states. Since very little is presently known about the spins and parities of these new levels, comparison would be useful only in the densities of levels.

It would be worthwhile to further test the unified model for the indium isotopes by calculating transition rates for the $\frac{1}{2}^- \rightarrow \frac{9}{2}^+$ isomeric decays, and also by calculating decay rates of the higher states.

ACKNOWLEDGMENT

One of us (B.I.A.) would like to thank Miss Elizabeth Stansell, student and friend, for assistance in many of the computer calculations.

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Complex Eigenvalues of a Coupled-Channels Model*

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(Received 21 October 1971)

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I. INTRODUCTION

Particles are "absorbed" from a channel of interest when they are lost to another channel. Such processes can be described either by a coupled-channels formulation, or equivalently, by an effective nonlocal single-channel optical model.

Recently we found that the spectrum of a conventional local optical model includes a set of "inner states."[1] These states, which apparently have not been discussed previously, are localized within the absorptive region and are the source of highly nonlinear effects in the familiar eigenstates. Their complex energy eigenvalues have a large imaginary part.

We have now obtained analogous results for a simple coupled-channels model. Thus it appears that they are a general feature of strongly absorptive interactions.

In the next section we summarize the optical-model calculations. In Sec. III, we discuss the coupled-channels description. Finally, in Sec. IV we briefly explore the significance of our results and their experimental implications.

II. OPTICAL MODEL

The usual particle-nucleus optical potential is of the form

$$V(r) = -(V_R + iV_I)\rho(r), \tag{1}$$

where ρ is the nuclear density. For K-mesonic atoms, Krell[2] discovered that when V_I was moderately large, the widths and shifts exhibited oscillations 90° out of phase with one another. Also, for large V_I, he found a net repulsive energy shift even when V_R was strongly attractive.

In order to understand the origin of these effects, we simulated the K-mesonic atom by a spherical box of radius r_A, and the nucleus by a complex square well of radius r_N within the box. We solved exactly for s states and found analogous oscillations and shifts.[1]

For $V_R = 0$ and large V_I, the low-lying eigenstates of the model split into "inner" and "outer" states, with eigenvalues $\epsilon_i - \frac{1}{2}i\Gamma_i$ and $\epsilon_o - \frac{1}{2}i\Gamma_o$, respectively; $\Gamma_i \gg \Gamma_o$. These correspond to standing waves mostly confined to the regions inside and outside the imaginary well. As V_R is in-