Perturbation Theory in Three-Particle Scattering*

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Two distinct types of perturbation theories for three-particle scattering which are based upon the decomposition of the two-particle transition operators into dominant and weak parts are investigated. The relationship of these two techniques and their usefulness in practical calculations are discussed.

I. INTRODUCTION

A few years ago, Alt, Grassberger, and Sandhas (AGS)[1] provided a general framework for a perturbative treatment of three-particle scattering. An essential aspect of their approach is the decomposition of the two-particle transition operators, t_α, into "dominant," t_α^d, and "weak," t_α^w, parts:

$$t_\alpha = t_\alpha^d + t_\alpha^w, \tag{1.1}$$

where the index α refers to that channel in which particle α (= 1, 2, 3) is asymptotically free. In the AGS approach t_α^d is incorporated in an exact fashion into the three-particle scattering problem, while t_α^w is to be treated perturbatively. The liberty in the choice of t_α^d and t_α^w, which arises from the ambiguity in the definition of weak and dominant, has given rise to a host of different realizations of the formalism of Ref. 1.[1-6]

A distinctive attribute of the perturbation theory of Ref. 1 is that even when t_α^w is incorporated into the formalism in lowest order, the resultant three-particle scattering amplitudes contain contributions from all orders of t_α^w, albeit approximately. This particular property makes possible the exploitation of the AGS theory as a unitarization technique.[2-4,7] This is all quite analogous to the use of the K matrix in two-particle scattering. For example, if one employs the Born approximation for K, one still generates a scattering amplitude which contains the potential to all orders. Needless to say, this attribute by no means guarantees the success of the approximation.

On the other hand, Sloan[8] has recently proposed a perturbation theory which is also based on the split (1.1) of t_α into dominant and weak parts, but which is somewhat more traditional in its handling of the contribution from t_α^w. For example, in lowest order in t_α^w the final (approximate) three-body amplitudes are a sum of the exact solution obtained with $t_\alpha^w = 0$ and a term of first order in t_α^w.

We propose to clarify the connection between these two theories. This turns out to be facilitated by generalizing the formalism in Ref. 8 and placing it in the canonical form of a scattering perturbation theory, namely as an iteration expansion of a specific integral equation. We demonstrate that the Sloan procedure can, in a certain sense, be regarded as a special case of the method of Ref. 1. The practical utility of both of these techniques is discussed.

II. PERTURBATION THEORY

We will use throughout this work the specific form of the three-body scattering integral equations introduced in Ref. 1.[9] The physical scattering amplitudes can be obtained from the matrix elements of the operators $F_{\beta\alpha}$ which satisfy

$$\begin{aligned} F &= B + BtF \\ &= B + FtB, \end{aligned} \tag{2.1}$$

where

$$B = \bar{\delta} G_0,$$

and G_0 is the free three-particle Green's function. We have employed the usual (cf. Ref. 4) matrix notation with respect to the channel indices $\alpha, \beta = 1, 2, 3$. Thus, F represents a 3×3 matrix whose elements are the operators $F_{\beta\alpha}$, t is a diagonal matrix with elements $t_\alpha \delta_{\beta\alpha}$, and the elements of $\bar{\delta}$ are $1 - \delta_{\beta\alpha}$. The physical significance of the operators $F_{\beta\alpha}$ is derived from the fact that

$$V + VGV = \sum_\alpha t_\alpha + \sum_{\beta,\alpha} t_\beta F_{\beta\alpha} t_\alpha,$$

where V and G are the full three-particle interaction and Green's function, respectively.[1,10]

If we write, corresponding to (1.1)

$$t = t^1 + t^2,$$

it follows that[1,4]

$$\begin{aligned} F &= \overline{F} + \overline{F} t^1 F \\ &= \overline{F} + F t^1 \overline{F}, \end{aligned} \tag{2.2a}$$

where

$$\overline{F} = B + Bt^2\overline{F}$$
$$= B + \overline{F}t^2B. \qquad (2.2b)$$

We will show that one can, using Eqs. (2.2), generate two distinct types of perturbation theories based upon the decomposition (1.1) depending on whether one identifies t^1 or t^2 with t^d. Before we demonstrate this a more definite characterization of t^d is required.

The practical utility of (2.2) ultimately depends upon the supposition that each t^d_α be of finite rank on the relevant two-particle subspace. We will assume, henceforth, that each t^d_α has this structure, which implies that in the three-body c.m. frame

$$t^d_\alpha = \sum_{n,m} \int d\vec{q}_\alpha \, |\alpha, n; \vec{q}_\alpha\rangle \, \tau^\alpha_{nm}(\vec{q}_\alpha{}^2) \langle \alpha, m; \vec{q}_\alpha| , \qquad (2.3)$$

where the sums over the discrete indices n and m are finite, and $\vec{q}_\alpha$ denotes the momentum of particle α in the c.m. system. In general the vertex states $\langle \alpha, n; \vec{q}_\alpha|$ and $|\alpha, n; \vec{q}_\alpha\rangle$ are not dual to each other. The vertex states as well as the scalar functions τ^α_{nm} depend, in general, on the three-body parametric energy.[9] We also assume that if in any channel α there are two-particle bound states, then some portion of each of these separable contributions to t_α be included in t^d_α. It may appear contradictory not to include anything but the full two-particle bound-state pole contributions in t^d_α. However, plausible arguments have been put forth for other choices.[2, 4, 7]

In the extreme case where $t^w = 0$, Eqs. (2.2) reduce to Eqs. (2.1). The reduction of (2.1) in this case to a set of multichannel quasi-two-particle equations is then easily demonstrated. If $t = t^d$, we obtain using (2.1) and (2.3),

$$F^{nm}_{\beta\alpha}(\vec{q}'_\beta|\vec{q}_\alpha) = B^{nm}_{\beta\alpha}(\vec{q}'_\beta|\vec{q}_\alpha) + \sum_{n'm',\gamma} \int d\vec{q}''_\gamma \, B^{nm'}_{\beta\gamma}(\vec{q}'_\beta|\vec{q}''_\gamma)$$
$$\times \tau^\gamma{}_{m'n'}(\vec{q}''_\gamma{}^2) F^{n'm}_{\gamma\alpha}(\vec{q}''_\gamma|\vec{q}_\alpha), \qquad (2.4)$$

where, for example,

$$F^{nm}_{\beta\alpha}(\vec{q}'_\beta|\vec{q}_\alpha) \equiv \langle \beta, n; \vec{q}'_\beta | F_{\beta\alpha} | \alpha, m; \vec{q}_\alpha\rangle \qquad (2.5)$$

are matrix elements of $F_{\beta\alpha}$ with respect to the vertex states appearing in (2.3). In the particular instance of nucleon-deuteron scattering the on-shell matrix elements (2.5) with respect to the deuteron-pole vertex states are simply the elastic N-d scattering amplitudes.[1] For the sake of definiteness in our subsequent discussion we will suppose that we are dealing specifically with the three-nucleon problem, and that we are interested only in elastic N-d scattering and breakup.

As Sloan[8] has pointed out, despite the enormous simplification achieved when one has reduced the problem to the solution of a set of integral equations of the form (2.4) even these equations become numerically unwieldly when the number of terms in (2.3) is too large. This particular circumstance, in point of fact, provided the specific motivation for the introduction of his form of perturbation theory.

We will now show that this perturbation theory is based upon the identification of t^2 in (2.2b) with t^d. Then

$$\overline{F} = B + Bt^d\overline{F}$$
$$= B + \overline{F}t^dB, \qquad (2.6)$$

and therefore, the vertex-state matrix elements of $\overline{F}$ satisfy a set of integral equations of the form (2.4). Provided that the number of terms in the expansion (2.3) is small enough we can regard these integral equations as practically soluble. We have then an exact solution of the N-d scattering problem with an approximate N-N dynamics.[11] $\overline{F}$ evidently can be regarded as the unperturbed transition operator. It is then natural to look upon (2.2a) in this case,

$$F = \overline{F} + \overline{F}t^wF, \qquad (2.7)$$

as a "two-potential formula" for the complete transition operator F with t^w in the role of the perturbing "interaction."

The perturbation expansion of Ref. 8 is, as we shall demonstrate below, simply the iteration solution of (2.7), viz.,

$$F = \overline{F} + \overline{F}t^w\overline{F} + \overline{F}t^w\overline{F}t^w\overline{F} + \cdots . \qquad (2.8)$$

The convergence of (2.8) depends on the properties of the kernel, $\overline{F}t^w$, of the integral equation (2.7); if t^w is "weak enough," only a few terms of (2.8) will be adequate.

Now for elastic N-d scattering we are only interested in the on-shell deuteron vertex-state matrix elements of F. If we call $F(n)$, where $n = 0, 1, \ldots$, the term in the expansion (2.8) which involves n occurrences of t^w, we see that except for $n = 0$ all $F(n)$ will involve non-vertex-state matrix elements of $\overline{F}$.[12] For $F(1)$ this presents no real problem, since by (2.6)

$$F(1) = (B + \overline{F}t^dB)\, t^w (B + Bt^d\overline{F}), \qquad (2.9)$$

and the on-shell deuteron vertex-state matrix elements of the right-hand side of (2.9) can be expressed entirely in terms of the half-off-shell solutions of the counterpart of (2.4) arising from (2.6). This information is obtained in calculating the vertex-state matrix elements of the unperturbed problem represented by $\overline{F} = F(0)$.

If $n > 1$, more information than would normally

be obtained in solving the unperturbed problem is, unfortunately, required. Obviously, since

$$F(0)=\overline{F}=B+Bt^{d}B+Bt^{d}\overline{F}t^{d}B,$$

any matrix element of $\overline{F}$ can be expressed in terms of its vertex-state matrix elements. However, in general, the latter are completely off shell and this requires much more preliminary computation than that for the input into $F(1)$. This additional computation is over and above that involved in the evaluation of the multidimensional integrals which are implicit in the matrix elements of $F(n)$ for $n>0$. Given the completely off-shell vertex-state matrix elements of $\overline{F}$ the evaluation of $F(2)$ is best begun from the easily derivable expression

$$F(2)=(B+\overline{F}t^{d}B)t^{w}(B+Bt^{d}B+Bt^{d}\overline{F}t^{d}B)\times t^{w}(B+Bt^{d}\overline{F}). \tag{2.10}$$

It seems evident that a perturbation theory based upon (2.7) and (2.8) will be quite impractical unless it suffices to neglect all terms in (2.8) for which $n\geq 2$.

The right-hand sides of (2.9) and (2.10) correspond to the correction terms obtained by Sloan[8] in his special case of a finite-rank t^{w}. Our main addition to the discussion of Ref. 8 has been to relate the perturbation procedure to the iteration solution of a specific integral equation (2.7). Thus, at least in principle, it is possible to discern whether or not (2.8) converges (and how fast) from the properties of $\overline{F}t^{w}$. However, the simplicity and generality of our derivation of the form of even the lowest-order correction, $F(1)$, may be more than of academic interest when considering the full spin complications of the *N-N* interaction, as well as the possibility that t^{w} is not of finite rank.

It is also interesting to inquire as to how one can compute the breakup scattering in a perturbative fashion. It is well known that the breakup scattering amplitudes are simply the on-shell matrix elements

$$\langle \text{p.w.}|G_0^{-1}F_{0\alpha}|\alpha;\text{v.s.}\rangle,$$

where $|\text{p.w.}\rangle$ is a plane-wave state, $|\alpha;\text{v.s.}\rangle$ is a deuteron vertex state, and

$$F_{0\alpha}=\overline{F}_{0\alpha}+\sum_{\gamma}\overline{F}_{0\gamma}t_{\gamma}^{w}F_{\gamma\alpha}, \tag{2.11}$$

where

$$\overline{F}_{0\alpha}\equiv G_0(1+\sum_{\gamma}t_{\gamma}^{d}\overline{F}_{\gamma\alpha})$$

yields the breakup amplitude for the unperturbed problem.[1,4] We note that if $F_{\gamma\alpha}$ in (2.11) is computed via (2.8) to order n in t^{w}, then $F_{0\alpha}$ is of order $n+1$ in t^{w}. We can avoid this disparity in powers of t^{w}, as well as simplify the calculation of breakup scattering, by neglecting the highest-order term in $F_{\gamma\alpha}$ when used in (2.11). This simplification is quite real in the (only practical) case where we have taken

$$F\simeq F(0)+F(1) \tag{2.12}$$

in computing elastic N-d scattering. If we use (2.12) in the right-hand side of (2.11) we evidently need completely off-shell vertex-state matrix elements of $\overline{F}$ in order to handle the contribution from $F(1)$. We can minimize this difficulty, as well as keep the corrections to the breakup and elastic scattering to equal orders of t^{w}, by using in this case,

$$F_{0\alpha}\simeq\overline{F}_{0\alpha}+\sum_{\gamma}\overline{F}_{0\gamma}t_{\gamma}^{w}\overline{F}_{\gamma\alpha},$$

which will involve a relatively restricted class of completely off-shell vertex-state matrix elements of $\overline{F}$.

We will next outline the perturbation theory of AGS[1,5] and compare it with the preceding technique. The AGS method is based on the identification of t^{1} with t^{d}. In this case Eqs. (2.2) become

$$F=\overline{F}^{A}+\overline{F}^{A}t^{d}F=\overline{F}^{A}+Ft^{d}\overline{F}^{A}, \tag{2.13a}$$

where

$$\overline{F}^{A}=B+Bt^{w}\overline{F}^{A}=B+\overline{F}^{A}t^{w}B. \tag{2.13b}$$

The perturbation theory derived from Eqs. (2.13) has been explained fully and applied in first order in Refs. 1 and 5. We will be concerned primarily with comparing this method with that based on (2.7).

The success of the AGS theory depends on the possibility of using only a few terms in the iteration solution of (2.13b):

$$\overline{F}^{A}=B+Bt^{w}B+Bt^{w}Bt^{w}B+\cdots. \tag{2.14}$$

Again, if t^{w} is "weak enough," only a few terms will suffice. Also, one can investigate the general convergence properties of (2.14) by studying the kernel Bt^{w}. However, unlike the case of the kernel of Eq. (2.7) this appears to be within the realm of practical execution with the relatively simple kernel Bt^{w}.[13] This feature of the AGS formalism can be regarded as a distinct advantage over the perturbation theory based on (2.7).

It is clear if (2.14) is truncated at some order n in t^{w}, and the resultant (approximate) $\overline{F}^{A}$ is employed as input in (2.13a), that the solution, F, of (2.13a) will nevertheless contain t^{w} to all orders. In order to see the connection in various orders of

t^w with the expansion (2.8) let us substitute the infinite expansion (2.14) into (2.13a). Then we write, formally,

$$F = \sum_{n=0}^{\infty} F^A(n), \tag{2.15}$$

where $F^A(n)$ has the same significance as $F(n)$. Equating powers of t^w on both sides of the resulting integral equation we obtain

$$F^A(0) = B + Bt^d F^A(0) = F(0),$$

$$\begin{aligned} F^A(1) &= B[t^w B + t^w B t^d F^A(0)] + Bt^d F^A(1) \\ &= F^A(0) t^w [B + Bt^d F^A(0)] \\ &= F(0) t^w F(0) = F(1), \end{aligned}$$

and in general,

$$F^A(n) = F(n), \tag{2.16}$$

as one would expect. It is also easy to demonstrate that if the expansion (2.14) for $\overline{F}^A$ is truncated at the term of order n' in t^w, that the formal expansion (2.15) will still be infinite, but the equality (2.16) will hold only for $n \leq n'$. Thus a truncation of order n in the AGS procedure yields the series (2.8) exactly to nth order plus an approximate representation of all higher-order terms. It is not clear which alternative is preferable, and perhaps some model calculations on the lines of those carried out in Ref. 5 could shed some light on this question.

One possible advantage of the AGS technique is that although one has a more difficult integral equation to solve in (2.13a) as compared to (2.6), one never has to compute complicated multidimensional integrals using the solutions of this integral equation if one is only interested in elastic scattering. We discussed the latter problem at length in connection with the two-potential theory and found that it limited that expansion, for all practical purposes, to the consideration of only the first-order terms. The multidimensional integrals implicit in the higher-order terms in (2.14) certainly present numerical problems, but, for a given dimensionality, they appear somewhat more tractable than the corresponding terms in (2.8). Finally, we recall that the prescription in the AGS formalism for calculating the perturbed breakup scattering involves only half-off-shell vertex-state matrix elements of the unperturbed scattering operator $\overline{F}$.

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[2]I. H. Sloan, Phys. Rev. 165, 1587 (1968).

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[4]K. L. Kowalski, Phys. Rev. 188, 2235 (1969); Phys. Rev. D 2, 812(E) (1970).

[5]E. O. Alt, P. Grassberger, and W. Sandhas, Phys. Rev. D 1, 2581 (1970).

[6]The varying reasons for a perturbation theory in performing practical three-particle calculations with realistic interactions are eloquently stated in almost all the references cited in this paper. There seems to be little need for repeating them here.

[7]J. Krauss and K. L. Kowalski, Phys. Rev. C 2, 1319 (1970).

[8]I. H. Sloan, to be published.

[9]We will follow, however, the notational conventions of Ref. 4 with one important exception. Namely, we, in the present paper, notationally suppress the dependence upon the (complex) three-particle parametric energy z (cf. Ref. 4), since this quantity plays no part in our considerations.

[10]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory* (Davey, New York, 1965). See also, T. A. Osborn and K. L. Kowalski, to be published; SLAC Report No. SLAC-PUB-812, 1970 (unpublished).

[11]It is assumed that the vertex states corresponding to the residue of the deuteron pole have not been approximated in any way. In other words, we are using "exact" deuteron wave functions throughout.

[12]This statement is still true even if t^w happens to be of finite rank. This is because the vertex states then appearing in the expansion for t^w will, in general, be entirely unrelated to those used in defining t^d.

[13]This has actually been done for some kernels of the form Bt by I. H. Sloan, Phys. Rev. 185, 1361 (1969).