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where $g_M = \mu_p - \mu_n = 4.70$, while for d we find

$$d = A[g_A \langle\beta\| i \sum_i \tau_i^+ \sigma_i \times L_i \|\alpha\rangle + g_{II} \langle\beta\| \sum_i \tau_i^+ \sigma_i \|\alpha\rangle],$$

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Simple Formula for the General Oscillator Brackets

L. Trlifaj

Nulcear Research Institute of Czechoslovak Academy of Sciences, Řež u Prahy, Czechoslovakia

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An explicit formula for Talmi-Moshinsky transformation brackets of unequal-mass particles is given which is the sum of simple expressions over five variables; it is especially suited for numerical calculations.

I. INTRODUCTION

Many papers[1-13] have been devoted to the study of the Talmi-Moshinsky transformation. Since the Talmi-Moshinsky brackets (TMB) find frequent and repeated application in programs for various model calculations of nuclear structure, one of the important aims of these studies was to derive as simple a formula for them as possible. Several excellent techniques have been developed for these purposes. Let us only mention the creation-operator technique for oscillator quanta which was introduced by Moshinsky[1,14] and then successfully applied in a number of works,[3,4,7] and the method of generating functions.[5,9,13] Only recently,[13] a computation of TMB by direct diagonalization of some special operator expressions has been suggested. An explicit and symmetrical expression of TMB for unequal-mass particles was first obtained by Kumar.[5] The other formulas[6,8,9] can be easily reexpressed in Kumar's form.

At the end of this note we demonstrate an explicit and simple expression for TMB. Since we have not succeeded in rewriting it from Kumar's form, we derive it directly. The first step of our approach consists in the derivation of the simple expression for the special $l=0$ TMB. Next, following the idea of Ref. 4 we look for the relation between the general and the $l=0$ brackets. Substituting then the explicit expression for the $l=0$ brackets into this relation we obtain the final formula. The computing time corresponding to this formula as compared with that for other existing programs (for example see Ref. 4) is shorter by a factor of 2.5–5.

II. PRELIMINARIES

It is convenient to describe the position of the moving oscillator mass by the product $\vec{x}$ of the square root of the respective mass and of the real position vector.[6,7,10] Taking, moreover, $\hbar\omega^{-1}$ as a unit, where ω is the common oscillator frequency, the harmonic-oscillator wave functions with n radial nodes and angular momentum l, q are then given by

$$\varphi_{nlq}(\vec{x}) = c_{nl}\, x^l e^{-(1/2)x^2} L_n^{l+1/2}(x^2) Y_{lq}\left(\frac{\vec{x}}{x}\right) \quad (|\vec{x}| = x). \tag{1}$$

Here the normalization constant

$$c_{nl} = \left(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\right)^{1/2} \tag{2}$$

and $Y_{lq}(\vec{\omega})$, $L_n^{l+1/2}(z)$, and $\Gamma(z)$ $[n! = \Gamma(n+1)]$ are the known normalized spherical harmonics, Laguerre polynomials, and Γ functions, respectively.

The orthogonal Talmi transformation[6,7,10] for two oscillators with masses m_1, m_2, and mass ratio $d = m_2/m_1$,

$$\begin{aligned} \vec{r} &= \left(\frac{d}{1+d}\right)^{1/2} \vec{r}_1 - (1+d)^{-1/2}\vec{r}_2, \\ \vec{R} &= (1+d)^{-1/2}\vec{r}_1 + \left(\frac{d}{1+d}\right)^{1/2}\vec{r}_2, \end{aligned} \tag{3}$$

is represented in the basis of their wave functions by the Talmi-Moshinsky transformation brackets according to the relation:

$$\sum_{q_i} (l_1q_1, l_2q_2|\lambda\mu)\varphi_{n_1l_1q_1}(\vec{r}_1)\varphi_{n_2l_2q_2}(\vec{r}_2) = \sum_{nlq,NLQ} \langle nl, NL, \lambda | n_1l_1, n_2l_2, \lambda\rangle_d (lq, LQ|\lambda\mu)\varphi_{nlq}(\vec{r})\varphi_{NLQ}(\vec{R}). \tag{4}$$

Because of the elementary conservation laws the basis wave functions are coupled by Clebsch-Gordon coefficients to the same angular momentum λ, and the energy in both the frames (3) is related by the equation

$$2n + l + 2N + L = 2n_1 + l_1 + 2n_2 + l_2. \tag{5}$$

The notation chosen for the TMB expresses their dependence on the mass ratio d and, in the case of $m_1 = m_2$, their identity with the brackets tabulated by Brody and Moshinsky.[2] From the definition (4) various properties of TMB are easily seen.[10,11] There is a full correspondence with the usual Moshinsky brackets[2] except for the necessary interchange of masses m_1 and m_2 in some cases. Phases are fixed within the scope of the relation of parity conservation which follows from Eq. (5).

III. SIMPLE FORMULA FOR THE $l=0$ BRACKET

The expansion (4) must be fulfilled by general vectors of Eqs. (3) and it is, therefore, valid also for $\vec{r}=0$, $\vec{R} = (1+d)^{1/2}\vec{x}$, $\vec{r}_2 = d^{1/2}\vec{r}_1 = d^{1/2}\vec{x}$. Since according to definition (1)

$$\varphi_{nlq}(\vec{r}=\vec{0}) = \frac{1}{\pi}\left(\frac{2\Gamma(n+\frac{3}{2})}{n!}\right)^{1/2} \delta_{l,0}\delta_{q,0} = \frac{2}{\pi c_{n0}}\,\delta_{l,0}\delta_{q,0}, \tag{6}$$

then

$$\sum_{q_i} (l_1q_1, l_2q_2|LQ)\varphi_{n_1l_1q_1}(\vec{x})\varphi_{n_2l_2q_2}(d^{1/2}\vec{x}) = \frac{2}{\pi}\sum_{nN} \langle n0, NL, L|n_1l_1, n_2l_2, L\rangle_d \frac{1}{c_{n0}}\varphi_{NLQ}((1+d)^{1/2}\vec{x}). \tag{7}$$

Let us multiply both sides of the equation by $\varphi^*_{NLQ}((1+d)^{1/2}\vec{x})$ and integrate over $\vec{x}$. On the right-hand side of Eq. (7) special TMB are projected out and on the left-hand side angular integrals are easily obtained using the definition (1) and the known formulas of Edmonds[15]:

$$\langle n0, NL, L|n_1l_1, n_2l_2, L\rangle_d = 2^{-2}\pi^{1/2}\delta_{2n+2N+L,\,2n_1+l_1+2n_2+2l_2}\times\frac{(2l_1+1)^{1/2}(2l_2+1)^{1/2}}{(2L+1)^{1/2}}$$

$$\times d^{l_2/2}(1+d)^{(L+3)/2}(l_10, l_20|L0)c_{n0}c_{NL}c_{n_1l_1}c_{n_2l_2}\int_0^\infty dx\, x^{L+l_1+l_2+2}$$

$$\times e^{-(1+d)x^2}L_N^{L+1/2}((1+d)x^2)L_{n_1}^{l_1+1/2}(x^2)L_{n_2}^{l_2+1/2}(dx^2)\,. \tag{8}$$

If we substitute $z=(1+d)x^2$ and write down the Laguerre polynomials $L_{n_1}^{l_1+1/2}$ and $L_{n_2}^{l_2+1/2}$ explicitly[16]

$$L_n^\alpha(z)=\sum_{p=0}^{n}\frac{\Gamma(n+\alpha+1)}{p!\,(n-p)!\,\Gamma(p+\alpha+1)}(-z)^p\,, \tag{9}$$

the last integration can be performed according to the formula[16]

$$\int_0^\infty dz\, z^{\beta-1}e^{-z}L_n^\alpha(z)=(-1)^n\frac{\Gamma(\beta)\Gamma(\beta-\alpha)}{n!\,\Gamma(\beta-\alpha-n)}\,. \tag{10}$$

Because of the definition (2), the result is

$$\langle n0, NL, L|n_1l_1, n_2l_2, L\rangle_d=(-1)^N 2^{-1}\pi^{1/2}(1+d)^{-l_1/2}\left(\frac{d}{1+d}\right)^{l_2/2}$$

$$\times(l_10, l_20|L0)\left[\frac{(2l_1+1)(2l_2+1)n!\,n_1!\,n_2!\,\Gamma(n_1+l_1+\frac{3}{2})\Gamma(n_2+l_2+\frac{3}{2})}{(2L+1)N!\,\Gamma(n+\frac{3}{2})\Gamma(N+L+\frac{3}{2})}\right]^{1/2}$$

$$\times\delta_{2n+2N+L,\,2n_1+l_1+2n_2+l_2}\sum_{t_i}(-1)^{t_1+t_2}(1+d)^{-t_1}\left(\frac{d}{1+d}\right)^{t_2}$$

$$\times\frac{[(l_1+l_2-L)/2+t_1+t_2]!\,\Gamma((l_1+l_2+L)/2+t_1+t_2+\frac{3}{2})}{t_1!\,t_2!\,(n_1-t_1)!\,(n_2-t_2)!\,[(l_1+l_2-L)/2+t_1+t_2-N]!\,\Gamma(t_1+l_1+\frac{3}{2})\Gamma(t_2+l_2+\frac{3}{2})}\,. \tag{11}$$

The restrictions on the sum over the variable t_i are those arising from factorial functions. The last factorial in the denominator of expression (11) may still be reexpressed using relation (5), but to no great advantage.

Formula (11) is simple compared with the corresponding expression in Ref. 4. It is entirely symmetrical under interchange of the quantities m_1, n_1, l_1 and m_2, n_2, l_2.

IV. RELATION BETWEEN THE GENERAL AND $l=0$ TMB

In Ref. 4 a useful relation was derived which relates the general oscillator brackets to the special $l=0$ brackets. We adapt, therefore, Baranger and Davies's procedure for unequal masses and translate it into our analytical language. We start with the expression

$$V_{nl,NL,\lambda\mu}(\vec{r},\vec{R})=\sum_{qQ}(lq, LQ\,|\,\lambda\mu)r^lY_{lq}\left(\frac{\vec{r}}{r}\right)\varphi_{n00}(\vec{r})\varphi_{NLQ}(\vec{R})\,. \tag{12}$$

The last product of the two harmonic-oscillator wave functions can be expanded according to Eq. (4)

$$V_{nl,NL,\lambda\mu}(\vec{r},\vec{R})=\sum_{s_ip_ik_iqQ}\langle n0, NL, L|s_1p_1, s_2p_2, L\rangle_d\,(lq, LQ\,|\,\lambda\mu)(p_1k_1, p_2k_2\,|\,LQ)\varphi_{s_1p_1k_1}(\vec{r}_1)\varphi_{s_2p_2k_2}(\vec{r}_2)r^lY_{lq}\left(\frac{\vec{r}}{r}\right). \tag{13}$$

Since the relations (3) hold, the decomposition of the solid spherical harmonic[1] is now written in the form

$$r^lY_{lq}\left(\frac{\vec{r}}{r}\right)=(1+d)^{-l/2}\sum_{\lambda_i\mu_i}^{l}(-1)^{\lambda_2}d^{-\lambda_1/2}\delta_{\lambda_1+\lambda_2,\,l}\left[\frac{4\pi(2l+1)}{(2\lambda_1+1)(2\lambda_2+1)}\binom{2l}{2\lambda_1}\right]^{1/2}$$

$$\times(\lambda_1\mu_1, \lambda_2\mu_2|lq)r_1^{\lambda_1}Y_{\lambda_1\mu_1}\left(\frac{\vec{r}_1}{r_1}\right)r_2^{\lambda_2}Y_{\lambda_2\mu_2}\left(\frac{\vec{r}_2}{r_2}\right), \tag{14}$$

where the symbol $\binom{a}{b}=a!/(a-b)!\,b!$. After having substituted the expression (1) for the wave functions

$\varphi_{s_i p_i k_i}$ and the decomposition (14) for the solid spherical harmonic into Eq. (13), we recouple the angular momenta with a 9-j symbol and form two coupled products of spherical harmonics with the same arguments $\vec{r}_i/r_i$. Next we reduce each of the coupled pairs to one spherical harmonic according to the formula in Ref. 15 and obtain

$$V_{nl,NL,\lambda\mu}(\vec{r},\vec{R})=(-1)^{\lambda+l+L}(4\pi)^{-1/2}(1+d)^{-l/2}(2l+1)(2L+1)^{1/2}\sum_{s_i p_i \lambda_i l_i q_i}(-1)^{\lambda_2}d^{\lambda_1/2}$$
$$\times\delta_{\lambda_1+\lambda_2,l}\left[(2p_1+1)(2p_2+1)\binom{2l}{2\lambda_1}\right]^{1/2}\langle n0,NL,L|s_1p_1,s_2p_2,L\rangle_d(p_10,\lambda_10|l_10)$$
$$\times(p_20,\lambda_20|l_20)\begin{Bmatrix}p_1 & p_2 & L\\ \lambda_1 & \lambda_2 & l\\ l_1 & l_2 & \lambda\end{Bmatrix}c_{s_1p_1}c_{s_2p_2}\exp[-(r_1^2+r_2^2)/2]\,r_1^{p_1+\lambda_1}r_2^{p_2+\lambda_2}$$
$$\times L_{s_1}^{p_1+1/2}(r_1^2)L_{s_2}^{p_2+1/2}(r_2^2)(l_1q_1,l_2q_2|\lambda\mu)Y_{l_1q_1}\left(\frac{\vec{r}_1}{r_1}\right)Y_{l_2q_2}\left(\frac{\vec{r}_2}{r_2}\right). \tag{15}$$

The other way of transforming the original expression (12) is to apply the recursion formula for Laguerre polynomials[16]

$$L_n^{1/2}(z)=\sum_{p=0}^{n}(-1)^p\binom{l}{p}L_{n-p}^{l+1/2}(z), \tag{16}$$

in the explicit formula (1) for the wave function $\varphi_{n00}(\vec{r})$ and to couple in reverse the Laguerre polynomials $L_{n-p}^{l+1/2}$ with the solid spherical harmonic $r^l Y_{lq}(\vec{r}_1/r_1)$ to new wave functions $\varphi_{(n-p)lq}(\vec{r})$ using the same definition (1). The result is

$$V_{nl,NL,\lambda\mu}(\vec{r},\vec{R})=(4\pi)^{-1/2}\sum_{qQ,p=0}^{n}(-1)^p\binom{l}{p}\frac{c_{n0}}{c_{n-p,l}}(lq,LQ|\lambda\mu)\varphi_{(n-p)lq}(\vec{r})\varphi_{NLQ}(\vec{R}). \tag{17}$$

We multiply both expressions (15) and (17) by $\sum_{q_i}(l_1q_1,l_2q_2|\lambda\mu)\varphi^*_{n_1l_1q_1}(\vec{r}_1)\varphi^*_{n_2l_2q_2}(\vec{r}_2)$ and perform the integration over $\vec{r}_1$ and $\vec{r}_2$ ($\int d\vec{r}_1 d\vec{r}_2=\int d\vec{r}d\vec{R}$). Using the energy relation (5) the sum in the expression (17) reduces to one term with $p=0$, and because of the expansion (4) the overlap integral defines the general TMB. In the expression (15) we perform the trivial angular integration using definition (1). Setting the results obtained equal, we write

$$\langle nl,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle_d=(-1)^{l+L+\lambda}(1+d)^{-l/2}(2l+1)(2L+1)^{1/2}\sum_{s_ip_i\lambda_i}(-1)^{\lambda_2}d^{\lambda_1/2}$$
$$\times\left[(2p_1+1)(2p_2+1)\binom{2l}{2\lambda_1}\right]^{1/2}\delta_{\lambda_1+\lambda_2,l}(c_{n0})^{-1}c_{nl}c_{s_1p_1}c_{s_2p_2}c_{n_1l_1}c_{n_2l_2}$$
$$\times\begin{Bmatrix}p_1 & p_2 & L\\ \lambda_1 & \lambda_2 & l\\ l_1 & l_2 & \lambda\end{Bmatrix}(p_10,\lambda_10|l_10)(p_20,\lambda_20|l_20)\langle n0,NL,L|s_1p_1,s_2p_2,L\rangle_d\int_0^\infty dr_1r_1^{p_1+\lambda_1+l_1+2}$$
$$\times e^{-r_1^2}L_{s_1}^{p_1+1/2}(r_1^2)L_{n_1}^{l_1+1/2}(r_1^2)\int_0^\infty dr_2r_2^{p_2+\lambda_2+l_2+2}e^{-r_2^2}L_{s_2}^{p_2+1/2}(r_2^2)L_{n_2}^{l_2+1/2}(r_2^2). \tag{18}$$

Both integrals of Laguerre polynomials are of the same type, so we show the calculation of only one of them. We express again the polynomial $L_s^{p+1/2}$ explicitly according to Eq. (9) and integrate using formula (10):

$$\int_0^\infty dr\,r^{p+\lambda+l+2}L_s^{p+1/2}(r^2)L_n^{l+1/2}(r^2)e^{-r^2}$$
$$=\tfrac{1}{2}\Gamma(s+p+\tfrac{3}{2})\sum_{t=0}^{s}(-1)^{n+t}\frac{[(\lambda+p-l)/2+t]!}{n!\,t!\,(s-t)!\,[(\lambda+p-l)/2+t-n]!}\frac{\Gamma((\lambda+p+l)/2+t+\frac{3}{2})}{\Gamma(t+p+\frac{3}{2})}$$
$$=\tfrac{1}{2}\Gamma(s+p+\tfrac{3}{2})(-1)^{(\lambda+p-l)/2}\sum_{t=0}^{s-n+(\lambda+p-l)/2}(-1)^t$$
$$\times\frac{(n+t)!\,\Gamma(n+l+t+\frac{3}{2})}{n!\,t!\,[(\lambda+p-l)/2+s-n-t]!\,[n+t-(\lambda+p-l)/2]!\,\Gamma(n+t+(p+l-\lambda)/2+\frac{3}{2})}. \tag{19}$$

In the last sum we introduced a new summation variable to involve only the terms with factorials of nonnegative arguments.

If we add the largest values of the summation variables,

$$\left(s_1 - n_1 + \frac{\lambda_1 + p_1 - l_1}{2}\right) + \left(s_2 - n_2 + \frac{\lambda_2 + p_2 - l_2}{2}\right) = n + N + \frac{L}{2} + \frac{l}{2} - n_1 - n_2 - \frac{l_1 + l_2}{2} = 0\,, \tag{20}$$

we find out that the sum must be zero according to the energy-conservation relation for TMB and the relation $\lambda_1 + \lambda_2 = l$ in formula (18) and according to relation (5). Because of their nonnegativity each of the parentheses in Eq. (20) must be zero. The sum (19) reduces to one term with $t_i = 0$ and also

$$s_i = n_i - \frac{\lambda_i + p_i - l_i}{2}. \tag{21}$$

We substitute the results obtained into Eq. (18) and write the final result:

$$\begin{aligned}\langle nl, NL, \lambda \,|\, n_1 l_1, n_2 l_2, \lambda\rangle_d &= (-1)^{\lambda + n + N + n_1 + n_2}(1+d)^{-l/2}(2l+1)(2L+1)^{1/2} \\ &\times \left[\frac{n_1!\, n_2!\, \Gamma(n+\frac{3}{2})\Gamma(n_1 + l_1 + \frac{3}{2})\Gamma(n_2 + l_2 + \frac{3}{2})}{\Gamma(n + l + \frac{3}{2})}\right]^{1/2} \sum_{p_i \lambda_i} (-1)^{\lambda_1 + (p_1 + p_2 + L)/2} d^{\lambda_1/2} \delta_{\lambda_1 + \lambda_2, l} \\ &\times \left[(2p_1+1)(2p_2+1)\binom{2l}{2\lambda_1}\right]^{1/2} (p_1 0, \lambda_1 0 \,|\, l_1 0)(p_2 0, \lambda_2 0 \,|\, l_2 0) \begin{Bmatrix} p_1 & p_2 & L \\ \lambda_1 & \lambda_2 & l \\ l_1 & l_2 & \lambda \end{Bmatrix} \\ &\times \left[\left(n_1 - \frac{\lambda_1 + p_1 - l_1}{2}\right)!\left(n_2 - \frac{\lambda_2 + p_2 - l_2}{2}\right)!\,\Gamma\left(n_1 + \frac{p_1 + l_1 - \lambda_1}{2} + \frac{3}{2}\right)\Gamma\left(n_2 + \frac{p_2 + l_2 - \lambda_2}{2} + \frac{3}{2}\right)\right]^{-1/2} \\ &\times \left\langle n0, NL, L \,\middle|\, \left(n_1 - \frac{\lambda_1 + p_1 - l_1}{2}\right)p_1, \left(n_2 - \frac{\lambda_2 + p_2 - l_2}{2}\right)p_2, L\right\rangle_d .\end{aligned} \tag{22}$$

This agrees with a similar formula in Ref. 4.

V. GENERAL FORMULA

Summing up the two formulas (11) and (22) we obtain the expression for the general TMB

$$\begin{aligned}\langle nl, NL, \lambda \,|\, n_1 l_1, n_2 l_2, \lambda\rangle_d &= (-1)^{n_1 + n_2 + n + \lambda}(1+d)^{-l/2} 2^{-1}(2l+1) \\ &\times \left[\frac{\pi\, n!\, n_1!\, n_2!\, \Gamma(n_1 + l_1 + \frac{3}{2})\Gamma(n_2 + l_2 + \frac{3}{2})}{N!\, \Gamma(n + l + \frac{3}{2})\Gamma(N + L + \frac{3}{2})}\right]^{1/2} \delta_{2n + l + 2N + L,\, 2n_1 + l_1 + 2n_2 + l_2} \\ &\times \sum_{p_i t_i \lambda_i} (-1)^{\lambda_1 + t_1 + t_2 + (p_1 + p_2 + L)/2} d^{t_2 + (\lambda_1 + p_2)/2} (1+d)^{-t_1 - t_2 - (p_1 + p_2)/2} \left[\binom{2l}{2\lambda_1}\right]^{1/2} \delta_{\lambda_1 + \lambda_2, l} \\ &\times (2p_1+1)(2p_2+1)(p_1 0, \lambda_1 0 \,|\, l_1 0)(p_2 0, \lambda_2 0 \,|\, l_2 0)(p_1 0, p_2 0 \,|\, L0) \begin{Bmatrix} p_1 & p_2 & L \\ \lambda_1 & \lambda_2 & l \\ l_1 & l_2 & \lambda \end{Bmatrix} \\ &\times \frac{\Gamma((p_1 + p_2 + L)/2 + t_1 + t_2 + \frac{3}{2})}{\Gamma(p_1 + t_1 + \frac{3}{2})\Gamma(p_2 + t_2 + \frac{3}{2})} \\ &\times \frac{[(p_1 + p_2 - L)/2 + t_1 + t_2]!}{t_1!\, t_2!\, [n_1 - (\lambda_1 + p_1 - l_1)/2 - t_1]!\, [n_2 - (\lambda_2 + p_2 - l_2)/2 - t_2]!\, [(p_1 + p_2 - L)/2 + t_1 + t_2 - N]!} .\end{aligned} \tag{23}$$

The values of the summation variables p_i, λ_i, t_i are restricted by the functions occurring in the sum: factorials, Clebsch-Gordan coefficients, 9-j symbols (and Kronecker δ's); only five of them are independent.

Now we compare expression (23) with the previous ones. In Kumar's formula[5] one sums over four pairs of quantum numbers n, l. Energy-conservation relations of the type (5), however, fix three out of the four nodal quantum numbers, which is not a symmetrical operation and thus was not carried out explicitly; that was done in various ways and on different footings in Refs. 6, 8, and 9. The sums over the remaining four angular momentum quantum numbers differ mainly in the sense that in Kumar's formula the 9-j symbol with the stretched configuration does not occur, in contrast to Eq. (23). In the important case of the quantities $n=0$, $l=0$, formula (23), or (11), agrees with the one given in Ref. 5 (see also Ref. 10).

Instead of the orthogonal transformation (3), generalized transformations of the U_2 group were investigated in Ref. 12, and again Kumar's form of TMB was obtained for the special transformation (3). It means that the specific form of formula (23) is closely tied to the (real and) orthogonal subgroup O_2 of the U_2 group.

Baranger and Davies apply different methods of calculating the Moshinsky brackets, i.e., TMB for $m_1=m_2$. Their alternative expression, for which the code was also written, contains the seven independent summation variables. When summed over two properly chosen variables it is very likely that expression (23) with $d=1$ will be obtained. A substantial simplification in expression (23) arises by use of the simple formula (11).

Programs of formula (23) were elaborated and examined by M. Gmitro and M. Sotona; in this context a useful formula for the calculation of the 9-j symbol with one stretched configuration[17] occurring also in Eq. (23) is to be mentioned. Their efficiency was compared with various similar programs.[3] As compared with Davies's widely used code, for example, the code corresponding to expression (23) was found to be roughly by a factor of 5 less time consuming for the $l=0$ TMB and by a factor of 2.5–4 less time consuming in the other cases.[18] It is also worth mentioning that formula (23) inclusive of the expression for the stretched 9-j symbol of Ref. 17 allows an easy computation of TMB tables in the form of powers of prime factors if only the mass fractions d are rational.

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