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# Recoil Effects in Allowed Nuclear Beta Decay

Barry R. Holstein*
*Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002*

and

W. Shanahan
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

and

S. B. Treiman†
*National Accelerator Laboratory, Batavia, Illinois 60510*

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Model-independent results, correct to first order in recoil terms $E/M$, are given for various spectrum and correlation effects in allowed $\beta$ decay with parent polarization and/or orientation and for $\beta$-$\gamma$ circular-polarization correlations.

## I. INTRODUCTION

Recently, experiments have begun to probe the effects of second-order forbidden terms in allowed ($\Delta j = 0, \pm 1$; "no") nuclear and hyperon $\beta$ decays.[1] Measurement of these terms, e.g., the familiar weak-magnetism contribution, can provide information concerning nuclear structure,[2] the validity of the conserved-vector-current (CVC) hypothesis,[3] the existence of second-class currents,[4] etc. Papers discussing allowed nuclear $\beta$ decays have generally omitted such effects or have given them in a model-dependent form, generally based on the impulse approximation.

We assume here the validity of the current-current form of the weak interaction and give, using elementary-particle methods,[5] model-*independent* forms for the decay spectra correct to first order in the "recoil" parameter $E/M$, where $E$ is the electron energy and $M$ is the nuclear mass. We make certain approximations, which we note at the beginning: (1) Electromagnetic effects are ignored except for final-state Coulomb interactions included in the standard Fermi function; (2) we adopt the CVC hypothesis, thus eliminating two (small) vector form factors.[6]

Section II defines notation and gives the Hamiltonian for allowed $\beta$ decay in terms of four reduced matrix elements. In Sec. III results in terms of these form factors are given for:
(i) spectra and correlations for decay of unpolarized nuclei,
(ii) the spectrum integrated over neutrino directions for decay of polarized (and oriented) nuclei,
(iii) $\beta$-particle correlation effects for a situation in which the daughter nucleus (from the $\beta$ decay of an unpolarized parent) itself undergoes a decay into a granddaughter nucleus and an additional (possibly polarized) particle.
Finally, in Sec. IV we discuss some implications of our results.

## II. DEFINITIONS

For definiteness we discuss the case of electron decay. Modifications appropriate to positron decay appear in the final formulas. We consider the reaction

$$\alpha \to \beta + e^- + \bar{\nu}_e .$$

Let $p_1$, $p_2$, $p$, and $k$ denote the respective four-momenta of parent nucleus, daughter nucleus, electron, and neutrino. The parent and daughter masses are $M_1$ and $M_2$. We also define

$$P = p_1 + p_2 , \qquad q = p_1 - p_2 = p + k ,$$
$$M = \tfrac{1}{2}(M_1 + M_2) , \quad \Delta = M_1 - M_2 .$$

We assume the weak-decay amplitude is given (for $\Delta S = 0$ decay) by

$$T = \frac{G_V \cos\theta_C}{\sqrt{2}} \langle \beta \,|\, V_\mu + A_\mu \,|\, \alpha \rangle l^\mu , \tag{1}$$

where $G_V$ is the usual weak-coupling constant ($G_V m_p^{\,2} \simeq 10^{-5}$), $\theta_C$ is the Cabibbo angle, and $l^\mu$ is the matrix element of the lepton current[7]

$$l^\mu = \bar{u}(p)\gamma^\mu (1+\gamma_5) v(k) .$$

For strangeness-changing decays we, of course, replace $\cos\theta_C$ by $\sin\theta_C$.

In the rest frame of the parent nucleus, let $E(\vec{p})$ denote the energy (three-momentum) of the electron and let $\hat{k}$ be a unit vector in the direction of the neutrino momentum. The maximum electron energy allowed by kinematics is $E_0$,

$$E_0 = \Delta(1 + m_e^{\,2}/2M\Delta)/(1+\Delta/2M) ,$$

where $m_e$ is the electron mass. Then, to first order in $E/M$ the decay spectrum is given by

$$d\Gamma = \frac{|T|^2}{(2\pi)^5}\left(1 + \frac{3E - E_0 - 3\vec{p}\cdot\hat{k}}{M}\right)(E_0 - E)^2 pEdEd\Omega_e d\Omega_\nu . \tag{2}$$

We first consider the familiar case of hyperon $\beta$ decay ($j = j' = \frac{1}{2}$). One can write for the matrix element of the weak current, correct to first order in recoil quantities and assuming CVC

$$\langle \beta | V_\mu + A_\mu | \alpha \rangle l^\mu = \bar{u}(p_2)\Big( g_V \gamma_\mu l^\mu - i\frac{g_M}{2M}\sigma_{\mu\nu} l^\mu q^\nu + g_A \gamma_\mu \gamma_5 l^\mu - i\frac{g_{II}}{2M}\sigma_{\mu\nu} l^\mu q^\nu \gamma_5 \Big) u(p_1) . \tag{3}$$

Here $g_V$ ($g_A$) are the conventional vector (axial-vector) coupling constants, $g_M$ is the weak magnetism contribution related (on the CVC hypothesis) to the anomalous magnetic moments of states $\alpha, \beta$, if $\alpha, \beta$ are members of the same isospin multiplet. The so-called induced tensor term, $g_{II}$, is an additional axial form factor. If $\alpha, \beta$ are members of a common isotopic multiplet then $g_{II} \neq 0$ requires the existence of second-class currents.[8] However, if $\alpha, \beta$ are *not* isotopic analogs, $g_{II}$ can receive contributions from both second-class and the usual first-class currents.[9] We now reduce the expression appearing in Eq. (3) to a form involving only two-component (Pauli) spinors and generalize this version to an arbitrary allowed nuclear $\beta$ decay.

Suppose parent (daughter) nucleus has spin $j$ ($j'$) and spin component $m$ ($m'$) along some axis of quantization. Then in the rest frame of the parent, to first order in recoil quantities and assuming CVC, we can write for the decay amplitude[10]

$$\begin{aligned}\langle \beta | V_\mu + A_\mu | \alpha \rangle l^\mu = {} & \frac{a}{2M} P \cdot l \delta_{jj'} \delta_{mm'} \\ & - \frac{i}{4M} (j'm'1k | j'1jm) \epsilon_{ijk} \\ & \times [2bl_i q_j + i\epsilon_{ij\lambda\eta} l^\lambda (cP^\eta \mp dq^\eta)] ,\end{aligned} \tag{4}$$

where the $-$ ($+$) sign preceding $d$ refers to electron (positron) decay.[11] Here repeated Latin indices are summed from 1 to 3, while repeated Greek indices imply a four-vector contraction.

All results in subsequent sections will be given in terms of the four form factors $a$, $b$, $c$, $d$ and will be *independent* of specific nuclear models. However, in order to express our form factors in more conventional terms, we give here the predictions for $a$, $b$, $c$, $d$ provided by the nuclear impulse approximation[12]

$$\begin{aligned} a &= g_V M_F\,, \\ b &= [g_V M_L + (g_V + g_M) M_{GT}]A\,, \\ c &= g_A[M_{GT} + A(\Delta/2M)M_{\sigma L}]\,, \\ d &= (\pm g_A M_{\sigma L} + g_{II} M_{GT})A\,, \end{aligned} \tag{5}$$

with

$$M_F = \langle\beta\|\sum_i \tau_i^+\|\alpha\rangle\,,\quad M_{GT} = \langle\beta\|\sum_i \tau_i^+\sigma_i\|\alpha\rangle\,,$$

$$M_L = \langle\beta\|\sum_i \tau_i^+ L_i\|\alpha\rangle\,,\quad M_{\sigma L} + \langle\beta\| i\sum_i \tau_i^+\sigma_i x L_i\|\alpha\rangle\,.$$

In Eq. (5) $g_V$, $g_A$, $g_M$, $g_{II}$ are the form factors at zero momentum transfer for neutron $\beta$ decay, as given in Eq. (3), $M_F$ ($M_{GT}$) is the usual Fermi (Gamow-Teller) matrix element, $A$ is the mass number, and $\langle\beta\|\theta\|\alpha\rangle$ indicates a reduced matrix element – for a tensor operator $\theta_\mu^\lambda$ of rank $\lambda$:

$$\langle\beta j'm'|\theta_\mu^{\lambda\dagger}|\alpha jm\rangle \equiv (j'm'\lambda\mu|j'\lambda jm)\langle\beta\|\theta^\lambda\|\alpha\rangle\,. \tag{6}$$

## III. RESULTS

Using the form factors defined in the previous section and the techniques sketched in Appendix A, we find the spectrum in electron and neutrino variables for $\beta$ decay from unpolarized nuclei:

$$d\Gamma = F_\mp(Z,E)\frac{G_V{}^2\cos^2\theta_C}{(2\pi)^5}(E_0-E)^2 pE\,dE\,d\Omega_e\,d\Omega_\nu\left\{f_1(E) + f_2(E)\frac{\vec{p}\cdot\hat{k}}{E} + f_3(E)\left[\left(\frac{\vec{p}\cdot\hat{k}}{E}\right)^2 - \frac{1}{3}\frac{p^2}{E^2}\right]\right\}, \tag{7}$$

where the dominant Coulomb effects are contained in the energy-dependent Fermi function $F_\mp(Z,E)$, the upper (lower) sign applying to electron (positron) decay. Similarly in the following expressions for the spectral functions $f_i$, upper signs refer to electron, lower signs to positron decay. We find

$$\begin{aligned} f_1(E) &= |a|^2 + |c|^2 - \frac{2}{3}\frac{E_0}{M}[|c|^2 \pm 2\,\mathrm{Re}\,c^*(b+d)] + \frac{2}{3}\frac{E}{M}(3|a|^2 + 5|c|^2 \pm 2\,\mathrm{Re}\,c^*b) - \frac{1}{3}\frac{m_e{}^2}{ME}[2|c|^2 \pm \mathrm{Re}\,c^*(2b+d)]\,, \\ f_2(E) &= |a|^2 - \tfrac{1}{3}|c|^2 + \frac{2}{3}\frac{E_0}{M}[|c|^2 \pm \mathrm{Re}\,c^*(b+d)] - \frac{4}{3}\frac{E}{M}(3|c|^2 \pm \mathrm{Re}\,c^*b)\,, \\ f_3(E) &= \frac{E}{M}(-3|a|^2 + |c|^2)\,. \end{aligned} \tag{8}$$

For polarized (or oriented) nuclei, we suppose the parent nuclei to form an incoherent ensemble with respect to the spin projection $m$ along an axis of quantization described by unit vector $\hat{n}$. The mean polarization vector is then $\vec{j}/j \equiv (\langle m\rangle/j)\hat{n}$, and we shall require an additional parameter $\Lambda_j$ which describes the deviation of $\langle m^2\rangle$ from the value $j(j+1)/3$ obtaining for an ensemble of randomly oriented spins[13]:

$$\Lambda_j = 1 - 3\langle m^2\rangle/j(j+1)\,.$$

The spectrum, integrated over neutrino directions, is[14]

$$d\Gamma = 2F_\mp(Z,E)\frac{G_V{}^2\cos^2\theta_C}{(2\pi)^4}(E_0-E)^2 pE\,dE\,d\Omega_e\left\{f_1(E) + f_4(E)\frac{\vec{j}}{j}\cdot\frac{\vec{p}}{E} + f_5(E)\Lambda_j\left[\left(\frac{\hat{n}\cdot\vec{p}}{E}\right)^2 - \frac{1}{3}\frac{p^2}{E^2}\right]\right\}. \tag{9}$$

Here the spectral function $f_1(E)$ has already been given, and for the other spectral functions we find

$$\begin{aligned} f_4(E) &= \delta_{j,j'}\left(\frac{j}{j+1}\right)^{1/2}\left[2\,\mathrm{Re}\,a^*c - \frac{2}{3}\frac{E_0}{M}\mathrm{Re}\,a^*(c \pm b \pm d) + \frac{2}{3}\frac{E}{M}\mathrm{Re}\,a^*(7c \pm b \pm d)\right] \\ &\quad \mp \frac{\gamma_{j,j'}}{j+1}\left\{|c|^2 - \frac{2}{3}\frac{E_0}{M}[|c|^2 \pm \mathrm{Re}\,c^*(b+d)] + \frac{E}{3M}[11|c|^2 \pm \mathrm{Re}\,c^*(5b-d)]\right\}, \\ f_5(E) &= \theta_{j,j'}\frac{E}{2M}[|c|^2 \pm \mathrm{Re}\,c^*(b-d)]\,, \end{aligned} \tag{10}$$

where

$$\gamma_{j,j'}=\begin{cases} j+1 & j=j'+1 \\ 1 & j=j' \\ -j & j=j'-1\,, \end{cases}$$

$$\theta_{j,j'}=\begin{cases} -(j+1)/(2j-1) & j=j'+1 \\ 1 & j=j' \\ -j/(2j+3) & j=j'-1\,. \end{cases}$$

Finally, we consider a situation in which the daughter nucleus produced in the $\beta$-decay process undergoes a subsequent transition to a final nucleus of spin $j''$ with emission of an accompanying $\alpha$ particle or (possibly polarized) photon. The latter is characterized by a unit vector $\hat{K}$ along its direction of motion in the laboratory frame (rest frame of the parent nucleus). The spectrum then contains certain kinematic-shift terms associated with the transformation to the lab frame from the rest frame of the $\beta$-decay daughter nucleus, wherein the subsequent transition is most simply characterized. In the following, it is the spectral function $g$ which expresses these kinematic effects. We find the spectrum for delayed $\alpha$ emission[15]:

$$d\Gamma = F_{\mp}(Z,E)\frac{G_V^2\cos^2\theta_C}{(2\pi)^5}(E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_K\left\{f_1(E)+g[E,v^*,\lambda^j_{j',j''}(L)]\frac{\hat{K}\cdot\vec{p}}{E}+\lambda^j_{j',j''}(L)f_6(E)\left[\left(\frac{\hat{K}\cdot\vec{p}}{E}\right)^2-\frac{1}{3}\frac{p^2}{E^2}\right]\right\}. \tag{11}$$

Here $f_1(E)$ has been given previously, and

$$g[E,v^*,\lambda^j_{j',j''}(L)]=\frac{2}{3}\frac{E_0}{Mv^*}\{-|a|^2+\tfrac{1}{3}|c|^2[1-\tfrac{1}{10}\lambda^j_{j',j''}(L)]\}-\frac{4}{3}\frac{E}{Mv^*}\{|a|^2+\tfrac{5}{3}|c|^2[1-\tfrac{1}{100}\lambda^j_{j',j''}(L)]\}, \tag{12}$$

$$f_6(E)=\frac{E}{20M}[|c|^2\pm\mathrm{Re}\,c^*(b-d)],$$

where $v^*$ is the velocity of the $\alpha$ particle in the center-of-mass frame of the $\beta$-decay daughter. $\lambda^j_{j',j''}(L)$ is a coefficient which depends upon the nuclear spins involved and on the angular momentum $L$ of the $\alpha$ particle with respect to the daughter nucleus. A general expression for $\lambda^j_{j',j''}(L)$ is given in Appendix B. Here we consider only the case of $p$-wave ($L=1$) or $d$-wave ($L=2$) emission. Then we have

$$\lambda^j_{j',j''}(L)=\eta_{j,j'}\tau_{j',j''}(L)\,, \tag{13}$$

where

$$\eta_{j,j'}=\begin{cases} -(j-1)/(2j+1) & j=j'+1 \\ 1 & j=j' \\ -(j+2)/(2j+1) & j=j'-1\,, \end{cases}$$

and

$$\tau_{j',j''}(L=1)=2\begin{cases} (2j'+3)/j' & j'=j''+1 \\ -(2j'+3)(2j'-1)/j'(j'+1) & j'=j'' \\ (2j'-1)/(j'+1) & j'=j''-1\,, \end{cases}$$

$$\tau_{j',j''}(L=2)=\tfrac{10}{7}\begin{cases} 2(2j'+3)/j' & j'=j''+2 \\ -(2j'+3)(j'-5)/j'(j'+1) & j'=j''+1 \\ -(2j'+5)(2j'-3)/j'(j'+1) & j'=j'' \\ -(2j'-1)(j'+6)/j'(j'+1) & j'=j''-1 \\ 2(2j'-1)/(j'+1) & j'=j''-2\,. \end{cases}$$

For circularly polarized photon emission, we have

$$d\Gamma=\tfrac{1}{2}F_{\mp}(Z,E)\frac{G_V^2\cos^2\theta_C}{(2\pi)^5}(E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_K\left\{f_1(E)+f_7(E)\,\frac{\hat{K}\cdot\vec{p}}{E}+f_8(E)\left[\left(\frac{\hat{K}\cdot\vec{p}}{E}\right)^2-\frac{1}{3}\frac{p^2}{E^2}\right]\right\}. \tag{14}$$

For $E(L)$ or $M(L)$ multipole radiation

$$\begin{aligned}f_7(E)&=g[E,1,\Gamma^j_{j',j''}(L)]+\sigma\Upsilon_{j',j''}(L)\Big(\sqrt{j(j+1)}\,\delta_{j,j'}\left[2\,\mathrm{Re}\,a^*c-\frac{2}{3}\frac{E_0}{M}\,\mathrm{Re}\,a^*(c\pm b\pm d)+\frac{2}{3}\frac{E}{M}\,\mathrm{Re}\,a^*(7c\pm b\pm d)\right]\\&\qquad\pm K_{j,j'}\left\{|c|^2-\frac{2}{3}\frac{E_0}{M}[|c|^2\pm\mathrm{Re}\,c^*(b+d)]+\frac{E}{3M}[11|c|^2\pm\mathrm{Re}\,c^*(5b-d)]\right\}\Big),\end{aligned} \tag{15}$$

$$f_8(E)=\Gamma^j_{j',j''}(L)f_6(E)-3\sigma\Upsilon_{j',j''}(L)\frac{E}{M}[\delta_{j,j'}\sqrt{j(j+1)}\,2\,\mathrm{Re}\,a^*c\pm K_{j,j'}|c|^2].$$

Here

$$K_{j,j'}=\begin{cases}-(j-1) & j=j'+1\\ 1 & j=j'\\ (j+2) & j=j'-1\,,\end{cases}$$

$$\sigma=\begin{cases}+1 & \begin{matrix}E(L)\text{ with right-hand circular polarization}\\ M(L)\text{ with left-hand circular polarization}\end{matrix}\\ -1 & \begin{matrix}E(L)\text{ with left-hand circular polarization}\\ M(L)\text{ with right-hand circular polarization.}\end{matrix}\end{cases}$$

General results for $\Upsilon_{j',j''}(L)$, $\Gamma^j_{j',j''}(L)$ are quoted in Appendix B. Here we note that for dipole radiation [$E(1)$ or $M(1)$]

$$\Gamma^j_{j',j''}(L=1)=-\tfrac{1}{2}\eta_{j,j'}\tau_{j',j''}(L=1)\,,$$

$$\Upsilon_{j',j''}(L=1)=\frac{1}{2j'(j'+1)}\begin{cases}j'+1 & j'=j''+1\\ 1 & j'=j''\\ -j' & j'=j''-1\,;\end{cases} \tag{16a}$$

while for quadrupole radiation [$E(2)$ or $M(2)$]

$$\Gamma^j_{j',j''}(L=2)=\tfrac{1}{2}\eta_{j,j'}\tau_{j',j''}(L=2)\,,$$

$$\Upsilon_{j',j''}(L=2)=\frac{1}{6j'(j'+1)}\begin{cases}2(j'+1) & j'=j''+2\\ (j'+3) & j'=j''+1\\ 3 & j'=j''\\ -(j'-2) & j'=j''-1\\ -2j' & j'=j''-2\,.\end{cases} \tag{16b}$$

## IV. SUMMARY

We have given results for the spectral functions $f_i(E)$ in terms of four nuclear form factors – $a$, $b$, $c$, $d$. In leading approximation – when recoil order terms are neglected – the spectral functions are independent of energy and depend only on $a$ and $c$. However, inclusion of $b$, $d$ produces corrections of order $AE/M=E/m$, $m$ being the *nucleon* mass. There exist also recoil terms in $a$, $c$ of order $E$ over the nuclear mass. Although we carried these along for completeness, these may usually be neglected for all but the lightest nuclei.

By careful experimental study of the spectral functions, as discussed in Refs. 8 and 9, we can measure the coefficients $b$, $d$ (we consider $a$, $c$ to be determined, as they dominate the spectra) and thus provide information concerning nuclear structure and/or the weak interactions. The size of $b$, of course, bears directly on the validity of the CVC hypothesis,[3] while $d$ provides a measure both of possible second-class axial-vector currents and of the validity of the impulse approximation.[4,9]

Except in $\beta$-$\gamma$ or $\beta$-$\alpha$ correlation coefficients recoil effects are generally small corrections to the leading terms and may only be isolated via their energy dependence. A place where such terms may be of importance is in the measurement of Coulomb mixing via the $\beta$-$\gamma$ circular-polarization correlation,[16] wherein one measures the coefficient $f_7(E)/f_1(E)$ and attempts to deduce the size of $a$, which in a nonanalog transition provides a direct measure of isospin mixing effects. Thus, for example, in the decays

$$\begin{aligned}&Na^{24}(Al^{24})(I=1,J^p=4^+)\\&\qquad\to Mg^{24}(I=0,J^p=4^+)+e^-(e^+)+\nu\\&\qquad\qquad\searrow Mg^{24}(I=0,J^p=2^+)+\gamma\,,\end{aligned}$$

the experimental results for right-hand circular polarization are[17]

$$\left(\frac{f_7(E)}{f_1(E)}\right)^{Na^{24}}=0.091\pm0.017\,,$$

$$\left(\frac{f_7(E)}{f_1(E)}\right)^{Al^{24}}=-0.086\pm0.054\,,$$

while the predicted value, in leading approximation, is

$$\frac{f_7(E)}{f_1(E)}=\pm\tfrac{1}{12}\left(1\pm 4\sqrt{5}\,\frac{a}{c}\right). \tag{17}$$

It is known from $ft$ values[18] that $c\sim 0.07$ (the decay is strongly $K$-forbidden); thus we find

$$a^{\mathrm{Na}^{24}}=(0.7\pm 1.6)\times 10^{-3}\,,\quad a^{\mathrm{Al}^{24}}=(-0.3\pm 5.1)\times 10^{-3}$$

providing rather strong limits on isospin mixing.

However, if we include recoil terms and average over the electron energy spectrum we find

$$\frac{f_7(E)}{f_1(E)}=\pm\frac{1}{12}\left[1\pm 4\sqrt{5}\,\frac{a}{c}\pm\frac{E_0}{6M}\left(\frac{b}{c}-5\frac{d}{c}\right)\right]. \tag{18}$$

Now the impulse approximation predicts $b/Ac\sim +5$ and Wilkinson's experiments comparing $ft$ values for mirror decays[4] imply according to one interpretation $d/Ac\sim -6$.[19] Then the experiments on the mass-24 system yield

$$\frac{1}{12}\left[4\sqrt{5}\,\frac{a}{c}\pm\frac{E_0}{6M}\left(\frac{b}{c}-5\frac{d}{c}\right)\right]=\begin{cases}-(0.3\pm 5.4)\times 10^{-2}\\ (0.8\pm 1.7)\times 10^{-2}\end{cases}.$$

Since $E_0^{\mathrm{Na}^{24}}/m\sim 2\times 10^{-3}$ and $E_0^{\mathrm{Al}^{24}}/m\sim 9\times 10^{-3}$, in either case a sizable $d$ coefficient, as suggested by the Wilkinson data, can mock up isospin mixing even if $a$ vanishes or can mask an actual mixing effect. Our purpose is not, however, to draw specific numerical conclusions but rather to indicate the importance of inclusion of recoil terms in the analysis of $\beta$-decay experiments.

## APPENDIX A

We assemble here a few details of the calculations whose results are reported above. We begin by noting that the weak-current matrix element given in Eq. (4),

$$\begin{aligned}\langle\beta|V_\mu+A_\mu|\alpha\rangle l^\mu=&\frac{a}{2M}P\cdot l\,\delta_{jj'}\delta_{mm'}\\&-\frac{i}{4M}(j'm'1k|j'1jm)\epsilon_{ijk}\\&\times[2bl_iq_j+i\epsilon_{ij\lambda\eta}(cP^\eta\mp dq^\eta)l^\lambda],\end{aligned} \tag{A1}$$

is split into two components according to transformation properties under spatial rotations – a term $(a/2M)P\cdot l$ transforming as a scalar and a second part, consisting of the term in brackets, transforming as a 3 vector. Then, for example, in calculating $\beta$-$\gamma$ correlations, we successively decompose parent and intermediate nuclear states into products of nuclear states and spherical harmonics relating to the weak current and the photon polarization. The work is simplified considerably by choosing the axis of quantization to be along the direction of photon momentum and specifying definite photon helicity. Thus, for the vector terms and $E(1)$ radiation we write

$$|jm\rangle=(j'm'1k|j'1jm)Y_1^k(S)|j'm'\rangle\,, \tag{A2}$$

where

$$|j'm'\rangle=(j''m''1r|j''1j'm')Y_1^r(\epsilon)|j''m''\rangle\,,$$

where $\epsilon$ represents the polarization of the photon, and where

$$(\vec{S})_l=-\frac{i}{4M}\epsilon_{ijl}[2bl_iq_j+i\epsilon_{ij\lambda\eta}l^\lambda(cP^\eta\mp dq^\eta)]\,.$$

A similar analysis may be carried through for the scalar component of the current.

One then squares, expresses the products of spherical harmonics in rotationally invariant form, and multiplies by phase space as in Eq. (2), yielding

$$\begin{aligned}d\Gamma\propto\ &\text{phase space}\\&\times\Big[A\vec{S}\cdot\vec{S}^*+B(\hat{Q}\cdot\vec{S}\,\hat{Q}\cdot\vec{S}^*-\tfrac{1}{3}\vec{S}^*\cdot\vec{S})+iC\hat{Q}\cdot\vec{S}^*\times\vec{S}\\&\quad+2D\,\mathrm{Re}\,\hat{Q}\cdot\vec{S}\,\frac{a^*}{2M}P\cdot l^*+E\frac{|a|^2}{4M^2}P\cdot l\,P\cdot l^*\Big].\end{aligned} \tag{A3}$$

Here $\hat{Q}$ is a unit vector describing the photon direction in the rest frame of the $\beta$-decay daughter nucleus, and $A, B, \ldots, E$ are combinations of Clebsch-Gordan coefficients. Finally, we substitude

$$l_\mu l_\nu^*=p_\mu k_\nu+k_\mu p_\nu-g_{\mu\nu}p\cdot k\pm i\epsilon_{\alpha\mu\beta\nu}p^\alpha k^\beta\,,$$

and normalize so that integration of $\hat{Q}$ over the sphere yields the standard allowed $\beta$-decay spectrum.

## APPENDIX B

We give here the forms of the coefficients $\lambda^j_{j',j''}(L)$, $\Gamma^j_{j',j''}(L)$, $\Upsilon_{j',j''}(L)$ for transitions of arbitrary multipolarity.[20] We find

$$\begin{aligned}\lambda^j_{j',j''}(L)&=\frac{1}{[1-3/L(L+1)]}\Gamma^j_{j',j''}(L)\\&=\eta_{j,j'}\left[\frac{L(L+1)(2L+1)}{(2L-1)(2L+3)}\right]^{1/2}\\&\quad\times\left[\frac{(2j'-1)(2j'+1)(2j'+3)}{j'(j'+1)}\right]^{1/2}\\&\quad\times W(2j'Lj'';j'L)\,,\end{aligned} \tag{B1}$$

and

$$\Upsilon_{j',j''}(L)=\left[\frac{2L+1}{L(L+1)}\right]^{1/2}\left[\frac{2j'+1}{j'(j'+1)}\right]^{1/2} \times W(1j'Lj'';j'L), \quad \text{(B2)}$$

where the $W$'s represent Racah coefficients.


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† Permanent address: Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540.

[1]A. Garcia, Phys. Rev. D 3, 2638 (1971); J. Lindquist *et al.*, Phys. Rev. Letters 27, 612 (1971); Y. K. Lee, L. W. Mo, and C. S. Wu, *ibid.* 10, 253 (1963); M. E. Nordberg, F. B. Morinigo, and C. A. Barnes, Phys. Rev. 125, 321 (1962).

[2]See, e.g., M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, Mass., 1962).

[3]M. Gell-Mann, Phys. Rev. 111, 362 (1958); J. Bernstein and R. R. Lewis, *ibid.* 112, 232 (1958).

[4]D. H. Wilkinson, Phys. Letters 31B, 447 (1970); S. Weinberg, Phys. Rev. 112, 1375 (1958).

[5]C. W. Kim and H. Primakoff, Phys. Rev. 139, B1447 (1965); 140, B566 (1965).

[6]That is, using the notation in Eq. (4), we could include form factors $e,f$ where

$$\langle\beta|V_\mu|\alpha\rangle=\frac{e}{2M}q.\,l\,\delta_{jj'}\delta_{mm'}+\frac{f}{2M}(j'm'2k|j'2jm) \times\sum_{n,n'}(1n1n'|112k)Y_1^n(l)Y_1^{n'}(q)\,.$$

However, their omission is probably not serious even without the CVC hypothesis, since $q_\mu l^\mu=m_l\bar{u}(p)(1+\gamma_5)v(k)$, where $m_l$ is the lepton mass and can in most cases be neglected, and $f$ must vanish (to order $E/M$) in the impulse approximation, since a tensor current of rank 2 cannot connect neutron to proton.

[7]We employ here the metric and conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964) except for a change in sign of $\gamma_5$ and $\epsilon_{\mu\nu\alpha\beta}$.

[8]B. R. Holstein and S. B. Treiman, Phys. Rev. C 3, 1921 (1971).

[9]B. R. Holstein, Phys. Rev. C 4, 740 (1971).

[10]In terms of the form factors for hyperon decay defined in Eq. (3), we have

$$a=g_V,\quad b=\sqrt{3}\,(g_V+g_M)\,,\quad c=\sqrt{3}\,g_A,\quad d=\sqrt{3}\,g_{\text{II}}\,.$$

[11]With this convention we have for mirror transitions (e.g., $B^{12}\to C^{12}$, $N^{12}\to C^{12}$)

$$d_{\text{II}}^{B^{12}\to C^{12}}=d_{\text{II}}^{N^{12}\to C^{12}},\quad d_{\text{I}}^{B^{12}\to C^{12}}=-d_{\text{I}}^{N^{12}\to C^{12}},$$

where $d_{\text{I}}$ ($d_{\text{II}}$) represent the first-class (second-class) contribution to the transition.

[12]The impulse approximation does not make a unique prediction for the second-class contribution to $d$, as has been pointed out by E. Henley and L. Wolfenstein, Phys. Letters 36B, 28 (1971); our value $d_{\text{II}}=g_{\text{II}}AM_{\text{GT}}$ is based on the assumption of a divergenceless axial second-class current, as discussed by J. Delorme and M. Rho, Phys. Letters 34B, 238 (1971).

[13]In terms of the conventional statistical population tensors

$$R_k=\sum_m(-)^{j-m}(jmj\,-m|jjk\,0)a(m)\,,$$

$a(m)$ being the population of the $m$th nuclear level

$$\frac{\langle m\rangle}{j}=\left(\frac{(2j+1)(j+1)}{3j}\right)^{1/2}R_1,$$

$$\Lambda_j=-\left(\frac{(2j-1)(2j+1)(2j+3)}{5j(j+1)}\right)^{1/2}R_2\,.$$

[14]The spectrum, for the case in which the neutrino direction is observed, is given for $j=j'$ transitions by B. Holstein, Phys. Rev. C 3, 764 (1971). Generalization to $j'=j\pm1$ decays is provided by multiplication of polarization terms by $\gamma_{j,j'}$ and of orientation terms by $\theta_{j,j'}$.

[15]Identical results hold for delayed proton or neutron transitions if $L$ is the angular momentum of the nucleon with respect to the daughter nucleus and the nucleon spin is unobserved.

[16]S. D. Bloom, L. G. Mann, and J. A. Miskel, Phys. Rev. 125, 2021 (1962).

[17]S. D. Bloom, L. G. Mann, R. Polichar, J. R. Richardson, and A. Scott, Phys. Rev. 134, B481 (1964).

[18]C. C. Bouchiat, Phys. Rev. Letters 3, 516 (1959).

[19]This value is based on the divergenceless hypothesis for the axial second-class current discussed by Delorme and Rho, Ref. 12.

[20]These results have been derived previously by Bernstein and Lewis, Ref. 3 and by M. Morita and R. S. Morita, Phys. Rev. 107, 1316 (1957).