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Sufficient Conditions for Saturation of Nuclear Forces

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Conditions are given and proven that are sufficient to insure the saturation property of nuclear binding energies. Interactions mediated by two-body local static potentials are considered. For some classes of potentials these sufficient conditions are also necessary for saturation.

1. INTRODUCTION

Recently, a rigorous approach to the old problem[1,2] of the saturation property of nuclear binding energies has been revived, and necessary conditions for saturation have been given, i.e., inequalities that the nuclear interaction must satisfy to be consistent with asymptotic saturation.[3] These rigorous conditions can be used to demonstrate that a given model of the nuclear force is inconsistent with saturation, or to pinpoint which features of a possibly saturating nuclear interaction are essential to bring about saturation. A remarkable outcome of these investigations is the observation that current models of the nuclear force do not comply with the requirements of saturation in an entirely satisfactory manner.[3] Moreover, these results indicate that it is unlikely that the empirical saturation displayed by the nuclear binding energies could be reconciled with a nuclear interaction whose static part, considered by itself, does not saturate. This conclusion, coupled with the remark that the more usual and simple nonstatic interactions (e.g., a local spin-orbit force) are essentially incompatible with saturation,[3(c)-3(e)] focuses attention on the static part of the nuclear force, and thus justifies the fact that our consideration is hereafter limited to a nuclear interaction mediated by a two-body static (local) potential.

Two basic features can be responsible for saturation: the exchange character of the nuclear force, and its predominantly repulsive nature at short range. While either one of these features can produce saturation, it is presumably a combination of them that is operative in the nuclear case. It is, nonetheless, quite interesting, to the extent that it is feasible, to elucidate the role that each feature is playing; this can to some extent be done, for a given model of the nuclear force, by modifying it so as to emphasize one or the other feature, and then testing whether saturation is, or is not, achieved. As noted above, a useful instrument to carry out this program is (necessary) conditions that the nuclear force must satisfy in order to be consistent with saturation[3]; obviously no less useful is the complementary tool consisting of (sufficient) conditions such that if the nuclear interaction satisfies them, saturation is guaranteed. It is the purpose of this paper to report such conditions,[4] together with their proofs.

Clearly sufficient conditions for saturation provide, moreover, an important tool to facilitate the search for a satisfactory model of the nuclear force, a problem that lies at the core of nuclear theory. It is indeed hoped that in the future such constraints, which display some of the information originating from the many-nucleon problem, will be taken into account by the researchers who produce models of the nucleon-nucleon force fitting the experimental data for the (bound and scattering) two-nucleon system.

As is well known, sufficient conditions for saturation are generally more difficult to obtain than necessary conditions, because they require the establishment of a *lower* (rather than an *upper*) bound for the ground-state energy of the many-nucleon system. Indeed, in nuclear physics the only result of this kind is the classical theorem, proved a generation ago by Wigner, which states that certain forces having a rather special exchange character [pure Majorana or pure Heisenberg, possibly mixed with a limited amount of nonexchange (Wigner) force] possess the property of saturation, provided the corresponding potentials belong to a class that includes only functions having a definite sign for all values of r.[1] Another, much more recent, development in this direction is the work by Fisher, Ruelle, and others on the stability problem in statistical mechanics.[5,6] These researches, while originally carried out in a somewhat different physical context, can rather easily be generalized and extended so that they apply to nuclear physics (indeed they could be taken over directly, were it not for the fact that Fisher and Ruelle,[5] although considering particles of different species, did not include exchange forces in their treatment).

In this paper, we use both the approach of Wigner[1] and that of Fisher and Ruelle[7] to obtain results relevant to nuclear physics. The first results consist of a generalization of the theorem of Wigner to more general mixtures of exchange forces. While the final sufficient conditions for saturation obtained in this manner are rather simple, their proof involves considerable algebraic complications, and requires some group-theoretical results relative to the characters of certain representations of the permutation group[8] that were not available when Wigner wrote his paper. The second results are obtained in a rather straightforward manner applying the technique of Fisher and Ruelle[7] to the nuclear interaction. Roughly speaking, the results of the first kind are more suited to test a model of the nuclear interaction that saturates due to the exchange nature of its forces; indeed, they are applicable only to potentials that do not change sign. The results of the second kind are instead applicable also to potentials that do change sign (and that may be quite different in each spin-isospin state), and are therefore more appropriate to test an interaction that saturates mainly because of the repulsive character of the forces at short range. Thus, these two sets of sufficient conditions for saturation are to some extent complementary, as witnessed by the existence of (saturating) models of the nuclear force that satisfy one set of conditions, but not the other. Moreover, they can be used in conjunction, as explained below.

In the following section a number of preliminaries are dealt with, including a precise definition of saturation. In Sec. 3, the generalized theorem of Wigner is proved, and in Sec. 4 the set of sufficient conditions for saturation is established that follows from a straightforward extension of the Fisher-Ruelle method.[7] Section 5 contains a statement of the final theorem that embodies the results proved in the previous sections. It has been written having in mind the convenience of a reader who is interested in using the results, but not in checking their proof; thus, it is completely self-contained (at the cost of some repetitiveness), and it can be read directly after Sec. 2. The concluding section contains also a discussion of the results, including a comparison with the necessary conditions for saturation. Some mathematical details are confined to the Appendix.

2. PRELIMINARIES AND DEFINITIONS

We write the nuclear interaction acting between the ith and jth nucleon in the form:

$$\begin{aligned} W_{ij} = {} & P_\sigma^{(+)}(i,j)P_\tau^{(+)}(i,j)U_{33}^C(r) + P_\sigma^{(+)}(i,j)P_\tau^{(-)}(i,j)U_{31}^C(r) \\ & + P_\sigma^{(-)}(i,j)P_\tau^{(+)}(i,j)U_{13}^C(r) \\ & + P_\sigma^{(-)}(i,j)P_\tau^{(-)}(i,j)U_{11}^C(r) \\ & + [P_\tau^{(+)}(i,j)U_{33}^T(r) + P_\tau^{(-)}(i,j)U_{31}^T(r)]S_T(i,j), \end{aligned} \tag{2.1a}$$

where $P_\sigma^{(\pm)}$ and $P_\tau^{(\pm)}$ are, respectively, the projection operators over the spin and isospin states (triplet or singlet) of the interacting nucleon pair, $\vec{r} = \vec{r}_i - \vec{r}_j$ is the internucleon distance, and S_T is the usual tensor operator

$$S_T(i,j) = -\vec{\sigma}_i \cdot \vec{\sigma}_j + 3(\vec{\sigma}_i \cdot \vec{r})(\vec{\sigma}_j \cdot \vec{r})/r^2 . \tag{2.2}$$

As is well known, this operator vanishes identically in the singlet spin state, and in the triplet spin state its expectation value ranges between the extrema −1 and 2, depending on the spatial configuration of the wave function. This same interaction can be written in either one of the following two equivalent ways:

$$W_{ij} = V_W^C(r) + V_B^C(r)P_\sigma(i,j) - V_H^C(r)P_\tau(i,j) - V_M^C(r)P_\sigma(i,j)P_\tau(i,j) + [V_W^T(r) - V_H^T(r)P_\tau(i,j)]S_T(i,j) \quad (2.1\text{b})$$

$$= V_d^C(r) + \vec{\sigma}_i\cdot\vec{\sigma}_j V_\sigma^C(r) + \vec{\tau}_i\cdot\vec{\tau}_j V_\tau^C(r) + (\vec{\sigma}_i\cdot\vec{\sigma}_j)(\vec{\tau}_i\cdot\vec{\tau}_j)V_{\sigma\tau}^C(r) + [V_d^T(r) + \vec{\tau}_i\cdot\vec{\tau}_j V_\tau^T(r)]\,S_T(i,j), \quad (2.1\text{c})$$

where $P_\sigma(i,j)$ and $P_\tau(i,j)$ are, respectively, the operators that exchange the spin and isospin coordinates of nucleons i and j. The connection between these different notations is implied by the relationships

$$P_\sigma^{(\pm)}(i,j) = \tfrac{1}{2}[1 \pm P_\sigma(i,j)] = \tfrac{1}{2} \pm \tfrac{1}{4}(1 + \vec{\sigma}_i\cdot\vec{\sigma}_j), \quad (2.3)$$

and by the analogous relationships for the isospin operators. Specifically:

$$V_W^C = \tfrac{1}{4}(U_{33}^C + U_{31}^C + U_{13}^C + U_{11}^C), \quad (2.4\text{a})$$

$$V_B^C = \tfrac{1}{4}(U_{33}^C + U_{31}^C - U_{13}^C - U_{11}^C), \quad (2.4\text{b})$$

$$V_H^C = \tfrac{1}{4}(-U_{33}^C + U_{31}^C - U_{13}^C + U_{11}^C), \quad (2.4\text{c})$$

$$V_M^C = \tfrac{1}{4}(-U_{33}^C + U_{31}^C + U_{13}^C - U_{11}^C), \quad (2.4\text{d})$$

$$V_W^T = \tfrac{1}{2}(U_{33}^T + U_{31}^T), \quad (2.4\text{e})$$

$$V_H^T = \tfrac{1}{2}(-U_{33}^T + U_{31}^T), \quad (2.4\text{f})$$

and

$$V_d^C = \tfrac{1}{16}(9U_{33}^C + 3U_{31}^C + 3U_{13}^C + U_{11}^C), \quad (2.5\text{a})$$

$$V_\sigma^C = \tfrac{1}{16}(3U_{33}^C + U_{31}^C - 3U_{13}^C - U_{11}^C), \quad (2.5\text{b})$$

$$V_\tau^C = \tfrac{1}{16}(3U_{33}^C - 3U_{31}^C + U_{13}^C - U_{11}^C), \quad (2.5\text{c})$$

$$V_{\sigma\tau}^C = \tfrac{1}{16}(U_{33}^C - U_{31}^C - U_{13}^C + U_{11}^C), \quad (2.5\text{d})$$

$$V_d^T = \tfrac{1}{4}(3U_{33}^T + U_{31}^T), \quad (2.5\text{e})$$

$$V_\tau^T = \tfrac{1}{4}(U_{33}^T - U_{31}^T). \quad (2.5\text{f})$$

The notation (2.1c) is employed in the treatise of Blatt and Weisskopf[2]; the notation (2.1b) is more old-fashioned, the potentials V_W, V_B, V_H, V_M being associated, respectively, with the names of Wigner, Bartlett, Heisenberg, and Majorana. We shall present our results using the notation (2.1a), not only because it is nowadays more fashionable, but because it has the distinct advantage of employing only positive definite spin-isospin operators (apart from the tensor operator that is intrinsically not positive definite), implying, therefore, validity of the following:

Remark. If the interaction W of Eq. (2.1a) saturates, so does any other interaction W′ such that the corresponding potentials satisfy, for all (nonnegative) values of r, the inequalities

$$U_{\sigma\tau}^{C\prime}(r) \geq U_{\sigma\tau}^C(r), \quad \sigma, \tau = 1, 3, \quad (2.6\text{a})$$

$$|U_{3\tau}^{T\prime}(r)| \leq |U_{3\tau}^T(r)|, \quad \tau = 1, 3. \quad (2.6\text{b})$$

Trivial as this result is, it is quite useful, as it may enlarge the scope of the sufficient conditions for saturation derived below. We shall return to this point in Sec. 5.

The restrictions on the potential functions that are required for the validity of the results given below shall be specified later. The only assumption that is understood hereafter is that all potentials vanish asymptotically at large interparticle separation, and that they are finite-valued functions for $r > 0$.

Throughout this paper the following definition of saturation is adopted. Let E_A be the ground-state energy of the system of A nucleons interacting via the nuclear interaction (2). The interaction is termed saturating if there exists a finite constant c, independent of A and such that, for all (positive) values of A,

$$E_A > -cA. \quad (2.7)$$

Of course this condition is restrictive only as far as the asymptotic value of E_A at large A is concerned (since c is an arbitrary constant), and it corresponds to the requirement that the binding energy $-E_A$ not grow faster than linearly with A. This definition of saturation (termed "asymptotic" saturation by Blatt and Weisskopf[2]) need not, of course, correspond to the "conditional" saturation manifested by the binding energies of real nuclei, which are always composed of a finite, if large, number of nucleons. The use of such a definition is the price that must be paid to obtain rigorous and simple results. It should, however, be emphasized that, far from being academic, the requirement of "asymptotic" saturation for the nuclear force is a very reasonable one, and, in fact, as A for heavy nuclei is indeed a large number, an asymptotically nonsaturating force (that typically produces a binding energy $-E_A$ that grows at large A quadratically rather than linearly in A) generally grossly overbinds heavy nuclei.

Another characteristic feature of the phenomenon of saturation is the lack of dependence of the (mean) density of the ground state of the many-nucleon system on the value of the nucleon number A at large A. Our analysis does not consider this facet of the problem; all we can say is that generally saturation of density and saturation of binding energy are associated, as they clearly originate from the same basic cause, namely the fact that, in the ground state of the many-nucleon system, each nucleon is close to, and interacts with, only a few other nucleons.

The nuclear interaction under consideration conserves the total spin, isospin, and angular momentum of the many-nucleon system, and therefore it should be considered separately in each of the sectors of the Hilbert space characterized by a given set of values of the corresponding quantum numbers (say, S, S_z, T, T_z, L, L_z). And indeed, even the very definition of saturation should include a specification of the sector of the Hilbert space it refers to; in fact, a nuclear interaction might saturate in one sector (for instance in the sector where the total spin S and isospin T are maximal, $S=T\approx\frac{1}{2}A\hbar$) and not saturate in another (for instance in the sector with $S=T\ll\frac{1}{2}A\hbar$). Although it would be easy to keep track of this refinement, we shall use the most stringent definition of saturation, corresponding to the requirement that the condition (2.7) holds in all sectors of the Hilbert space (or equivalently, for all possible values of the set of conserved quantities). Equivalently it may be stated that the ground-state energy E_A introduced above is the energy of the lowest possible state of the A-nucleon system, not the lowest state for some given value of the conserved quantities. The justification for this decision is that the cases of actual physical interest correspond to values of the conserved quantities S, S_z, T, T_z, L, L_z close to zero, and that the conditions of saturation valid in this sector coincide essentially with the conditions of saturation for the whole Hilbert state. This is due to the assumed asymptotic vanishing of the forces at large interparticle separation, which implies the possibility of forming clusters. It is then obviously impossible for an interaction to saturate in the state with, say, $S\approx 0$, and not in the state with $S\approx\frac{1}{2}\hbar A$; for, nonsaturation in the latter case implies for large A the existence of a state composed of $A/2$ nucleons and having $S\approx\frac{1}{4}\hbar A$ and with a binding energy growing more than linearly with A. However, two such clusters, kept apart so that they do not interact, can always combine into a nonsaturating state with $S\approx 0$. The converse, of course, is not true, but the case of a nuclear force that saturates only in states with S and/or T and/or L of the order of $\hbar A$ is not sufficiently interesting to justify a separate treatment. In any case the recovery of the results apposite to some of these cases from those given below is an easy task, as it requires only retracing the steps involved in the proofs given in the following sections.

The theoretical framework of our treatment is nonrelativistic quantum mechanics. However, because our purpose, as indicated by Eq. (2.7), is to obtain a lower bound for the ground-state energy of the many-nucleon system [or rather, to establish conditions that, if satisfied by the nuclear interaction (2.1), guarantee the existence of the lower bound (2.7)], we shall hereafter systematically neglect the kinetic energy part of the Hamiltonian, since it is an essentially positive operator, and limit our consideration to the potential energy. In this sense, therefore, the results that we give remain valid even for a classical system. A more picturesque, if less precise, way to phrase this point is to recall that both the Heisenberg uncertainty and the Pauli exclusion principles raise the ground-state energy of a system, and therefore favor saturation. Thus, for a given nuclear interaction, saturation in the classical case (i.e., with the requirements of the Heinsenberg and Pauli principles not enforced and the nucleons considered as pointlike particles obeying the equations of classical dynamics) implies saturation in the quantal case. The results given below have been obtained neglecting the Heisenberg principle; the Pauli principle has been used to obtain the results of Sec. 3, but not to obtain those of Sec. 4. It should, however, be noted that inclusion of these additional restrictions will not always permit the establishment of a less stringent set of sufficient conditions. This is indeed demonstrated by the fact that, in some cases (see below), the set of sufficient conditions for saturation that we derive also constitutes a set of necessary conditions for saturation.

3. GENERALIZED WIGNER THEOREM

In this section we consider the special class of nuclear interactions that obtains if in Eq. (2.1a)

$$U_{\sigma\tau}^{C,T}(r)=U_{\sigma\tau}^{C,T}u(r)\,, \tag{3.1}$$

with $U_{\sigma\tau}^{C,T}$ constant and with the function $u(r)$ characterized by the integral representation:

$$u(r)=\int_0^\infty d\rho\, g(\rho) f(r/\rho)\,, \tag{3.2}$$

where $g(\rho)$ is nonnegative and integrable (over the range 0 to ∞) but otherwise arbitrary, and $f(x)$ is the common volume of two spheres of unit radius whose centers are at a distance x:

$$f(x)=\tfrac{1}{12}\pi(16-12x+x^3)\theta(2-x)\,. \tag{3.3}$$

The principal restriction that this integral representation implies for $u(r)$ is that it is a positive semidefinite function.

For such a nuclear interaction we now prove that validity of the following set of inequalities:

$$9U_{33}+3U_{31}+3U_{13}+U_{11}\geq 0\,, \tag{3.4a}$$

$$3U_{33}+U_{31}\geq 0\,, \tag{3.4b}$$

$$3U_{33}+U_{13}\geq 0\,, \tag{3.4c}$$

$$3U_{33}+U_{11}\geq 0\,, \tag{3.4d}$$

$$U_{33}\geq 0\,, \tag{3.4e}$$

where

$$\begin{aligned} &U_{33}=U_{33}^C+\eta_{33}U_{33}^T\,, \quad U_{31}=U_{31}^C+\eta_{31}U_{31}^T\,, \\ &U_{13}=U_{13}^C\,, \quad U_{11}=U_{11}^C\,, \end{aligned} \tag{3.5}$$

and each of the constants η_{33} and η_{31} can take the values 2 or −1, is sufficient for saturation.

This set of conditions consists of 5 inequalities if only central forces are present, and of 14 inequalities if tensor forces are present in addition to central forces. A discussion of these conditions is given below, after their proof.

Our method of proof is closely patterned after Wigner's treatment.[1] Let $\Psi(\vec{r}_i, \sigma_i, \tau_i)$, $i=1, 2, \ldots, A$, be the wave function associated with the many-nucleon system; $\vec{r}_i, \sigma_i, \tau_i$ are the space, spin, and isospin coordinates of the ith nucleon. In order to fulfill the Pauli exclusion principle, the function $\Psi(\vec{r}_i, \sigma_i, \tau_i)$ must of course be antisymmetric under exchange of the (space, spin, and isospin) coordinates of any two nucleons.

It is convenient to begin the proof considering a (fictitious) nuclear interaction that differs from Eqs. (2.1) and (3.1) in two respects: We assume $u(r)=1$ (so that the potential functions are constant), and $U_{3\tau}^T=0$, $\tau=1,3$ (so that only central forces are present).

For central forces, the total spin S, its z component S_z, the total isospin T, and its z component T_z are conserved. Thus, each state of the many-nucleon system can be characterized by a given set of these quantum numbers.

The wave function of the many-nucleon system can be decomposed as follows:

$$\Psi(\vec{r}_i, \sigma_i, \tau_i)=\sum_{\lambda,\alpha} b_{\lambda\alpha}N_\lambda^{-1/2}\sum_{\mu}\Psi_{\tilde{\lambda}\tilde{\mu}}^{\alpha}(\vec{r}_i)\Phi_{\lambda\mu,\lambda_S\lambda_T}^{\alpha}(\sigma_i, \tau_i), \tag{3.6}$$

where the spin-isospin part can in its turn be written as follows:

$$\Phi_{\lambda\mu,\lambda_S\lambda_T}^{\alpha}(\sigma_i, \tau_i)=\sum_{\mu_S\mu_T} C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha\lambda\mu}\varphi_{\lambda_S\mu_S}(\sigma_i)\varphi_{\lambda_T\mu_T}(\tau_i)\,. \tag{3.7}$$

Here we have introduced the many-nucleon spin function $\varphi_{\lambda_S\mu_S}(\sigma_i)$, that corresponds to a definite Young tableau (characterized by the index λ_S) with Yamanuchi index μ_S. For a given spin S the index λ_S is fixed. The notation for the many-nucleon isospin function $\varphi_{\lambda_T\mu_T}(\tau_i)$ is analogous. The Clebsch-Gordan coefficients of the permutation group $C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha\lambda\mu}$ in Eq. (3.7) combine the spin and isospin functions into a spin-isospin function $\Phi_{\lambda\mu,\lambda_S\lambda_T}^{\alpha}(\sigma_i, \tau_i)$ with definite symmetry (characterized by the indices λ and μ) under simultaneous permutation of spin and isospin variables. The index α is a repetition index that distinguishes different possibilities of constructing the spin-isospin Young tableau characterized by the indices λ and μ from the two (spin and isospin) Young tableaux characterized, respectively, by the indices λ_S and λ_T. N_λ is the dimension of the representation characterized by the index λ. The totally antisymmetric wave function $\Psi(\vec{r}_i, \sigma_i, \tau_i)$ is expressed in Eq. (3.6) as a superposition (characterized by the coefficients $b_{\lambda\alpha}$) of products of the spin-isospin functions $\Phi_{\lambda\mu,\lambda_S\lambda_T}^{\alpha}(\sigma_i, \tau_i)$ times functions $\Psi_{\tilde{\lambda}\tilde{\mu}}^{\alpha}(\vec{r}_i)$ of the space coordinates that transform under permutation of these coordinates according to the representation of the permutation group of A particles conjugate to that characterized by the indices λ and μ. As usual, this representation is indicated by the indices $\tilde{\lambda}$ and $\tilde{\mu}$.

All the functions are assumed to be normalized, and the following orthogonality relations hold:

$$(\varphi_{\lambda_S\mu_S}, \varphi_{\lambda_S'\mu_S'})=\delta_{\lambda_S\lambda_S'}\delta_{\mu_S\mu_S'}\,, \tag{3.8a}$$

$$(\varphi_{\lambda_T\mu_T}, \varphi_{\lambda_T'\mu_T'})=\delta_{\lambda_T\lambda_T'}\delta_{\mu_T\mu_T'}\,, \tag{3.8b}$$

$$(\Phi_{\lambda\mu,\lambda_S\lambda_T}^{\alpha}, \Phi_{\lambda'\mu',\lambda_S\lambda_T}^{\alpha'})=\delta_{\alpha\alpha'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\,, \tag{3.8c}$$

$$(\Psi_{\lambda\mu}^{\alpha}, \Psi_{\lambda'\mu'}^{\alpha'})=c_{\alpha\alpha'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\,, \quad c_{\alpha\alpha}=1\,. \tag{3.8d}$$

The Clebsch-Gordan coefficients of the permutation group are real and satisfy the orthogonality conditions[9]:

$$\sum_{\mu_S,\mu_T} C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha\lambda\mu}C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha'\lambda'\mu'}=\delta_{\alpha\alpha'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\,, \tag{3.9a}$$

$$\sum_{\mu,\mu_S} C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha\lambda\mu}C_{\lambda_S\mu_S,\lambda_T'\mu_T'}^{\alpha'\lambda\mu}=\frac{N_\lambda}{N_{\lambda_T}}\delta_{\alpha\alpha'}\delta_{\lambda_T\lambda_T'}\delta_{\mu_T\mu_T'}\,, \tag{3.9b}$$

$$\sum_{\mu,\mu_T} C_{\lambda_S\mu_S,\lambda_T\mu_T}^{\alpha\lambda\mu}C_{\lambda_S'\mu_S',\lambda_T\mu_T}^{\alpha'\lambda\mu}=\frac{N_\lambda}{N_{\lambda_S}}\delta_{\alpha\alpha'}\delta_{\lambda_S\lambda_S'}\delta_{\mu_S\mu_S'}\,. \tag{3.9c}$$

These orthogonality conditions imply for the coefficients $b_{\lambda\alpha}$ the normalization condition:

$$(\Psi, \Psi)=\sum_{\lambda,\alpha}|b_{\lambda\alpha}|^2=1\,. \tag{3.10}$$

The ground-state energy E_A of the A nucleon system is given by

$$E_A=\min\left(\Psi, \left(\sum_{i=1}^{A}T_i+\sum_{i>j=1}^{A}W_{ij}\right)\Psi\right), \tag{3.11}$$

where the minimum is to be taken over all functions of the type (3.6). Of course, since total spin and isospin are conserved, one can restrict the minimization procedure only to the class of wave functions that have definite values of S, S_z, T, and T_z, and the corresponding ground-state energy E_A is then that appropriate to the sector of Hilbert space characterized by the given values of S and T [it is clearly independent of S_z and T_z; indeed, degenerate with multiplicity $(2S+1)(2T+1)$]. Minimization over all the (allowed) values of S and T can be subsequently performed to obtain the ground-state energy for the whole Hilbert space, i.e., without specifying the values of the total spin and isospin.

In Eq. (3.11), W_{ij} is the potential energy of Eqs. (2.1) and T_i is the kinetic-energy operator for the ith nucleon. Since the kinetic energy is certainly positive, Eq. (3.11) implies for E_A the lower bound

$$E_A > \min(\overline{W}) = \min\left(\Psi, \sum_{i>j=1}^{A} W_{ij}\Psi\right). \qquad (3.12)$$

It is now convenient to use the notation (2.1b) for the potential energy, which under the assumptions introduced above reduces to

$$W_{ij} = V_W^C + V_B^C P_\sigma(i,j) - V_H^C P_\tau(i,j) - V_M^C P_\sigma(i,j) P_\tau(i,j), \qquad (3.13)$$

with V_W, V_B, V_H, V_M constant. Inserting this expression in Eq. (3.12) we get

$$\overline{W} = \tfrac{1}{2}A(A-1)V_W^C + V_B^C \sum_{i>j=1}^{A} (\Psi, P_\sigma(i,j)\Psi) - V_H^C \sum_{i>j=1}^{A} (\Psi, P_\tau(i,j)\Psi) - V_M^C \sum_{i>j=1}^{A} (\Psi, P_\sigma(i,j)P_\tau(i,j)\Psi). \qquad (3.14)$$

Let us now evaluate the expectation value of the operator of transposition of the spin coordinates $P_\sigma(i,j)$. Using Eq. (3.6) we get

$$(\Psi, P_\sigma(i,j)\Psi) = \sum b^*_{\lambda'\alpha'} b_{\lambda\alpha} (N_{\lambda'} N_\lambda)^{-1/2} C^{\alpha'\lambda'\mu'}_{\lambda_S\mu_S', \lambda_T\mu_T'} \times C^{\alpha\lambda\mu}_{\lambda_S\mu_S, \lambda_T\mu_T} (\varphi_{\lambda_S\mu_S'}, P_\sigma(i,j)\varphi_{\lambda_S\mu_S}) \times (\varphi_{\lambda_T\mu_T'}, \varphi_{\lambda_T\mu_T})(\Psi^{\alpha'}_{\tilde\lambda'\tilde\mu'}, \Psi^{\alpha}_{\tilde\lambda\tilde\mu}), \qquad (3.15)$$

the sum extending over α, α', λ, λ', μ, μ', μ_T, μ_T', μ_S, μ_S' [but not λ_S, λ_T, since clearly the total wave function Ψ is characterized by a fixed value of the total spin S and isospin T, and therefore by unique values of λ_S and λ_T (see above)].

Using the orthogonality relations given above, this reduces to

$$(\Psi, P_\sigma(i,j)\Psi) = \sum_{\lambda\alpha} |b_{\lambda\alpha}|^2 N_{\lambda_S}{}^{-1} \sum_{\mu_S} D^{\lambda_S}_{\mu_S\mu_S}(i,j), \qquad (3.16)$$

where

$$D^{\lambda_S}_{\mu_S'\mu_S}(i,j) = (\varphi_{\lambda_S\mu_S'}, P_\sigma(i,j)\varphi_{\lambda_S\mu_S}) \qquad (3.17)$$

is the matrix element of the permutation operator $P_\sigma(i,j)$ in the irreducible representation characterized by the index λ_S. To get Eq. (3.16) one must use Eq. (3.8b) to sum over μ_T', Eq. (3.8d) to sum over μ' and λ', and Eq. (3.9c) to sum first over μ and μ_T, and then over μ_S' and α'.

Note that only the diagonal elements $D^{\lambda_S}_{\mu_S\mu_S}(i,j)$ enter in this expression. One can now introduce the character

$$\chi^{\lambda_S} = \sum_{\mu_S} D^{\lambda_S}_{\mu_S\mu_S}(i,j), \qquad (3.18)$$

of the irreducible representation λ_S. As is well known, it is independent of the particular indices i, j. Thus one can write

$$\sum_{i>j=1}^{A} (\Psi, P_\sigma(i,j)\Psi) = \tfrac{1}{2}A(A-1)\sum_{\lambda\alpha} |b_{\lambda\alpha}|^2 \frac{\chi^{\lambda_S}}{N_{\lambda_S}}. \qquad (3.19)$$

In a completely analogous fashion one obtains

$$\sum_{i>j=1}^{A} (\Psi, P_\tau(i,j)\Psi) = \tfrac{1}{2}A(A-1)\sum_{\lambda\alpha} |b_{\lambda\alpha}|^2 \frac{\chi^{\lambda_T}}{N_{\lambda_T}}, \qquad (3.20)$$

and

$$\sum_{i>j=1}^{A} (\Psi, P_\sigma(i,j)P_\tau(i,j)\Psi) = \tfrac{1}{2}A(A-1)\sum_{\lambda\alpha} |b_{\lambda\alpha}|^2 \frac{\chi^{\lambda}}{N_{\lambda}}, \qquad (3.21)$$

where

$$\chi^\lambda = \sum_\mu (\varphi^\alpha_{\lambda\mu,\lambda_S\lambda_T}, P_\sigma(i,j)P_\tau(i,j)\varphi^\alpha_{\lambda\mu,\lambda_S\lambda_T}) \qquad (3.22)$$

is independent of α.

Inserting these expressions into Eq. (3.14) one obtains finally

$$\overline{W} = \tfrac{1}{2}A(A-1)\sum_\lambda |b_\lambda|^2 \left[V_W^C + V_B^C \frac{\chi^{\lambda_S}}{N_{\lambda_S}} - V_H^C \frac{\chi^{\lambda_T}}{N_{\lambda_T}} - V_M^C \frac{\chi^\lambda}{N_\lambda}\right], \qquad (3.23)$$

where we have defined

$$|b_\lambda|^2 = \sum_\alpha |b_{\lambda\alpha}|^2, \qquad (3.24)$$

and of course we have

$$\sum_\lambda |b_\lambda|^2 = 1. \qquad (3.25)$$

The sum over λ in this equation and in Eq. (3.23) extends over all the Young tableaux λ that can be formed with a spin Young tableau λ_S and an isospin Young tableau λ_T; in symbolic notation,

$$\{\lambda_S\} \otimes \{\lambda_T\} = \sum_\lambda \{\lambda\}. \qquad (3.26)$$

The characters appearing in Eq. (3.23) are known[8]:

$$\tfrac{1}{2}A(A-1)\frac{\chi^{\lambda_S}}{N_{\lambda_S}}=S(S+1)+\tfrac{1}{4}A(A-4)\,, \qquad (3.27a)$$

$$\tfrac{1}{2}A(A-1)\frac{\chi^{\lambda_T}}{N_{\lambda_T}}=T(T+1)+\tfrac{1}{4}A(A-4)\,, \qquad (3.27b)$$

$$\tfrac{1}{2}A(A-1)\frac{\chi^{\lambda}}{N_{\lambda}}=\tfrac{1}{2}\sum_{i=1}^{4}\lambda_i(\lambda_i-2i+1)\,. \qquad (3.28)$$

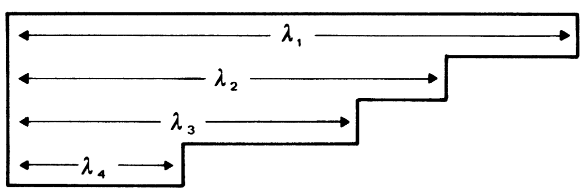
Here S and T are, respectively, the quantum numbers characterizing the total spin and total isospin [specifically, $\hbar^2 S(S+1)$ equals the square of the total spin]; and the four indices λ_i, satisfying the inequalities

$$\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1\,, \qquad (3.29)$$

and the normalization condition

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=A\,, \qquad (3.30)$$

characterize the length of the ith line of the spin-isospin Young tableau:



The Young tableaux λ that are compatible with a given total spin S and total isospin T are also known.[8] To characterize them it is convenient to introduce the three independent coordinates

$$\alpha=2(\lambda_1-\lambda_2)/A\,, \qquad (3.31a)$$

$$\beta=2(\lambda_2-\lambda_3)/A\,, \qquad (3.31b)$$

$$\gamma=2(\lambda_3-\lambda_4)/A\,, \qquad (3.31c)$$

that must satisfy the inequalities (independent of S and T)

$$\alpha\ge 0\,, \qquad (3.32a)$$

$$\beta\ge 0\,, \qquad (3.32b)$$

$$\gamma\ge 0\,, \qquad (3.32c)$$

$$\alpha+2\beta+3\gamma\le 2\,. \qquad (3.32d)$$

Then any Young tableau characterized by the four indices λ_i is compatible with total spin S and total isospin T provided the following five inequalities hold:

$$\beta+\tfrac{1}{2}(\alpha+\gamma)\ge y\,, \qquad (3.33a)$$

$$\beta+\min(\alpha,\gamma)\ge y-x\,, \qquad (3.33b)$$

$$\alpha+\beta+\gamma\ge x+y\,, \qquad (3.33c)$$

$$\beta+\tfrac{1}{2}(\alpha+\gamma)\ge x\,, \qquad (3.33d)$$

$$\beta+\min(\alpha,\gamma)\ge x-y\,. \qquad (3.33e)$$

Here we have introduced the two convenient variables

$$x=2S/A\,, \qquad (3.34a)$$

$$y=2T/A\,, \qquad (3.34b)$$

that are obviously restricted in the range

$$0\le x\le 1\,, \qquad (3.35a)$$

$$0\le y\le 1\,. \qquad (3.35b)$$

The inequalities (3.33) and (3.35) correspond, for a given Young tableau, i.e., for a given values of α, β, and γ, to the requirement that the two variables x and y of Eq. (3.34) lie inside or on the boundary of the heptagon $OPQRR'Q'P'$ in the xy plane, whose vertices have the coordinates

O: $x=0$, $y=0$,

P: $x=0$, $y=\beta+\min(\alpha,\gamma)$,

Q: $x=\tfrac{1}{2}|\alpha-\beta|$, $y=\beta+\tfrac{1}{2}(\alpha+\gamma)$,

R: $x=\tfrac{1}{2}(\alpha+\beta)$, $y=\beta+\tfrac{1}{2}(\alpha+\gamma)$,

with the coordinates P', Q', R' obtained from P, Q, R by exchanging x with y (see Fig. 1).

It is now convenient to write $\overline{W}$ in terms of these variables. Using Eqs. (3.23), (3.25), (3.27), (3.28), (3.31), and (3.34) we get

$$\overline{W}=\tfrac{1}{2}A^2F(x,y;\alpha,\beta,\gamma)-\tfrac{1}{2}A[V_W^C+(2-x)V_B^C-(2-y)V_H^C-\tfrac{1}{2}(8-3\alpha-4\beta-3\gamma)V_M^C]\,, \qquad (3.36)$$

where

$$F(x,y;\alpha,\beta,\gamma)=V_W^C+\tfrac{1}{2}(x^2+1)V_B^C-\tfrac{1}{2}(y^2+1)V_H^C-\tfrac{1}{4}[1+f(\alpha,\beta,\gamma)]V_M^C\,, \qquad (3.37)$$

and

$$f(\alpha,\beta,\gamma)=\tfrac{1}{4}[\alpha^2+\beta^2+\gamma^2+(\alpha+\beta)^2+(\beta+\gamma)^2+(\alpha+\beta+\gamma)^2]\,. \qquad (3.38)$$

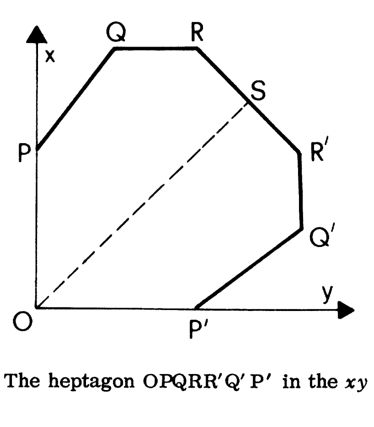


FIG. 1. The heptagon OPQRR′Q′P′ in the xy plane.

Now a lower bound to the ground-state energy E_A^{ST} of the many-nucleon system with given values of S and T (or, equivalently, x and y) is given by

$$E_A^{ST} > \min_{\alpha\beta\gamma}(\overline{W}), \tag{3.39}$$

the minimum being taken over all the values of α, β, and γ that are consistent with the inequalities (3.32) and (3.33). A lower bound to the ground-state energy E_A without restrictions on S and T is given by

$$E_A > \min_{ST}(E_A^{ST}) = \min_{\alpha\beta\gamma xy}(\overline{W}), \tag{3.40}$$

the minimum of $\overline{W}$ being now taken over all values of α, β, γ, x, and y consistent with the inequalities (3.32), (3.33), and (3.35).

It may be recalled at this point that the system under consideration is that of A nucleons interacting via the (fictitious) interaction (3.13). Clearly a sufficient condition for saturation for this system is given by the inequality

$$\min_{\alpha\beta\gamma xy}[F(x, y; \alpha, \beta, \gamma)] \geq 0\,. \tag{3.41}$$

Indeed, for the (fictitious) system under consideration, this condition[10] guarantees that, for large A, the ground-state energy of the system diverges quadratically to positive infinity. This anomalous result is due to the assumed constancy of the potentials, which do not vanish at large distances. In fact, the physical meaning of the above result (in the realistic case with nonconstant potential functions, to which we shall return below) is that if Eq. (3.41) holds,[10] a situation where all nucleons are so close to each other that every nucleon is within the interaction range of every other nucleon is energetically unfavored; namely the condition (3.41) implies that collapse is energetically disadvantageous.

In Appendix I we prove that the condition (3.41) is completely equivalent to the following set of five inequalities:

$$4V_W^C + 2V_B^C - 2V_H^C - V_M^C \geq 0\,, \tag{3.42a}$$

$$2V_W^C + 2V_B^C - V_H^C - V_M^C \geq 0\,, \tag{3.42b}$$

$$2V_W^C + V_B^C - 2V_H^C - V_M^C \geq 0\,, \tag{3.42c}$$

$$2V_W^C + V_B^C - V_H^C - 2V_M^C \geq 0\,, \tag{3.42d}$$

$$V_W^C + V_B^C - V_H^C - V_M^C \geq 0\,. \tag{3.42e}$$

The proof is accomplished by considering all the possible values that F, Eq. (3.37), can take, as α, β, γ, x, and y vary in their allowed ranges (3.32), (3.33), and (3.35), or equivalently, as the many-nucleon system assumes all possible spin-isospin configurations. It is of interest to note the five configurations that correspond to the five (extremal) conditions (3.42): Eq. (3.42a) obtains from the most antisymmetrical Young tableau for the spin-isospin wave function, characterized by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \tfrac{1}{4}A\,, \quad S = T = 0\,; \tag{3.43a}$$

Eq. (3.42b) obtains from

$$\lambda_1 = \lambda_2 = \tfrac{1}{2}A\,, \quad \lambda_3 = \lambda_4 = 0\,, \quad S = \tfrac{1}{2}A\,, \quad T = 0\,; \tag{3.43b}$$

Eq. (3.42c), from

$$\lambda_1 = \lambda_2 = \tfrac{1}{2}A\,, \quad \lambda_3 = \lambda_4 = 0\,, \quad S = 0\,, \quad T = \tfrac{1}{2}A\,; \tag{3.43c}$$

Eq. (3.42d) from

$$\lambda_1 = A\,, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0\,, \quad S = T = 0\,; \tag{3.43d}$$

and Eq. (3.42e) from

$$\lambda_1 = A\,, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0\,, \quad S = T = \tfrac{1}{2}A\,. \tag{3.43e}$$

Using the connection between the potentials V_W^C, V_B^C, V_H^C, and V_B^C and the potentials $U_{\sigma\tau}^C$, Eqs. (2.4), one can rewrite the conditions (3.42) in the following form:

$$9U_{33}^C + 3U_{31}^C + 3U_{13}^C + U_{11}^C \geq 0\,, \tag{3.44a}$$

$$3U_{33}^C + U_{31}^C \geq 0\,, \tag{3.44b}$$

$$3U_{33}^C + U_{13}^C \geq 0\,, \tag{3.44c}$$

$$3U_{33}^C + U_{11}^C \geq 0\,, \tag{3.44d}$$

$$U_{33}^C \geq 0\,. \tag{3.44e}$$

To complete the proof of the generalized Wigner theorem for central potentials one must now obtain, from the above result valid for constant potentials, a similar result valid for potentials of type (3.1)–(3.3). The procedure is identical to that given by Wigner,[1] and need not be repeated here.

The second extension required to complete the proof of the generalized Wigner's theorem, as stated at the beginning of this section, is to interactions containing a tensor component. This is quite trivial; indeed the result of Eqs. (3.4)–(3.5) obtains immediately from the above result and the remarks that the tensor potential operates only between nucleons in a triplet spin state, and that the extremal values of the tensor operator S_T are 2 and –1.

As the method of proof indicates, if no tensor potentials are present, validity of the five inequalities (3.4) is not only sufficient, but also necessary for saturation. This can be proved as follows. Suppose that one of the five conditions (3.4) is violated, say the inequality (3.4x). Assuming this, use the Raleigh-Ritz principle to obtain an upper bound for the ground-state energy of the many-nucleon system, employing a trial wave function whose spin-isospin structure is characterized by the Young tableau of Eq. (3.43x), and whose space structure represents A free nucleons all enclosed within a sphere whose radius is very small relative to the range of the forces. Then effectively in place of $u(r)$ in Eq. (3.1) we can substitute the constant $u(o)$ (note that this quantity cannot vanish, except in the trivial case of free nucleons), and therefore the evaluation of the expectation value of the potential energy, as performed in the proof of the generalized Wigner theorem, is now applicable, and yields a result that, at large A, tends to negative infinity proportionally to A^2. As for the expectation value of the kinetic energy, it cannot balance this behavior, because it grows only proportionally to $A^{5/3}$. We may therefore conclude that, in the case without tensor forces, if any one of the five inequalities (3.4) is violated, an upper bound to the ground-state energy E_A of the many-nucleon system can be established which at large A diverges quadratically to negative infinity; and this implies, of course, no saturation.

The fact that, if no tensor forces are present, the conditions (3.4) are both necessary and sufficient for saturation, indicates that for the class of potentials (3.1)–(3.3) these are the best conditions that can be given.

It is interesting to note that the set of "conditions for saturation" given in the literature (for the case without tensor forces)[2] differs from that given above. The difference consists in the replacement of Eq. (3.4d) with the inequality

$$5U_{33}^C + U_{31}^C + U_{13}^C + U_{11}^C \geq 0 \,. \tag{3.45}$$

The resulting set of conditions is less stringent than that obtained here, Eqs. (3.44), since the inequality (3.45) is implied by (3.44a) and (3.44d). Thus, the assertion that the set (3.44a), (3.44b), (3.44c), (3.45), (3.44e) provides necessary conditions for saturation is correct, although these conditions are not the most stringent that can be given[11]; the assertion that it provides sufficient conditions for saturation is incorrect,[12] as demonstrated by the existence of a stronger set of necessary conditions.

In the more general case with tensor forces, the 14 inequalities (3.4) have been obtained assuming that the tensor interaction can yield maximal attraction for every nucleon pair occuring in the triplet spin state. While this assumption is certainly adequate to provide sufficient conditions for saturation, it is not likely to yield the best possible conditions. Indeed, one might conjecture that a less stringent, but still sufficient, set of conditions would obtain performing the substitution (3.5) only in the inequalities (3.4b) and (3.4e), setting $U_{\sigma\tau} = U_{\sigma\tau}^C$ in the other three inequalities [that originate from configurations with vanishing total spin; see Eqs. (3.43)], and moreover restricting the two constants η_{33} and η_{31} to coincide, i.e., to both take either the value 2 or the value −1. The proof or disproof of this conjecture does not appear to be an entirely trivial exercise.

The conditions for saturation provided by the generalized Wigner's theorem refer only to the very special class of nuclear interactions characterized by the potentials (3.1)–(3.3). Although the very special character of this interaction corresponding to the existence of one and the same potential functions for all the spin-isospin states can be relaxed using the Remark of Sec. 2, the fact that the function $u(r)$ of Eqs. (3.1)–(3.2) is positive semidefinite is a major limitation, and in particular it excludes the possibility of treating the case of nuclear interactions that saturate mainly because of the presence of repulsion at short range, rather than because of the exchange nature of the forces. In the following section a different set of sufficient conditions for saturation is derived that are especially suited to test interactions where the repulsive nature of the forces at short range plays an important role in securing saturation. As discussed in Sec. 5, these conditions are complementary to those proved here, and can be used in conjunction with them.

4. RESULTS YIELDED BY THE FISHER-RUELLE APPROACH

We return now to the consideration of the most general (static) nuclear interaction, Eqs. (2.1). We assume, however, that the potential functions $U_{\sigma\tau}^{C,T}(r)$ are finite (nondivergent) at $r=0$, and that their Fourier transforms,

$$\hat{U}_{\sigma\tau}^{C,T}(p) = \int d\vec{r}\, e^{-i\vec{p}\cdot\vec{r}}\, U_{\sigma\tau}^{C,T}(r) \tag{4.1a}$$

$$= 4\pi \int_0^\infty dr\, r^2 \frac{\sin(pr)}{pr}\, U_{\sigma\tau}^{C,T}(r) \,, \tag{4.1b}$$

are also finite for all nonnegative values of p. The extent to which these restrictions can be relaxed is discussed at the end of this section.

It is actually more convenient, in this section, to work with the potentials of Eq. (2.1c). We accordingly introduce their Fourier transforms writing

$$V_\nu^{C,T}(r) = (2\pi)^{-3} \int d\vec{p}\, e^{i\vec{p}\cdot\vec{r}}\, \hat{V}_\nu^{C,T}(p) \,, \tag{4.2}$$

where ν stands for d, σ, τ, or $\sigma\tau$ in the central case, and for d or τ in the tensor case. Of course, the functions $\hat{V}_\nu^{C,T}(p)$ are related to the functions $\hat{U}_{\sigma\tau}^{C,T}(p)$ by Eqs. (2.5). It is also convenient to introduce the modified Fourier transforms $\check{V}_\nu^T(p)$

through

$$r^{-2}V_v^T(r)=(2\pi)^{-3}\int d\vec{p}\,e^{i\vec{p}\cdot\vec{r}}\,\check{V}_v^T(p). \qquad (4.3)$$

These modified Fourier transforms are more convenient for expressing the tensor operator, since clearly

$$V_v^T(r)S_T=(2\pi)^{-3}\int d\vec{p}\,e^{i\vec{p}\cdot\vec{r}}\sigma_m\sigma_n\Delta_{mn}\check{V}_v^T(p), \qquad (4.4)$$

where

$$\Delta_{mn}=-3\frac{\partial}{\partial p_m}\frac{\partial}{\partial p_n}+\delta_{mn}\Delta_p. \qquad (4.5)$$

Here and in the following equations the indices m and n are used to label the three Cartesian coordinates of a vector, and the sum over these indices is understood whenever they are repeated. To obtain Eq. (4.4) an integration by parts has been performed. Δ_p is the Laplace operator acting in p space. Clearly the relation between the usual Fourier transforms of Eq. (4.2) and the modified Fourier transforms of Eq. (4.3) is

$$\Delta_p\check{V}_v^T(p)=-\hat{V}_v^T(p). \qquad (4.6)$$

We now apply the Fisher-Ruelle technique. Let Ψ be the ground-state wave function of the A-nucleon system, so that

$$E_A=\left(\Psi,\left(\sum_{i=1}^{A}T_i+\sum_{i>j=1}^{A}W_{ij}\right)\Psi\right), \qquad (4.7)$$

with W_{ij} defined by Eqs. (2.1). Using the positivity of the kinetic energy we obtain from this formula the inequality

$$E_A>\left(\Psi,\sum_{i>j=1}^{A}W_{ij}\Psi\right). \qquad (4.8)$$

This can be rewritten in the form[13]

$$E_A>\tfrac{1}{2}\left(\Psi,\sum_{i=1}^{A}\sum_{j=1}^{A}W_{ij}\Psi\right)-\tfrac{1}{2}A(\Psi,W_{ii}\Psi), \qquad (4.9)$$

where we have used the fact that the diagonal terms,

$$W_{ii}=V_d^C(0)+3V_\sigma^C(0)+3V_\tau^C(0)+9V_{\sigma\tau}^C(0), \qquad (4.10)$$

are independent of the particle index i. It is therefore clear that a sufficient condition for saturation is[13]

$$\left(\Psi,\sum_{i=1}^{A}\sum_{j=1}^{A}W_{ij}\Psi\right)\geq 0. \qquad (4.11)$$

Using the Fourier transforms (4.2) and (4.4), we can rewrite this inequality in the form

$$\int d\vec{p}\{(\Phi^d,\Phi^d)\hat{V}_d^C(p)+(\Phi_m^\sigma,\Phi_n^\sigma)[\delta_{mn}\hat{V}_\sigma^C(p)+\Delta_{mn}\check{V}_d^T(p)] + (\Phi_m^\tau,\Phi_m^\tau)\hat{V}_\tau^C(p)+(\Phi_{ml}^{\sigma\tau},\Phi_{nl}^{\sigma\tau}) \times[\delta_{mn}\hat{V}_{\sigma\tau}^C(p)+\Delta_{mn}\check{V}_\tau^T(p)]\}\geq 0, \qquad (4.12)$$

having introduced the 16 wave functions

$$\Phi^d=\sum_{i=1}^{A}e^{i\vec{p}\cdot\vec{r}_i}\Psi, \qquad (4.13a)$$

$$\Phi_m^\sigma=\sum_{i=1}^{A}e^{i\vec{p}\cdot\vec{r}_i}\sigma_{im}\Psi, \qquad (4.13b)$$

$$\Phi_m^\tau=\sum_{i=1}^{A}e^{i\vec{p}\cdot\vec{r}_i}\tau_{im}\Psi, \qquad (4.13c)$$

$$\Phi_{mn}^{\sigma\tau}=\sum_{i=1}^{A}e^{i\vec{p}\cdot\vec{r}_i}\sigma_{im}\tau_{in}\Psi. \qquad (4.13d)$$

These wave functions depend on the (space, spin, and isospin) coordinates of all the nucleons, and on the vector $\vec{p}$; the scalar products (Φ,Φ) depend, of course, only on the vector $\vec{p}$.

Clearly a sufficient condition for nonnegativity of the left-hand side of Eq. (4.12) is that, for all (nonnegative) values of p,

$$\hat{V}_d^C(p)\geq 0, \qquad (4.14a)$$

$$\hat{V}_\tau^C(p)\geq 0, \qquad (4.14c)$$

and that the two symmetric and real 3×3 matrices

$$\delta_{mn}\hat{V}_\sigma^C(p)+\Delta_{mn}\check{V}_d^T(p), \qquad (4.15a)$$

$$\delta_{mn}\hat{V}_{\sigma\tau}^C(p)+\Delta_{mn}\check{V}_\tau^T(p), \qquad (4.15b)$$

be nonnegative, i.e., that their three eigenvalues be nonnegative. These last two conditions are most easily displayed noting that, since they must be independent of the orientation of the Cartesian frame, this can be chosen so that $\vec{p}$ be parallel to one coordinate axis. Then these matrices diagonalize, and the conditions that they be positive correspond to the inequalities

$$\hat{V}_\sigma^C(p)-\eta\Delta_p\check{V}_d^T(p)\geq 0, \qquad (4.16a)$$

$$\hat{V}_{\sigma\tau}^C(p)-\eta\Delta_p\check{V}_\tau^T(p)\geq 0, \qquad (4.16b)$$

where η can take the values 2 and -1. Using finally the relationship (4.6) these can be rewritten in the form

$$\hat{V}_\sigma^C(p)+\eta\hat{V}_d^T(p)\geq 0, \qquad (4.14b)$$

$$\hat{V}_{\sigma\tau}^C(p)+\eta\hat{V}_\tau^T(p)\geq 0. \qquad (4.14d)$$

The inequalities (4.14a)–(4.14d) are the final set of sufficient conditions for saturation. In terms of the potential $U^{C;T}_{\sigma\tau}$ they can be rewritten in the form

$$9\hat{U}^C_{33}(p)+3\hat{U}^C_{31}(p)+3\hat{U}^C_{13}(p)+\hat{U}^C_{11}(p)\geq 0\,, \tag{4.17a}$$

$$3\hat{U}^C_{33}(p)+\hat{U}^C_{31}(p)-3\hat{U}^C_{13}(p)-\hat{U}^C_{11}(p)+4\eta[3\hat{U}^T_{33}(p)+\hat{U}^T_{31}(p)]\geq 0\,, \tag{4.17b}$$

$$3\hat{U}^C_{33}(p)-3\hat{U}^C_{31}(p)+\hat{U}^C_{13}(p)-\hat{U}^C_{11}(p)\geq 0\,, \tag{4.17c}$$

$$\hat{U}^C_{33}(p)-\hat{U}^C_{31}(p)-\hat{U}^C_{13}(p)+\hat{U}^C_{11}(p)+4\eta[\hat{U}^T_{33}(p)-\hat{U}^T_{31}(p)]\geq 0\,, \tag{4.17d}$$

the constant η taking the values 2 and -1.

The theorem we have just proved asserts that validity of all these inequalities for all (nonnegative) values of p is a sufficient condition for saturation for any nuclear interaction of type (2.1), such that the functions $U^{C;T}_{\sigma\tau}(r)$ are finite at $r=0$ and that their Fourier transforms (4.1) are defined (nondivergent) for all nonnegative values of p.

The results can be extended to functions that are not finite at the origin and/or that vanish at infinity more slowly than r^{-3}, so that their Fourier transforms do not exist (the integrals defining them diverge). We refer to the literature for a more precise treatment of this point,[14] limiting here our discussion to a qualitative analysis and a statement of the results more relevant to the nuclear case.

If the potential functions vanish faster than r^{-3} at infinity, then validity of the conditions (4.17) is sufficient for saturation. This statement refers both to the case of potentials that are less divergent at the origin than r^{-3}, so that their Fourier transforms are well defined and finite for all values of p, and to the case of potentials that diverge at the origin faster than r^{-3}, so that the corresponding Fourier transforms are divergent. In the latter case the condition to be verified is that the left-hand sides of Eqs. (4.17), if unbounded, diverge to positive infinity.

If instead the Fourier transforms diverge due to an asymptotic vanishing of the potential functions that is slower than r^{-3}, then even if the left-hand sides of Eqs. (4.17) diverge to positive infinity, saturation is not guaranteed (as is physically clear, since an interaction repulsive at long range and attractive at short range generally does not saturate). In such a case significant conditions can be recovered eliminating the long-range slowly-vanishing repulsion by means of the Remark of Sec. 2.

Attention should be called to the special role played by the central potential U^C_{33} acting in the triplet odd state. This is, of course, due to the fact that this is the potential that acts between identical particles. For instance, it is clear that if this potential is altogether missing, the sufficient conditions (4.17) cannot be satisfied. This is as it should be, for it can be proved that in this case saturation is indeed essentially unattainable.[15]

The fact that in some of the inequalities (4.17) some potentials enter with a negative sign is surprising, since it might appear to imply that the addition of a purely repulsive potential makes the fulfillment of the conditions (4.17) more difficult. But it should not be forgotten that positivity of a function does not necessarily imply positivity of its Fourier transform. It is, however, true that the method used to obtain Eqs. (4.17) does not necessarily imply that these are the best conditions of this kind that could be obtained. In fact, in the step leading from Eq. (4.12) to Eqs. (4.14), the implicit assumption has been used that the 16 wave functions (4.13) can be varied independently. Thus, Eqs. (4.14) certainly imply Eqs. (4.12) (and are therefore sufficient for saturation), but the converse need not be true. It is clear that the conditions of Eqs. (4.17) are not necessary for saturation; it is easy to invent a model of the nuclear interaction that violates them and yet saturates (see below). However, for special classes of nuclear interactions (see below), validity of the inequalities (4.17) is indeed both sufficient and necessary for saturation.

5. SUMMARY OF RESULTS AND DISCUSSION

In this section we summarize and combine[15] the results of the previous sections, and make the following statement:

Theorem: Consider a (two-body static local) nuclear interaction that may be decomposed in the following way:

$$W={}^{(1)}W+{}^{(2)}W+{}^{(3)}W\,, \tag{5.1}$$

with

$${}^{(i)}W=P^{(+)}_\sigma P^{(+)}_\tau\,{}^{(i)}U^C_{33}(r)+P^{(+)}_\sigma P^{(-)}_\tau\,{}^{(i)}U^C_{31}(r)+P^{(-)}_\sigma P^{(+)}_\tau\,{}^{(i)}U^C_{13}(r)+P^{(-)}_\sigma P^{(-)}_\tau\,{}^{(i)}U_{11}(r)+[P^{(+)}_\tau\,{}^{(i)}U^T_{33}(r)+P^{(-)}_\tau\,{}^{(i)}U^T_{31}(r)]S_T\,, \tag{5.2}$$

where $\vec{r}$ is the internucleon distance, $P^{(\pm)}_\sigma$ and $P^{(\pm)}_\tau$ are, respectively, the projection operators over the spin and isospin states of the interacting nucleon pair (+ = triplet, − = singlet), and S_T is the usual tensor operator,

$$S_T=-\vec{\sigma}_1\cdot\vec{\sigma}_2+3(\vec{r}\cdot\vec{\sigma}_1)(\vec{r}\cdot\vec{\sigma}_2)/r^2\,, \tag{5.3}$$

that vanishes identically in the singlet spin state and whose expectation value ranges between −1 and +2.

Let the potential functions ${}^{(1)}U^{C,T}_{\sigma\tau}(r)$ *satisfy, for all (nonnegative) values of r, the inequalities*

$$ {}^{(1)}U^{C}_{\sigma\tau}(r)+\eta_{\sigma\tau}\,{}^{(1)}U^{T}_{\sigma\tau}(r)\geq 0\,, \tag{5.4}$$

where each of the constants η_{33} *and* η_{31} *can take the values −1 or +2, while* $\eta_{13}=\eta_{11}=0$.

Let the potential functions ${}^{(2)}U^{C,T}_{\sigma\tau}(r)$ *have the special form,*

$$ {}^{(2)}U^{C,T}_{\sigma\tau}(r)={}^{(2)}U^{C,T}_{\sigma\tau}u(r)\,, \tag{5.5}$$

the constants ${}^{(2)}U^{C,T}_{\sigma\tau}$ *satisfying the 14 inequalities*

$$9\,{}^{(2)}U^{C}_{33}+3\,{}^{(2)}U^{C}_{31}+3\,{}^{(2)}U^{C}_{13}+{}^{(2)}U^{C}_{11} +9\eta_{33}\,{}^{(2)}U^{T}_{33}+3\eta_{31}\,{}^{(2)}U^{T}_{31}\geq 0\,, \tag{5.6a}$$

$$3\,{}^{(2)}U^{C}_{33}+{}^{(2)}U^{C}_{31}+3\eta_{33}\,{}^{(2)}U^{T}_{33}+\eta_{31}\,{}^{(2)}U^{T}_{31}\geq 0\,, \tag{5.6b}$$

$$3\,{}^{(2)}U^{C}_{33}+{}^{(2)}U^{C}_{13}+3\eta_{33}\,{}^{(2)}U^{T}_{33}\geq 0\,, \tag{5.6c}$$

$$3\,{}^{(2)}U^{C}_{33}+{}^{(2)}U^{C}_{11}+3\eta_{33}\,{}^{(2)}U^{T}_{33}\geq 0\,, \tag{5.6d}$$

$${}^{(2)}U^{C}_{33}+\eta_{33}\,{}^{(2)}U^{T}_{33}\geq 0\,, \tag{5.6e}$$

where each of the constants η_{33} *and* η_{31} *can take the values −1 or 2, and the function u(r) being expressible by the integral representation*

$$u(r)=\int_{r/2}^{\infty}d\rho\;g(\rho)f(r/\rho)\,, \tag{5.7}$$

where g(ρ) is nonnegative and integrable,

$$g(\rho)\geq 0\,,\quad \int_0^{\infty}d\rho\;g(\rho)<\infty\,, \tag{5.8}$$

but otherwise arbitrary, and f(x) is the common volume of two spheres of unit radius whose centers are at a distance x,

$$f(x)=\tfrac{1}{12}\pi(16-12x+x^3)\theta(2-x)\,. \tag{5.9}$$

[*Note that the integral representation (5.7), (5.8) implies that u(r) is positive definite and that it vanishes asymptotically.*]

Let the potential functions ${}^{(3)}U^{C,T}(r)$ *vanish asymptotically faster than* r^{-3},

$$\lim_{r\to\infty}[r^{3+\epsilon}U^{C,T}_{\sigma\tau}(r)]=0\,,\quad \epsilon>0 \tag{5.10}$$

and be finite valued for $r>0$. *Let their Fourier transforms,*

$${}^{(3)}\hat{U}^{C,T}_{\sigma\tau}(p)=\int d\vec{r}\,e^{i\vec{p}\cdot\vec{r}}\,{}^{(3)}U^{C,T}_{\sigma\tau}(r) \tag{5.11a}$$

$$=4\pi\int_0^{\infty}dr\,r^2\frac{\sin(pr)}{pr}\,{}^{(3)}U^{C,T}_{\sigma\tau}(r)\,, \tag{5.11b}$$

satisfy for all (nonnegative) values of p the six inequalities

$$9\,{}^{(3)}\hat{U}^{C}_{33}(p)+3\,{}^{(3)}\hat{U}^{C}_{31}(p)+3\,{}^{(3)}\hat{U}^{C}_{13}(p)+{}^{(3)}\hat{U}^{C}_{11}(p)\geq 0\,, \tag{5.12a}$$

$$3\,{}^{(3)}\hat{U}^{C}_{33}(p)+{}^{(3)}\hat{U}^{C}_{31}(p)-3\,{}^{(3)}\hat{U}^{C}_{13}(p)-{}^{(3)}\hat{U}^{C}_{11}(p) +4\eta[3\,{}^{(3)}\hat{U}^{T}_{33}(p)+{}^{(3)}\hat{U}^{T}_{31}(p)]\geq 0\,, \tag{5.12b}$$

$$3\,{}^{(3)}\hat{U}^{C}_{33}(p)-3\,{}^{(3)}\hat{U}^{C}_{31}(p)+{}^{(3)}\hat{U}^{C}_{13}(p)-{}^{(3)}\hat{U}^{C}_{11}(p)\geq 0\,, \tag{5.12c}$$

$${}^{(3)}\hat{U}^{C}_{33}(p)-{}^{(3)}\hat{U}^{C}_{31}(p)-{}^{(3)}\hat{U}^{C}_{13}(p)+{}^{(3)}\hat{U}^{C}_{11}(p) +4\eta[{}^{(3)}\hat{U}^{T}_{33}(p)-{}^{(3)}\hat{U}^{T}_{31}(p)]\geq 0\,, \tag{5.12d}$$

where the constant η takes the values −1 and +2. [*Note that the Fourier transforms are not required to be finite; they can diverge, so long as their divergence does not violate the inequalities (5.12).*]

Then there is saturation, i.e., the ground-state energy E_A *of an assembly of A nucleons interacting via (5.1) satisfies the inequality*

$$E_A>-cA\,, \tag{5.13}$$

where c is a finite constant (independent of A).

This conclusion remains valid even if the interaction ${}^{(i)}W$ *does not satisfy the conditions stated above, but another interaction* ${}^{(i)}W'$ *exists, that satisfies the conditions stated above and such that, for all (nonnegative) values of r,*

$${}^{(i)}U^{C}_{\sigma\tau}(r)\geq {}^{(i)}U^{C\prime}_{\sigma\tau}(r)\,,\qquad |{}^{(i)}U^{T}_{3\tau}(r)|\leq|{}^{(i)}U^{T\prime}_{3\tau}(r)|\,. \tag{5.14}$$

This theorem states that a nuclear interaction saturates if it can be decomposed into three parts, each of which saturates: the first one trivially, as it corresponds to a purely repulsive interaction; the second, because of the generalized Wigner theorem (proved in Sec. 3); the third, because of the results obtained using the Fisher-Ruelle method[7] of Sec. 4.[16] It should be emphasized that these three criteria are not completely overlapping, as is demonstrated by the existence of (saturating) models of the nuclear force that satisfy one criterion, but violate the other two. We give below three such examples.

The interaction characterized by

$$\begin{aligned} U^{C}_{33}(r)&=U^{T}_{33}(r)=\theta(1-r)\,,\\ U^{C}_{31}(r)&=U^{T}_{31}(r)=2\theta(2-r)\,, \end{aligned} \tag{5.15}$$

all the other potentials being zero, belongs to the class ${}^{(1)}W$, but does not belong to either ${}^{(2)}W$ or ${}^{(3)}W$.[17]

The interaction characterized by

$$\begin{aligned} U^{C}_{33}(r)&=-U^{C}_{31}(r)=\theta(1-r)\,,\\ U^{C}_{13}(r)&=100\,\theta(1-r)\,, \end{aligned} \tag{5.16}$$

all the other potentials being zero, belongs to the class ${}^{(2)}W$, but does not belong to either ${}^{(1)}W$ or ${}^{(3)}W$.[17]

The interaction characterized by

$$U^C_{33}(r)=2e^{-r}-e^{-2r}\,, \tag{5.17}$$

all the other potentials being zero, belongs to the class ${}^{(3)}W$, but does not belong to either ${}^{(1)}W$ or ${}^{(2)}W$.[17]

It is interesting to compare these sufficient conditions for saturation with an equally rigorous set of necessary conditions.[3(a),3(d),3(e)] Considering for simplicity the case with central forces only, such a set is provided by the five inequalities[3(e)]

$$9\tilde{U}^C_{33}+3\tilde{U}^C_{31}+3\tilde{U}^C_{13}+\tilde{U}^C_{11}\geq 0\,, \tag{5.18a}$$

$$3\tilde{U}^C_{33}+\tilde{U}^C_{31}\geq 0\,, \tag{5.18b}$$

$$3\tilde{U}^C_{33}+\tilde{U}^C_{13}\geq 0\,, \tag{5.18c}$$

$$3\tilde{U}^C_{33}+\tilde{U}^C_{11}\geq 0\,, \tag{5.18d}$$

$$\tilde{U}^C_{33}\geq 0\,, \tag{5.18e}$$

where

$$\tilde{U}^C_{\sigma\tau}=\int\int d\vec{r}_1\,d\vec{r}_2\,\mu(\vec{r}_1)\mu(\vec{r}_2)U^C_{\sigma\tau}(|\vec{r}_1-\vec{r}_2|)\,, \tag{5.19}$$

the function $\mu(\vec{r})$ being nonnegative, but otherwise arbitrary. For instance, the especially simple choice

$$\mu(\vec{r})=(2p/\pi)^{3/2}e^{-2p^2r^2}\,, \tag{5.20}$$

yields

$$\tilde{U}^C_{\sigma\tau}(p)=\int d\vec{r}\,e^{-p^2r^2}U^C_{\sigma\tau}(r) \tag{5.21a}$$

$$=4\pi\int_0^\infty dr\,r^2e^{-p^2r^2}U^C_{\sigma\tau}(r)\,. \tag{5.21b}$$

Since validity of each of the five inequalities (5.18) is a necessary condition for saturation, this set of conditions can never be more stringent than the conditions, stated in the theorem given above, that are sufficient to guarantee saturation. However, for certain classes of interactions, the two sets of conditions coincide, and provide therefore a set of necessary and sufficient conditions for saturation. One such example (as already noted in Sec. 3) is an interaction of the type ${}^{(2)}W$. Another such example, belonging to the class ${}^{(3)}W$, obtains noting that, for $p=0$, both $\tilde{U}^C_{\sigma\tau}(p)$ of Eq. (5.21b) and $\hat{U}^{C;T}_{\sigma\tau}(p)$ of Eqs. (5.11) reduce to the volume integral of the corresponding potential:

$$\tilde{U}^C_{\sigma\tau}(0)=\hat{U}^C_{\sigma\tau}(0)=\int d\vec{r}\,U^C_{\sigma\tau}(r)\,. \tag{5.22}$$

Thus, for instance, for the special class of nuclear interactions containing only the (central) triplet odd potential $U^C_{33}(r)$,[18] with the additional restriction that this potential belong to the class of functions such that $\hat{U}^C_{33}(0)\geq 0$ implies $\hat{U}^C_{33}(p)\geq 0$ for all values of p,[19] it can be asserted that the condition that the volume integral of the potential $\hat{U}^C_{33}(0)$ be nonnegative is necessary and sufficient for saturation.

The conditions for saturation of nuclear forces summarized by the Theorem given above require some ingenuity in any practical test of the saturation properties of the static part of a given nuclear-force model. This follows because of the arbitrariness of the decomposition (5.1), and also because of the flexibility afforded by the last paragraph of the statement of the Theorem given above. General prescriptions cannot be given, but a useful rule of thumb is to begin by testing the validity of Eq. (5.12a), with ${}^{(3)}W=W$, obtaining in the process an estimate of the relative importance of the contributions of the four central potentials acting in the four spin-isospin states. It should be emphasized that the conditions of the Theorem, although they are not necessary for saturation in the case of the most general model of a (static) nuclear interaction, are by and large rather realistic, so that if a given model of the nuclear interaction cannot be made to comply with them, the most likely guess is that this happens because the interaction in question just does not saturate. To settle the matter, turning this conjecture into a proof, one must resort to the necessary conditions for saturation, Eqs. (5.18), trying to show that the interaction in question violates (at least one of) them. Here a certain amount of ingenuity is again called for, in view of the flexibility afforded by the arbitrariness in the choice of the (nonnegative) function $\mu(\vec{r})$ in Eq. (5.19).

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APPENDIX

In this Appendix we prove that validity of the five inequalities (3.42) is a necessary and sufficient condition for validity of the inequality (3.41), with the function F defined by Eqs. (3.37), (3.38) and with the five variables α, β, γ, x, and y constrained by the inequalities (3.32), (3.33), and (3.35).

It is clearly convenient to define

$$F(x,y;\alpha,\beta,\gamma)=V_W^C+\tfrac{1}{2}V_B^C-\tfrac{1}{2}V_H^C-\tfrac{1}{4}V_M^C+G(x,y;\alpha,\beta,\gamma)\,, \tag{A1}$$

with

$$G(x,y;\alpha,\beta,\gamma)=\tfrac{1}{2}V_B^C x^2-\tfrac{1}{2}V_H^C y^2-\tfrac{1}{4}V_M^C f(\alpha,\beta,\gamma)\,, \tag{A2}$$

and to search for the minimum of G.

We begin minimizing x and y for fixed α, β, and γ. The variables x and y are then constrained to lie inside or on the boundary of the heptagon of Fig. 1. Since everything is symmetrical in x and y (i.e., in spin and isospin), we need only consider the pentagon $OPQRS$, and then at the end symmetrize the results in S and T.

The simple dependence of the function G on x and y immediately implies that its minima cannot occur inside the pentagon $OPQRS$, but only on its boundary, and specifically only at the points O, R, Q or on the line PQ. These (possible) minima are (with obvious notation)

$$G_0=-\tfrac{1}{4}V_M^C f(\alpha,\beta,\gamma)\,, \tag{A3a}$$

$$G_R=-\tfrac{1}{4}V_M^C f(\alpha,\beta,\gamma)+\tfrac{1}{8}V_B^C(\alpha+2\beta+\gamma)^2-\tfrac{1}{8}V_H^C(\alpha+\gamma)^2\,, \tag{A3b}$$

$$G_Q=-\tfrac{1}{4}V_M^C f(\alpha,\beta,\gamma)+\tfrac{1}{8}V_B^C(\alpha+2\beta+\gamma)^2-\tfrac{1}{8}V_H^C(\alpha-\gamma)^2\,, \tag{A3c}$$

$$G_{PQ}=-\tfrac{1}{4}V_M^C f(\alpha,\beta,\gamma)-\frac{1}{2}\frac{V_B^C V_H^C}{V_B^C-V_H^C}[\beta+\min(\alpha,\gamma)]\,,$$

if

$$0\leqslant-\frac{V_B^C}{V_B^C-V_H^C}[\beta+\min(\alpha,\gamma)]\leqslant\tfrac{1}{2}|\alpha-\gamma|\,. \tag{A3d}$$

The minima of these four functions must now be investigated, as α, β, and γ vary in the ranges (3.32), i.e., in the $\alpha\beta\gamma$ space, inside or on the surface of the tetrahedron $OIJK$ (see Fig. 2). Since these are quadratic forms in α, β, γ, the minima can occur only at $\alpha=\beta=\gamma=0$ or on the triangle IJK.

Using these remarks and some geometrical considerations (and/or straightforward, if tedious, algebra), one can show that validity of the following 16 conditions is sufficient to insure that the function $F(x,y;\alpha,\beta,\gamma)$ be nonnegative (to simplify the writing, we replace here V_W^C with W, V_B^C with B, etc.):

$$W+\tfrac{1}{2}B-\tfrac{1}{2}H-\tfrac{1}{4}M\geqslant0\,, \tag{A4a}$$

$$W+\tfrac{1}{2}B-\tfrac{1}{2}H-M\geqslant0\,, \tag{A4b}$$

$$W+B-H-M\geqslant0\,, \tag{A4c}$$

$$W+B-\tfrac{1}{2}H-\tfrac{1}{2}M\geqslant0\,, \tag{A4d}$$

$$W+\tfrac{1}{2}B-H-\tfrac{1}{2}M\geqslant0\,, \tag{A4e}$$

$$C-\frac{1}{4}\frac{(B-H-M)M}{B-H-3M}\geqslant0\quad\text{if}\quad\tfrac{1}{3}\leqslant-\frac{M}{B-H-3M}\leqslant1\,, \tag{A5a}$$

$$C-\frac{1}{4}\frac{(2H+M)(B-M)}{B-4H-3M}\geqslant0\quad\text{if}\quad0\leqslant-\frac{2H+M}{B-4H-3M}\leqslant1\,, \tag{A5b}$$

$$C+\frac{1}{4}\frac{(2B-M)(H-M)}{4B-H-3M}\geqslant0\quad\text{if}\quad0\leqslant\frac{2B-M}{4B-H-3M}\leqslant1\,, \tag{A5c}$$

$$C+\tfrac{1}{16}(2B-2H-3M)(\alpha^2+\gamma^2)+\tfrac{1}{4}(2B-M)\beta(\alpha+\beta+\gamma)+\tfrac{1}{8}(2B-2H-M)\alpha\gamma\geqslant0$$
$$\text{if}\ \alpha=\frac{2H(2B-M)}{D}\geqslant0\,,\quad\beta=\frac{2M(H+B)}{D}\geqslant0\,,\quad\gamma=-\frac{2(2B-M)(H+M)}{D}\geqslant0\,, \tag{A5d}$$

$$C+\tfrac{1}{16}(2B-2H-3M)(\alpha^2+\gamma^2)+\tfrac{1}{4}(2B-M)\beta(\alpha+\beta+\gamma)+\tfrac{1}{8}(2B+2H-M)\alpha\gamma\geqslant0$$
$$\text{if}\ \alpha=-\frac{2H(2B-M)}{D}\geqslant0\,,\quad\beta=\frac{2B(2H+M)}{D}\geqslant0\,,\quad\gamma=-\frac{2(2B-M)(H+M)}{D}\geqslant0\,, \tag{A5e}$$

$$C-\tfrac{1}{8}M[\alpha^2+\gamma^2+\tfrac{1}{2}(\alpha+\gamma)^2]-\frac{1}{2}\frac{BH}{B-H}[\min(\alpha,\gamma)]^2\geqslant0$$
$$\text{with}\ \alpha=-\frac{2BH}{D_1},\quad\text{and if}\quad0\leqslant\gamma=-\frac{2M(B-H)}{D_1}\leqslant\frac{B-H}{B-2H}, \tag{A5f}$$

$$C-\tfrac{1}{16}M[\alpha^2+2\beta^2+2(\alpha+\beta)^2]-\frac{1}{2}\frac{BH}{B-H}\beta^2\geqslant0$$
$$\text{with}\ \alpha=-\frac{2BH}{D_2}\quad\text{and if}\quad0\leqslant\beta=-\frac{M(B-H)}{D_2}\leqslant\frac{H-B}{B}. \tag{A5g}$$

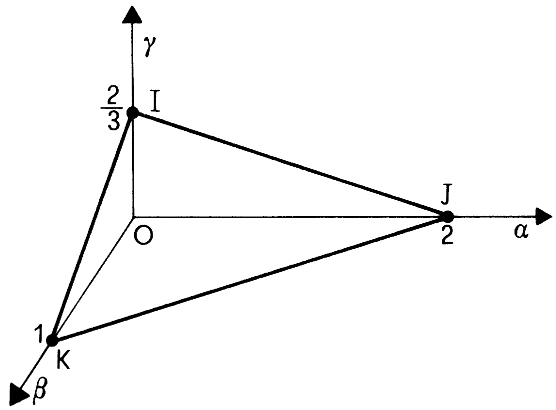

FIG. 2. The tetrahedron OIJK in the $\alpha\beta\gamma$ space.

In Eqs. (A5),

$$C = W + \tfrac{1}{2}B - \tfrac{1}{2}H - \tfrac{1}{4}M. \tag{A6}$$

In Eqs. (A5d) and (A5e),

$$D = 3M^2 - 4BM + 4HM - 4BH. \tag{A7}$$

In Eqs. (A5f) and (A5g),

$$D_1 = 3M(H - B) - BH, \tag{A8a}$$

$$D_2 = M(H - B) - BH. \tag{A8b}$$

Finally, four more conditions, hereafter referred to as Eqs. (A5h)–(A5m), obtain exchanging B and $-H$ in Eqs. (A5d)–(A5g).

The next step is to prove that the 11 inequalities (A5) are implied by the 5 inequalities (A4). The canonical technique to achieve this end is to find a linear combination *with nonnegative coefficients* of the left-hand sides of Eqs. (A4), that minorates or coincides with the left-hand sides of the Eq. (A5) under consideration. For instance, one may note that a restriction that is stronger than Eq. (A5a) reads

$$c_6 = C + \tfrac{1}{2}\lambda(B - H - M) \geq 0 \quad \text{if} \quad \tfrac{1}{6} \leq \lambda \leq \tfrac{1}{2}, \tag{A9}$$

and it is easily seen that this holds necessarily if Eqs. (A4) hold, since, if we indicate with c_i, $i = 1, 2, 3, 4, 5$ their left-hand sides, clearly

$$c_6 = \sum_{i=1}^{5} x_i c_i \tag{A10}$$

with

$$x_1 = 1 - 2\lambda, \quad x_2 = x_3 = 0, \quad x_4 = x_5 = \lambda, \tag{A11}$$

and these coefficients are all nonnegative for all values of λ in its allowed range.

Similar proofs of dependence can be performed for all the other inequalities (A5), although for the last ones they are not as elementary as the example given here. In any case the labor involved in this task is considerably less than that required to obtain the 16 inequalities (A4) and (A5).

It remains to check that the five conditions (A4), which have now been shown to be sufficient to guarantee nonnegativity of the function $F(x, y; \alpha, \beta, \gamma)$ when the variables are in the allowed region, are indeed independent. The most straightforward way to achieve this aim is to provide five set of values of the four potentials V_W, V_B, V_H, and V_M such that for each value one and only one of the five inequalities (A5) is violated. Such sets of values are, for instance, (−2, 1, −1, −2), (1, 2, −2, 4), (7, −4, 4, 0), (5, −8, 0, −4), (5, 0, 8, −4).

The inequalities (3.42) coincide[20] with the conditions (A4). As for the identification of the Young tableau that corresponds to each inequality, it obtains from the values of the variables x, y, α, β, and γ for which the function $F(x, y; \alpha, \beta, \gamma)$ takes the five extremal values that provide the left-hand sides of the five inequalities (A4) or, equivalently, (3.42) or (3.44).

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[7]We have borrowed this technique from the paper by Fisher and Ruelle (Ref. 5), and we therefore refer to it as the Fisher-Ruelle approach.

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[10]For simplicity, in the discussion we neglect the mar-

ginal case in which the equality sign holds in Eq. (3.41).

[11]An explanation of why a less stringent set of necessary conditions was previously obtained is given in Ref. 3(e), where the more stringent set is proved.

[12]The assertion in the treatise by Blatt and Weisskopf [Ref. 2, p. 150, after Eq. (4.40)] is actually incorrect on two accounts. The Wigner proof (Ref. 1) does not consider the most general mixture of exchange forces, and therefore it does not purport to provide any information for that case. Further, the set of conditions given is *not* sufficient for saturation.

[13]Here we use the assumption that the potential functions are finite (nondivergent) at zero separation.

[14]The discussion given in preceding papers (Ref. 6) can be extended without change to our case.

[15]Even if all the other potentials were characterized by hard cores, in this case there would be no saturation (unless all the potentials are everywhere attractive, in which case of course there is no binding whatsoever). This conclusion is implied by Theorem 1 of Ref. 3(a) [see also Ref. 3(e)].

[16]The possibility of combining the different conditions follows trivially from their proofs.

[17]Even if these classes are enlarged according to the prescription mentioned in the last part of the statement of the Theorem.

[18]This is of course by no means a realistic potential. In this connection, however, the possible usefulness of testing separate pieces of the nuclear force as regards their compatibility with saturation should be reemphasized [see Sec. 1 of this paper, and Refs. 3(c)–3(e)].

[19]Many potentials belong to this class, for instance the sum of two Yukawa or of two Gaussian functions (if the longer-range term is attractive).

[20]Except for their ordering, which in Eq. (3.42) has been chosen to correspond to the "traditional" one (Ref. 2).

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Scattering of Neutrons by Carbon from 3 to 7 MeV*

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Angular distributions of neutrons elastically scattered from natural carbon have been measured at 32 neutron energies between 3 and 7 MeV with energy spreads ranging from 37 to 165 keV. The angular distributions of neutrons inelastically scattered from ^{12}C have also been measured at the nine highest neutron energies. A phase-shift analysis of the elastic scattering cross sections yields the following spin and parity assignments for the excited states of ^{13}C (E_x,J): 8.3 MeV, $\frac{3}{2}^+$; 8.88 MeV, $\frac{1}{2}^-$; 9.50 MeV, $(\frac{1}{2}, \frac{3}{2})^-$; 9.90 MeV, $\frac{3}{2}^-$; 10.75 MeV, $\frac{7}{2}^-$; 11.00 MeV, $(\frac{1}{2}^+)$. Neutron polarizations calculated from the extracted phase shifts agree fairly well with most of the available polarization measurements. A polarization contour map is given showing that carbon may be useful as a polarization analyzer between 4.5 and 7 MeV.

I. INTRODUCTION

Measurements at many laboratories[1–5] of the ^{12}C + n total cross section in the energy range from 3.0 to 7.0 MeV reveal resonances at neutron bombarding energies of 3.5, 4.23, 4.93, 5.37, 6.29, and 6.6 MeV. These resonances correspond to excited states of ^{13}C with excitation energies E_x of 8.3, 8.86, 9.50, 9.90, 10.75, and 11.0 MeV, respectively. Differential cross sections for the elastic scattering of neutrons from ^{12}C have been measured and analyzed in the energy range from 2.4 to 3.65 MeV by Meier, Scherrer, and Trumpy (MST)[6]; from 1.5 to 4.1 MeV by Wills *et al.*[7]; and from 3.0 to 4.7 MeV by Lister and Sayres.[8]

Since the ground-state spin of ^{12}C is 0^+, the channel spin has only the value $S=\frac{1}{2}$, which simplifies the problem of performing the phase-shift analysis of differential cross sections. Analyses of these cross sections by the above three groups[6–8] show reasonable agreement with each other in the regions of overlap of bombarding energy and give a relatively consistent set of phase shifts for the whole energy range of the measurements. It is found from these analyses that the very broad level at $E_x = 8.3$ MeV in ^{13}C has a spin and parity of $\frac{3}{2}^+$. An assignment of $\frac{1}{2}^-$ for the level at 8.9 MeV is given by Lister and Sayres.[8] From the total-cross-section measurements of Fossan *et al.*,[1] spin limitations have been placed on the excited states between 9- and 11-MeV excitation as follows: $E_x = 9.50$ MeV, $J \geq \frac{1}{2}$; $E_x = 9.90$ MeV, $J \geq \frac{3}{2}$; $E_x = 10.75$ MeV, $J \geq \frac{7}{2}$.

Very recently Perey and Kinney[9] have reported ^{12}C(n, n)^{12}C angular distributions measured at eight bombarding energies between 4.6 and 7.0 MeV. Angular distributions of ^{12}C(n, n)^{12}C have also been measured with good energy resolution