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[32]There are well-known analytical expressions of the scattering length and effective range for a square-well potential. In the π^--mesonic problem it has been demonstrated for the s-wave interaction, where the local part of the potential is strongest, that the choice of the square-well potential form is practically immaterial as far as the potential plays the role of an intermediary between the scattering length and the energy-level shift and width [R. Seki, Phys. Rev. Letters 23, 1000 (1969)].

[33]For example, S. Frauchi, *Regge Poles and S-Matrix* (Benjamin, New York, 1963), p. 105.

[34]R. Seki, Phys. Rev. C 3, 454 (1971).

[35]We note that because of the arbitrary phase factor of the wave function ImU can have the oscillating behavior instead of ReU. Both cases correspond to the same physical state. D. Kelly, B. S. thesis, San Fernando Valley State College, 1971 (unpublished); M. Krell, private communication.

[36]We would like to thank Professor M. Krell for clarifying this point.

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Description of ^{4}He(d, d)^{4}He Polarization-Transfer Experiments*

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A convenient parametrization of experiments in which a polarized spin-1 particle bombards a target and a polarized spin-1 particle emerges is given. The parameters defined are a generalization of the "Wolfenstein parameters" long used in the description of spin-$\frac{1}{2}$ polarization-transfer experiments. A specific M matrix for a spin-0 target is used to find the relations between the various observables in several coordinate systems. The formalism is valid for the general $1+0 \rightarrow 1'+0'$ problem, where 1 and 1′ may indicate different spin-1 particles and 0 and 0′ may indicate different spin-0 particles, but where the product of initial channel and final channel parities is unchanged. The constraints imposed by time-reversal invariance when the reaction is specialized to elastic scattering are discussed. The prospects for a complete determination of the M matrix and for possible time-reversal tests are briefly touched upon.

I. INTRODUCTION

In this paper we will discuss the manner in which polarization-transfer experiments with a polarized spin-1 particle incoming and a polarized spin-1 particle outgoing may be parametrized. The parameters used have been given previously[1] and are analogous to the Wolfenstein parameters,[2] long used for the description of spin-$\frac{1}{2}$ to spin-$\frac{1}{2}$ polarization transfer. We restrict ourselves to the usual case where the product of initial intrinsic parities is the same as the product of the final intrinsic parities, which includes, of course, elastic scattering. We will then specialize to the case of an elastic scattering of spin-1 particles from a spin-0 target, as in $d+{}^4$He or in $\pi+d$ scattering. This study was motivated by the experimental studies of d-^{4}He polarization-transfer phenomena which are in progress at the Los Alamos Scientific Laboratory (LASL), by the general applicability of the formalism if target spin is neglected, as is done in most spin-1 optical-model studies, and by the intrinsic interest of this relatively simple spin system. A brief treatment of the related $1+1 \rightarrow 0+0$ problem has been given by Köhler and Fick.[3]

II. COORDINATE SYSTEMS

At the Third Polarization Symposium in 1970, certain conventions were adopted[4] for the parametrization of the simpler types of experiments involving spin-1 particles. The conventions did not include the polarization-transfer phenomena of interest in the present paper, but covered a special case of the more complex situation. In particular, for the rectangular tensor description of spin-1 polarization effects, which will be the basis of the present description, it was agreed that A's should be used for analyzing tensors and p's for polarizations of ensembles of particles. We will follow this notation as far as possible. We will use P's to indicate the polarization functions, that is, the polarization which would be produced by an unpolarized incident beam, and K's to indicate polarization-transfer coefficients of any rank.

The definition of the coordinate systems in terms of which the reaction is to be described is very important. Five distinct coordinate frames are of interest in this discussion.

First, we define the projectile helicity frame, with its y axis along $\hat{n}=(\vec{k}_{in}\times\vec{k}_{out})/|\vec{k}_{in}\times\vec{k}_{out}|$, its z axis along $\hat{k}_{in}=\vec{k}_{in}/|\vec{k}_{in}|$, and the x axis chosen to make a right-handed system. $\vec{k}_{in}$ and $\vec{k}_{out}$ are, of course, the incident and outgoing particle momentum vectors.

Second and third, we define the "ordinary" and "laboratory" helicity frames for the outgoing particle, in which the y axis is again along $\hat{n}$, but the z axis is along the unit vector $\hat{k}_{out}$(c.m.) or the unit vector $\hat{k}_{out}$(lab), respectively. Clearly the system using $\hat{k}_{out}$(lab) is the one with respect to which outgoing spin polarizations can be actually measured, but the system with z along $\hat{k}_{out}$(c.m.) has some convenient formal properties with respect to time reversal, as we shall see.

Fourth, consider the system with y still along $\vec{k}_{in}\times\vec{k}_{out}$, but with x along $\vec{k}_{out}-\vec{k}_{in}$ (we define a unit vector $\hat{k}_-$ parallel to this direction), and with z axis along $\hat{k}_+$ $(\equiv\hat{k}_-\times\hat{n})$. This system is inconvenient for the description of experiments, but will be useful in our discussion of time reversal. We refer to it as the "natural coordinate system," since nuclear-reaction models usually indicate a dependence on the momentum transfer $\vec{k}_{out}-\vec{k}_{in}$. Note that this coordinate system is the same whether we consider $\vec{k}_{out}$ and $\vec{k}_{in}$ to be lab or c.m. quantities.

Finally, we consider the coordinate system natural to a polarized beam from an ion source, where we assume a z axis along the axis of spin symmetry. We reserve the letters X, Y, Z for this system. In this system the polarized beam is completely characterized by its vector and tensor polarization, p_Z and p_{ZZ}, where we use the Goldfarb normalization of the Cartesian spin operators so that $-1<p_Z<1$ and $-2<p_{ZZ}<1$ (see Sec. IV and Appendix I).

We will need to describe such a polarized beam in the projectile helicity frame. Referring to Fig. 1, let β be the angle between the quantization axis and $\hat{k}_{in}$; i.e., $\cos\beta=\hat{S}\cdot\hat{k}_{in}$, where $\hat{S}$ is a unit vector along the beam quantization axis. Next, consider the projection of $\hat{S}$ on the x, y plane. The angle ϕ is defined to be the angle between this projection and the y axis. This can be expressed as $\cos\phi=(\hat{S}\times\hat{k}_{in})\cdot(\hat{n}\times\hat{k}_{in})$. The sense of ϕ is very important; it is assumed to be positive in the sense of a right-handed screw with x going into y (see Fig. 1). Thus, if the projection of $\hat{S}$ is parallel to the y axis, $\phi=0°$; if it is parallel to the x axis, $\phi=-90°$; if it is antiparallel to the y axis, $\phi=180°$; and if it is antiparallel to the x axis, $\phi=90°$. Since in our definition $\vec{k}_{out}$ is always in the x, z half plane with positive x, we summarize by saying that left, right, up, and down scattering, with respect to the vector $\hat{S}$, correspond to $\phi=0$, 180, -90, and 90°, respectively.

We can write the first-rank components of the beam polarization in the projectile helicity frame by resolving $\hat{S}$ into x, y, and z components and multiplying by the polarization magnitude, p_Z:

$$p_x=-p_Z\sin\beta\sin\phi\,,\quad p_y=p_Z\sin\beta\cos\phi\,,\quad p_z=p_Z\cos\beta\,. \tag{1}$$

The second-rank components are found to be[5]

$$\begin{aligned}
p_{xy}&=-\tfrac{3}{2}p_{ZZ}\sin^2\beta\cos\phi\sin\phi\,,\\
p_{yz}&=\tfrac{3}{2}p_{ZZ}\sin\beta\cos\beta\cos\phi\,,\\
p_{xz}&=-\tfrac{3}{2}p_{ZZ}\sin\beta\cos\beta\sin\phi\,,\\
p_{xx}&=\tfrac{1}{2}p_{ZZ}(3\sin^2\beta\sin^2\phi-1)\,,\\
p_{yy}&=\tfrac{1}{2}p_{ZZ}(3\sin^2\beta\cos^2\phi-1)\,,\\
p_{zz}&=\tfrac{1}{2}p_{ZZ}(3\cos^2\beta-1)\,.
\end{aligned} \tag{2}$$

All possible components of beam polarization in the projectile helicity frame may be prepared by suitable selection of p_Z, p_{ZZ}, β, and ϕ, although one may not always isolate a particular single component. Notice that $p_{xx}+p_{yy}+p_{zz}=0$, so that one quantity is redundant. The pair of components $\frac{1}{2}(p_{xx}-p_{yy})$ and p_{zz} are often convenient, where in terms of β and ϕ,

$$\tfrac{1}{2}(p_{xx}-p_{yy})=-\tfrac{3}{4}p_{ZZ}\sin^2\beta\cos2\phi\,. \tag{3}$$

However, this redundancy of the Cartesian description offers considerable flexibility which can often be exploited to the experimentalist's advantage.

III. FORM OF CROSS SECTIONS AND OUTGOING POLARIZATION EXPRESSIONS

In this section we make use of the fact that the cross section, $I(\theta)$, and polarization of a scattered

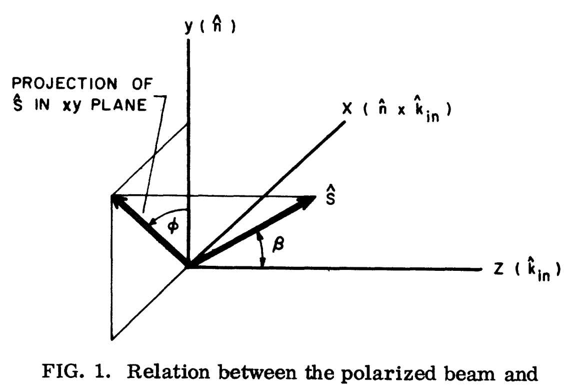

FIG. 1. Relation between the polarized beam and the projectile helicity coordinate systems.

spin-1 particle cannot depend on particular choices of coordinate systems, and therefore must have the form of invariant contractions of tensors. One set of tensors, denoted by p_j and p_{jk}, describe the incident polarized beam, and the other set, denoted by P's, A's, and K's as described below, are functions of the energy and of the angle θ between $\vec{k}_{in}$ and $\vec{k}_{out}$ (the scattering angle) and are characteristic of the nuclear interaction. We will specialize to experimentally convenient coordinate systems and deduce certain systematics associated with the observables for reflection of the scattering angle, $\theta \to -\theta$, for conservation of parity, and for time-reversal invariance.

The general form[1] of the final polarization (described in the x', y', z' system) in terms of the initial polarization (described in the x, y, z system), for any pair of coordinate systems which have their y axis along $\vec{k}_{in}\times\vec{k}_{out}$, may be written:

$$
\begin{aligned}
I(\theta) &= I_0(1+\tfrac{3}{2}p_y A_y+\tfrac{2}{3}p_{xz}A_{xz}+\tfrac{1}{3}p_{xx}A_{xx}+\tfrac{1}{3}p_{yy}A_{yy}+\tfrac{1}{3}p_{zz}A_{zz}),\\
p_{x'}I &= I_0(\tfrac{3}{2}p_x K_x^{x'}+\tfrac{3}{2}p_z K_z^{x'}+\tfrac{2}{3}p_{xy}K_{xy}^{x'}+\tfrac{2}{3}p_{yz}K_{yz}^{x'}),\\
p_{y'}I &= I_0(P^{y'}+\tfrac{3}{2}p_y K_y^{y'}+\tfrac{2}{3}p_{xz}K_{xz}^{y'}+\tfrac{1}{3}p_{xx}K_{xx}^{y'}+\tfrac{1}{3}p_{yy}K_{yy}^{y'}+\tfrac{1}{3}p_{zz}K_{zz}^{y'}),\\
p_{z'}I &= I_0(\tfrac{3}{2}p_x K_x^{z'}+\tfrac{3}{2}p_z K_z^{z'}+\tfrac{2}{3}p_{xy}K_{xy}^{z'}+\tfrac{2}{3}p_{yz}K_{yz}^{z'}),\\
p_{x'y'}I &= I_0(\tfrac{3}{2}p_x K_x^{x'y'}+\tfrac{3}{2}p_z K_z^{x'y'}+\tfrac{2}{3}p_{xy}K_{xy}^{x'y'}+\tfrac{2}{3}p_{yz}K_{yz}^{x'y'}),\\
p_{x'z'}I &= I_0(P^{x'z'}+\tfrac{3}{2}p_y K_y^{x'z'}+\tfrac{2}{3}p_{xz}K_{xz}^{x'z'}+\tfrac{1}{3}p_{xx}K_{xx}^{x'z'}+\tfrac{1}{3}p_{yy}K_{yy}^{x'z'}+\tfrac{1}{3}p_{zz}K_{zz}^{x'z'}),\\
p_{y'z'}I &= I_0(\tfrac{3}{2}p_x K_x^{y'z'}+\tfrac{3}{2}p_z K_z^{y'z'}+\tfrac{2}{3}p_{xy}K_{xy}^{y'z'}+\tfrac{2}{3}p_{yz}K_{yz}^{y'z'}),\\
p_{x'x'}I &= I_0(P^{x'x'}+\tfrac{3}{2}p_y K_y^{x'x'}+\tfrac{2}{3}p_{xz}K_{xz}^{x'x'}+\tfrac{1}{3}p_{xx}K_{xx}^{x'x'}+\tfrac{1}{3}p_{yy}K_{yy}^{x'x'}+\tfrac{1}{3}p_{zz}K_{zz}^{x'x'}),\\
p_{y'y'}I &= I_0(P^{y'y'}+\tfrac{3}{2}p_y K_y^{y'y'}+\tfrac{2}{3}p_{xz}K_{xz}^{y'y'}+\tfrac{1}{3}p_{xx}K_{xx}^{y'y'}+\tfrac{1}{3}p_{yy}K_{yy}^{y'y'}+\tfrac{1}{3}p_{zz}K_{zz}^{y'y'}),\\
p_{z'z'}I &= I_0(P^{z'z'}+\tfrac{3}{2}p_y K_y^{z'z'}+\tfrac{2}{3}p_{xz}K_{xz}^{z'z'}+\tfrac{1}{3}p_{xx}K_{xx}^{z'z'}+\tfrac{1}{3}p_{yy}K_{yy}^{z'z'}+\tfrac{1}{3}p_{zz}K_{zz}^{z'z'}),
\end{aligned}
\tag{4}
$$

where I_0 is the cross section for an initially unpolarized beam. The general principles which allow one to write this expression are discussed in previous work.[5] As noted, these equations are of the nature of contractions of tensors and can be written in the form

$$
\begin{aligned}
I &= I_0(1+\tfrac{3}{2}p_j A_j+\tfrac{1}{3}p_{jk}A_{jk}),\\
Ip_{j'} &= I_0(P^{j'}+\tfrac{3}{2}p_j K_j^{j'}+\tfrac{1}{3}p_{jk}K_{jk}^{j'}),\\
Ip_{j'k'} &= I_0(P^{j'k'}+\tfrac{3}{2}p_j K_j^{j'k'}+\tfrac{1}{3}p_{jk}K_{jk}^{j'k'}),
\end{aligned}
\tag{5}
$$

where here the repeated indices represent a summation over x, y, z.[6] The expression analogous to the first of Eq. (5) in terms of spherical tensors is

$$I=I_0\sum_{k,q}(-1)^q t_{kq}T_{k\,-q}=\sum_{k,q}t_{kq}T_{kq}^{*};$$

this is clearly also an invariant contraction of tensors.[4,7,8] Similar expressions may be written for the outgoing polarization components.

No assumption is built into Eq. (4) as to the choice of the x, z axes or the x', z' axes. If z is chosen along $\hat{k}_{in}$ and z' is chosen along $\hat{k}_{out}$(lab), e.g., if the laboratory helicity frames are used, we will refer to the K's as laboratory polarization-transfer coefficients. These would be the quantities directly analogous to the Wolfenstein R, A, R', and A' parameters. If z is along $\hat{k}_{in}$ and z' along $\hat{k}_{out}$(c.m.), we will refer to the K's as center-of-mass polarization-transfer coefficients, and for other frames we will use enough words to make the assumptions clear. Notice that conceptually the transfer coefficients connect two coordinate systems, and cannot properly be said to be "in" any coordinate system. Notice also that various subsets of the K's form second-, third-, and fourth-rank tensors, since the superscript quantities, as well as the subscript quantities, are tensor indices.

There are three helpful rules about the transfer coefficients which we will state here without proof. These rules are discussed in Ref. 5 and are extensions of the concepts put forth by Csonka and Moravcsik.[9] Let N_x, N_y, and N_z denote the number of x's, y's, and z's (with or without primes), respectively, that appear in a particular coefficient. We have

(1) if N_x+N_y is odd, the coefficient is an odd function of the scattering angle, θ;

(2) if N_x+N_z is odd, the transfer coefficient is required to be zero by parity conservation.

Rules (1) and (2) apply for any of the coordinate-system choices used in the present paper, and, in particular, we emphasize that they apply in the laboratory helicity frames. Next, we have a rule which holds only if certain coordinate-system choices are made[9,10]:

(3) If N_x is odd, the corresponding polarization-transfer coefficient in the inverse reaction reverses sign. For example: $K_{xz}^{xx}=-\bar{K}_{xx}^{xz}$ (inverse reaction). For elastic scattering only, the forward and inverse reactions are identical, so this statement would reduce to $K_{xz}^{xx}=-K_{xx}^{xz}$.

This type of relation holds in two coordinate systems as follows: (a) if both incoming and outgoing particles are described in the natural $(\hat{k}_-, \hat{n}, \hat{k}_+)$ coordinate system, or (b) if each particle is described in its own (c.m.) helicity frame. (This rule applies only to terms which are allowed by parity conservation. A more general form of the rule can be written for parity-violating terms.)

It is important to emphasize the generality of Eq. (4). The only restriction in going from Eq. (5) (which is more general) to Eq. (4) is that the two coordinate systems (initial and final) have $\vec{k}_{in}\times\vec{k}_{out}$ as their common y axis, and that parity is conserved. The form of the expression for the cross section, $I(\theta)$, is the same for any nuclear reaction with a polarized spin-1 particle in the entrance channel, and is independent of the target spin, of the spins of the particles appearing in the exit channel, and of the intrinsic parities of the various particles. Similarly, the form of the expressions for the outgoing particle polarizations, $p_{j'}$ and $p_{j'k'}$, is independent of the target spin, the residual particle spin, and the intrinsic parities of the particles.

IV. FORM OF THE SCATTERING MATRIX

The amplitude for scattering a particle with spin from the direction $\vec{k}_{in}$ through an angle θ to the direction $\vec{k}_{out}$ is a matrix in spin space and is referred to here as the scattering matrix, M, or as the M matrix. We now consider the most general form of the scattering matrix which describes the $1+0\to 1'+0'$ spin system. Since it must be a 3×3 matrix, it can be expanded in terms of any complete set of 3×3 matrices. We will use two such representations, the first of which is the set of Cartesian tensors $\mathcal{P}_j=S_j$ and $\mathcal{P}_{jk}=\frac{3}{2}(S_jS_k+S_kS_j)-2\delta_{jk}$, where the S_j are the Cartesian components of the spin operator $\vec{S}$. The second representation is in terms of products of "spin creation and annihilation operators" which annihilate spin in the entrance channel and create spin in the exit channel. The relations between the two representations will be shown. Also, we give the relations between the M matrix elements which obtain when the scattering is described in various coordinate frames.

The scattering matrix is first expanded in terms of 9 independent operators, with complex coefficients, so that there are 17 independent real numbers to be measured at each energy and angle (the over-all phase is not considered). Conservation of parity eliminates four coefficients, so that the number of independent numbers reduces to nine. For elastic scattering, time-reversal invariance eliminates one more coefficient, so that there are then only seven independent numbers to be determined experimentally.

The M matrix can be expanded in terms of the overcomplete set of 3×3 matrices, I, $\mathcal{P}_i$, $\mathcal{P}_{ij}$ (see Appendix I). Consider the three coordinate directions along the unit vectors $\hat{q}$, $\hat{n}$, $\hat{k}$, where $\hat{n}=(\vec{k}_{in}\times\vec{k}_{out})/|\vec{k}_{in}\times\vec{k}_{out}|$ and (for the moment) $\hat{q}$ and $\hat{k}$ are any two orthogonal vectors in the scattering plane, which will be identified with x and z axes in the following. We can then write

$$M=B_0+B_x\mathcal{P}_x+B_y\mathcal{P}_y+B_z\mathcal{P}_z+B_{xy}\mathcal{P}_{xy}+B_{yz}\mathcal{P}_{yz}+B_{xz}\mathcal{P}_{xz}+B_{xx}\mathcal{P}_{xx}+B_{yy}\mathcal{P}_{yy}+B_{zz}\mathcal{P}_{zz}. \qquad (6)$$

We have written the M matrix in this form so that the effect of parity can be easily discussed. Under the parity transformation, all vectors of the problem are changed into their negatives. That is $\vec{k}_{in}\to-\vec{k}_{in}$, $\vec{k}_{out}\to-\vec{k}_{out}$, so that $\vec{k}_{in}\times\vec{k}_{out}\to\vec{k}_{in}\times\vec{k}_{out}$. Note that neither first- nor second-rank spin operators change sign under the parity operation (because they transform as $\vec{r}\times\vec{p}$). Thus, since x (along $\hat{q}$) and z (along $\hat{k}$) are directions defined by linear combinations of $\vec{k}_{in}$ and $\vec{k}_{out}$, the parity operation when applied to our M matrix causes $x\to-x$, $y\to y$, and $z\to-z$. Keeping only the terms which do not change sign, and hence are allowed by parity, M becomes

$$M=B_0+B_y\mathcal{P}_y+B_{xz}\mathcal{P}_{xz}+B_{xx}\mathcal{P}_{xx}+B_{yy}\mathcal{P}_{yy}+B_{zz}\mathcal{P}_{zz}. \qquad (7)$$

Note that the last three matrices in Eq. (7) satisfy the equation

$$\mathcal{P}_{xx}+\mathcal{P}_{yy}+\mathcal{P}_{zz}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}. \qquad (8)$$

Introducing some new letters for the coefficients of the matrices, we may therefore write M in the form

$$M=a+b\mathcal{P}_y+c\mathcal{P}_{xz}+d(\mathcal{P}_{xx}-\mathcal{P}_{yy})+e\mathcal{P}_{zz}. \qquad (9)$$

Inserting the explicit form of the spin-1 matrices,

$$\mathcal{P}_y=\frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},$$

$$\mathcal{P}_{xz}=\frac{3}{2\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&-1\\0&-1&0\end{pmatrix},$$

$$\mathcal{P}_{xx}-\mathcal{P}_{yy}=\begin{pmatrix}0&0&3\\0&0&0\\3&0&0\end{pmatrix}, \tag{10}$$

$$\mathcal{P}_{zz}=\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix},$$

this becomes

$$M=\begin{pmatrix}a+e & \frac{1}{\sqrt{2}}(-ib+\frac{3}{2}c) & 3d\\ \frac{1}{\sqrt{2}}(ib+\frac{3}{2}c) & a-2e & \frac{1}{\sqrt{2}}(-ib-\frac{3}{2}c)\\ 3d & \frac{1}{\sqrt{2}}(ib-\frac{3}{2}c) & a+e\end{pmatrix}, \tag{11}$$

which may be written in the form,

$$M=\begin{pmatrix}a'&c'&e'\\-d'&b'&d'\\e'&-c'&a'\end{pmatrix}, \tag{12}$$

where

$$a'=a+e,$$
$$b'=a-2e,$$
$$c'=\frac{1}{\sqrt{2}}(-ib+\tfrac{3}{2}c), \tag{13}$$
$$d'=\frac{1}{\sqrt{2}}(-ib-\tfrac{3}{2}c),$$
$$e'=3d.$$

This is the form which is obtained in derivations of M suitable for phase-shift analysis.

We now consider the effect of time-reversal invariance on the form of the M matrix for the special case of elastic scattering, i.e., where the reaction is its own inverse. It is convenient to use the choice (so far arbitrary) $\hat{q}=(\vec{k}_{out}-\vec{k}_{in})/|\vec{k}_{out}-\vec{k}_{in}|\equiv\hat{k}_-$ and $\hat{k}=\hat{q}\times\hat{n}\equiv\hat{k}_+$. Notice that $\hat{k}_+=(\vec{k}_{out}+\vec{k}_{in})/|\vec{k}_{out}+\vec{k}_{in}|$ if nonrelativistic kinematics is assumed. In the center of mass, the time-reversal transformation is $\vec{k}_{in}\to-\vec{k}_{out}$, $\vec{k}_{out}\to-\vec{k}_{in}$, so that $\hat{q}\to\hat{q}$, $\hat{k}\to-\hat{k}$, $\hat{n}\to-\hat{n}$, and $\vec{\mathcal{P}}\to-\vec{\mathcal{P}}$. Thus, the term in $\mathcal{P}_{xz}$ changes sign under time reversal, and because we require $M(\vec{k}_{out},\vec{k}_{in},\vec{S})=M(-\vec{k}_{in},-\vec{k}_{out},-\vec{S})$, it must be excluded. That is, we require $c=0$ in Eq. (9). This result holds only for this particular coordinate system; the form of the constraint in other systems will be discussed below.

We now turn to a second representation of the M matrix (following Csonka, Moravcsik, and Scadron[11]). This form has a number of computational advantages over the more usual form given above, as follows: (a) The calculation of the relevant traces which define polarization-transfer coefficients (observables) will be somewhat simplified; (b) we may choose a representation in which the initial and final particle polarizations are referred to different axes; and (c) the resulting expressions for observables will be simpler.

The idea of this representation is to build 3×3 matrices out of the spin-1 spinors. We will use a purely formal construction and turn to the more conventional representation to identify the conditions imposed by time-reversal and parity invariance. Consider the spherical vectors which are proportional to rank-1 spherical harmonics, in terms of Cartesian components of a vector $\vec{R}$:

$$R_+=-\frac{1}{\sqrt{2}}(R_x+iR_y),$$
$$R_0=R_z, \tag{14}$$
$$R_-=\frac{1}{\sqrt{2}}(R_x-iR_y).$$

(We choose the Condon-Shortley phase convention[12] on the spherical harmonics.) These can be solved for R_x, R_y, R_z:

$$R_x=-\frac{1}{\sqrt{2}}(R_+-R_-),$$
$$R_y=\frac{i}{\sqrt{2}}(R_++R_-), \tag{15}$$
$$R_z=R_0.$$

We identify the spin-1 spinors χ_+, χ_0, χ_- with R_+, R_0, R_-, where

$$\chi_+=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \chi_0=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \chi_-=\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{16}$$

so that we can deduce the three Cartesian components of a vector operator $\vec{\chi}$:

$$\chi_x = -\frac{1}{\sqrt{2}}(\chi_+ - \chi_-) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

$$\chi_y = \frac{i}{\sqrt{2}}(\chi_+ + \chi_-) = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \tag{17}$$

$$\chi_z = \chi_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

The $\chi^\dagger$ operators are defined to be the transpose conjugate of the $\vec{\chi}$'s, having Cartesian components:

$$\chi_x^\dagger = \frac{1}{\sqrt{2}}(-1\ \ 0\ \ 1),$$

$$\chi_y^\dagger = -\frac{i}{2}(1\ \ 0\ \ 1), \tag{18}$$

$$\chi_z^\dagger = (0\ \ 1\ \ 0).$$

The quantity $(\vec{\chi}\cdot\vec{R})^\dagger$, where $\vec{R}$ is a general vector with complex components, would be

$$\begin{aligned}(\vec{\chi}\cdot\vec{R})^\dagger = \vec{\chi}^\dagger\cdot\vec{R} &= \frac{1}{\sqrt{2}}(-1\ \ 0\ \ 1)R_x - \frac{i}{\sqrt{2}}(1\ \ 0\ \ 1)R_y \\ &\quad + (0\ \ 1\ \ 0)R_z \\ &= \left[-\frac{1}{\sqrt{2}}(R_x + iR_y)\ \ R_z\ \ \frac{1}{\sqrt{2}}(R_x - iR_y)\right] \\ &= (R_+ R_0 R_-),\end{aligned} \tag{19}$$

and similarly, $\vec{\chi}\cdot\vec{Q}$, where $\vec{Q}$ is a general vector with complex components, would be

$$\begin{aligned}\vec{\chi}\cdot\vec{Q} &= \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}Q_x + \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}Q_y + \begin{pmatrix}0\\1\\0\end{pmatrix}Q_z \\ &= \begin{pmatrix}-\frac{1}{\sqrt{2}}(Q_x - iQ_y)\\ Q_z \\ \frac{1}{\sqrt{2}}(Q_x + iQ_y)\end{pmatrix} \\ &= \begin{pmatrix}Q_+^*\\Q_0^*\\Q_-^*\end{pmatrix}.\end{aligned} \tag{20}$$

In practice, the vectors $\vec{R}$ and $\vec{Q}$ of interest will always be real.

We can form nine 3×3 matrices by forming outer products of these quantities:

$$\begin{aligned}
\chi_x\chi_x^\dagger &= \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}(-1\ \ 0\ \ 1) = \frac{1}{2}\begin{pmatrix}1&0&-\\0&0&0\\-1&0&1\end{pmatrix},\\
\chi_y\chi_y^\dagger &= \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}(1\ \ 0\ \ 1) = \frac{1}{2}\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix},\\
\chi_z\chi_z^\dagger &= \begin{pmatrix}0\\1\\0\end{pmatrix}(0\ \ 1\ \ 0) = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\\
\chi_x\chi_z^\dagger &= \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}(0\ \ 1\ \ 0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-1&0\\0&0&0\\0&1&0\end{pmatrix},\\
\chi_z\chi_x^\dagger &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}(-1\ \ 0\ \ 1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\-1&0&1\\0&0&0\end{pmatrix},\\
\chi_x\chi_y^\dagger &= \frac{-i}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}(1\ \ 0\ \ 1) = \frac{i}{2}\begin{pmatrix}1&0&1\\0&0&0\\-1&0&-1\end{pmatrix},\\
\chi_y\chi_x^\dagger &= \frac{i}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}(-1\ \ 0\ \ 1) = \frac{i}{2}\begin{pmatrix}-1&0&1\\0&0&0\\-1&0&1\end{pmatrix},\\
\chi_y\chi_z^\dagger &= \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}(0\ \ 1\ \ 0) = \frac{i}{\sqrt{2}}\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix},\\
\chi_z\chi_y^\dagger &= -\frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}(1\ \ 0\ \ 1) = \frac{i}{\sqrt{2}}\begin{pmatrix}0&0&0\\-1&0&-1\\0&0&0\end{pmatrix}.
\end{aligned} \tag{21}$$

These matrices span the 3×3 space, so that an arbitrary matrix can be expanded in terms of them. We may therefore write the most general M matrix as follows:

$$\begin{aligned}M &= C_{xx}\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{q})^\dagger + C_{yy}\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{n})^\dagger + C_{zz}\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{k})^\dagger \\ &\quad + C_{xz}\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{q})^\dagger + C_{zx}\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{k})^\dagger + C_{yx}\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{n})^\dagger \\ &\quad + C_{xy}\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{q})^\dagger + C_{zy}\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{k})^\dagger + C_{yz}\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{n})^\dagger.\end{aligned} \tag{22}$$

Again, parity excludes the last four terms for any definition of $\hat{q}$ and $\hat{k}$, so long as both are in the scattering plane. The M matrix then simplifies to

$$M = A'\chi_x\chi_x^\dagger + B'\chi_y\chi_y^\dagger + C'\chi_z\chi_z^\dagger + D'\chi_z\chi_x^\dagger + E'\chi_x\chi_z^\dagger. \tag{23}$$

(So far we are not making use of the possibility of using separate coordinate systems for the initial and final particles.) The M matrix can now be written with the aid of Eq. (21), as

$$M=\begin{pmatrix} \frac{1}{2}(A'-B') & -\frac{1}{\sqrt{2}}E' & -\frac{1}{2}(A'+B') \\ -\frac{1}{\sqrt{2}}D' & C' & \frac{1}{\sqrt{2}}D' \\ -\frac{1}{2}(A'+B') & \frac{1}{\sqrt{2}}E' & \frac{1}{2}(A'-B') \end{pmatrix}. \quad (24)$$

Thus, if Eqs. (9) and (23) are expressed in the same coordinate system,

$$\begin{aligned} A'&=a+e-3d\,, \\ B'&=-a-e-3d\,, \\ C'&=a-2e\,, \\ D'&=-ib-\tfrac{3}{2}c\,, \\ E'&=ib-\tfrac{3}{2}c\,. \end{aligned} \quad (25)$$

If the natural $(\hat{k}_-, \hat{n}, \hat{k}_+)$ coordinate system is chosen, as noted above, time-reversal invariance implies (for elastic scattering) $c=0$. Thus, for this special coordinate system, $D'=-E'$ is the time-reversal condition.

We now generalize the M matrix to allow different coordinate systems for the initial and final particles. Consider the typical term of the M matrix, $E'\chi_x\chi_z^\dagger$. Notice that the $\chi_z^\dagger$ plays the role of a spin annihilation operator. That is, when $\chi_z^\dagger$ operates on a spin-1 spinor, a one-component object (spin 0) is generated. For example, we have

$$(\chi_z^\dagger)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (0\ 1\ 0)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = b\,. \quad (26)$$

When this one-component object is operated on by the χ_x, a three-component (spin-1) object is generated, as required:

$$\chi_x b = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}(b) = \frac{1}{\sqrt{2}}\begin{pmatrix} -b \\ 0 \\ b \end{pmatrix}; \quad (27)$$

i.e., χ_x acts as a spin creation operator. Thus, the $\chi^\dagger$ quantities should be expressed in terms of the initial coordinate system (to annihilate incident spin-1 particles), and the χ quantities in the outgoing particle system (to create spin-1 particles). We therefore write M in the form

$$M=A\vec{\chi}\cdot\hat{q}'(\vec{\chi}\cdot\hat{q})^\dagger + B\vec{\chi}\cdot\hat{n}'(\vec{\chi}\cdot\hat{n})^\dagger + C\vec{\chi}\cdot\hat{k}'(\vec{\chi}\cdot\hat{k})^\dagger + D\vec{\chi}\cdot\hat{k}'(\vec{\chi}\cdot\hat{q})^\dagger + E\vec{\chi}\cdot\hat{q}'(\vec{\chi}\cdot\hat{k})^\dagger\,. \quad (28)$$

This form will apply for any choice of $\hat{q}'$, $\hat{k}'$, and in particular for $\hat{k}'$ along either the outgoing laboratory or outgoing center-of-mass direction. If $\hat{k}'$ is chosen along the center-of-mass direction, time-reversal invariance implies $D=-E$, as will now be shown. The M matrix in terms of the $\hat{k}_-$, $\hat{n}$, $\hat{k}_+$ vectors is, as previously stated in Eq. (23),

$$M=A'\vec{\chi}\cdot\hat{k}_-(\vec{\chi}\cdot\hat{k}_-)^\dagger + B'\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{n})^\dagger + C'\vec{\chi}\cdot\hat{k}_+(\vec{\chi}\cdot\hat{k}_+)^\dagger + D'\vec{\chi}\cdot\hat{k}_+(\vec{\chi}\cdot\hat{k}_-)^\dagger + E'\vec{\chi}\cdot\hat{k}_-(\vec{\chi}\cdot\hat{k}_+)^\dagger\,, \quad (29)$$

where $D'=-E'$ for elastic scattering. We can express $\hat{k}_-$ and $\hat{k}_+$ in terms of either $\hat{q}$, $\hat{k}$ or $\hat{q}'$, $\hat{k}'$:

$$\begin{aligned} \hat{k}_-&=\hat{q}\cos\tfrac{1}{2}\theta-\hat{k}\sin\tfrac{1}{2}\theta\,, \\ \hat{k}_+&=\hat{q}\sin\tfrac{1}{2}\theta+\hat{k}\cos\tfrac{1}{2}\theta\,, \end{aligned} \quad (30)$$

or

$$\begin{aligned} \hat{k}_-&=\hat{q}'\cos\tfrac{1}{2}\theta+\hat{k}'\sin\tfrac{1}{2}\theta\,, \\ \hat{k}_+&=-\hat{q}'\sin\tfrac{1}{2}\theta+\hat{k}'\cos\tfrac{1}{2}\theta\,, \end{aligned} \quad (31)$$

where θ is the center-of-mass scattering angle. These relations are evident from Fig. 2. Substitution of Eq. (31) into the first factor and of Eq. (30) into the second factor of each term of Eq. (28) as appropriate, we find

$$\begin{aligned} A&=A'\cos^2(\tfrac{1}{2}\theta)-C'\sin^2(\tfrac{1}{2}\theta)+(-D'+E')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,, \\ B&=B'\,, \\ C&=-A'\sin^2(\tfrac{1}{2}\theta)+C'\cos^2(\tfrac{1}{2}\theta)+(-D'+E')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,, \\ D&=(A'+C')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+D'\cos^2(\tfrac{1}{2}\theta)+E'\sin^2(\tfrac{1}{2}\theta)\,, \\ E&=-(A'+C')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+D'\sin^2(\tfrac{1}{2}\theta)+E'\cos^2(\tfrac{1}{2}\theta)\,. \end{aligned} \quad (32)$$

If $D'=-E'$, inspection of the above equations shows that $D=-E$; that is, the time-reversal condition in the (c.m.) helicity frames is $D=-E$, as was to be shown.

We may also deduce the time-reversal condition which holds if both the incident and final particle spins are expressed in the projectile helicity frame. In that case, we use only Eq. (30) in re-

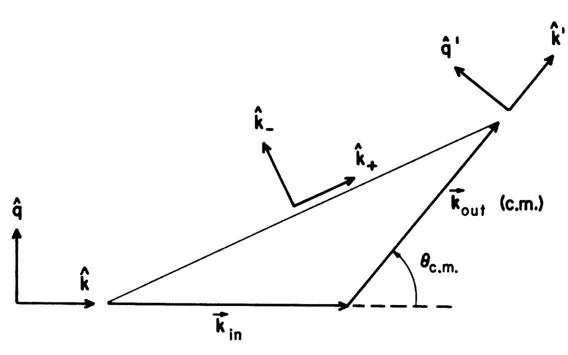

FIG. 2. Relation between coordinate systems.

writing Eq. (29). Writing

$$M = A''\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{q})^\dagger + B''\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{n})^\dagger + C''\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{k})^\dagger + D''\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{q})^\dagger + E''\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{k})^\dagger, \quad (33)$$

we find

$$A'' = A'\cos^2(\tfrac{1}{2}\theta) + C'\sin^2(\tfrac{1}{2}\theta) + (D'+E')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta,$$

$$B'' = B',$$

$$C'' = A'\sin^2(\tfrac{1}{2}\theta) + C'\cos^2(\tfrac{1}{2}\theta) - (D'+E')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta, \quad (34)$$

$$D'' = (-A'+C')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta + D'\cos^2(\tfrac{1}{2}\theta) - E'\sin^2(\tfrac{1}{2}\theta),$$

$$E'' = (-A'+C')\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - D'\sin^2(\tfrac{1}{2}\theta) + E'\cos^2(\tfrac{1}{2}\theta),$$

whence, if $E' = -D'$, we find

$$(D''+E'') = (-A''+C'')\tan\theta. \quad (35)$$

V. CALCULATION OF THE OBSERVABLES

Techniques for calculating the cross section and polarization are discussed in detail in Appendixes I and II. These observables will involve bilinear products of M matrix elements, such as ReAB^*, ImBC^*, etc. The results of these calculations appear in Tables I and II. It is through these relations that the independence of experiments and effects of time-reversal invariance will be studied.

The observables are given by

$$I_0 = \tfrac{1}{3}\mathrm{Tr}MM^\dagger \quad \text{(cross section)},$$

$$A_\alpha = \mathrm{Tr}M\mathcal{P}_\alpha M^\dagger/\mathrm{Tr}MM^\dagger \quad \text{(analyzing tensors)}, \quad (36)$$

$$P^\alpha = \mathrm{Tr}MM^\dagger\mathcal{P}_\alpha/\mathrm{Tr}MM^\dagger \quad \text{(polarization tensors)},$$

$$K_\alpha^\beta = \mathrm{Tr}M\mathcal{P}_\alpha M^\dagger\mathcal{P}_\beta/\mathrm{Tr}MM^\dagger \quad \text{(polarization-transfer tensors)},$$

where α or β may even mean any of the quantities x, y, z, xy, xz, yz, xx, yy, or zz. Many of these quantities vanish because of parity conservation, as previously noted.

Evaluation of these traces in terms of an M matrix of the form of Eq. (7) is straightforward. However, in the $\vec{\chi}$ representation, we can carry out the trace operations in an even simpler manner. If we let the initial and final operators be denoted by Ω_i and Ω_f, respectively, the most general quantities we need to evaluate are of the form

TABLE I. ^{4}He$(d,d)^4$He cross section and polarization for a polarized beam.

Class 1	$\|A\|^2$	$\|B\|^2$	$\|C\|^2$	$\|D\|^2$	$\|E\|^2$
$3I_0$	1	1	1	1	1
$3I_0A_{xx}$	−2	1	1	−2	1
$3I_0A_{yy}$	1	−2	1	1	1
$3I_0A_{zz}$	1	1	−2	1	−2
$3I_0P^{xx}$	−2	1	1	1	−2
$3I_0P^{yy}$	1	−2	1	1	1
$3I_0P^{zz}$	1	1	−2	−2	1
$3I_0K_{xx}^{xx}$	4	1	1	−2	−2
$3I_0K_{yy}^{xx}$	−2	−2	1	1	−2
$3I_0K_{zz}^{xx}$	−2	1	−2	1	4
$3I_0K_{xx}^{yy}$	−2	−2	1	−2	1
$3I_0K_{yy}^{yy}$	1	4	1	1	1
$3I_0K_{zz}^{yy}$	1	−2	−2	1	−2
$3I_0K_{xx}^{zz}$	−2	1	−2	4	1
$3I_0K_{yy}^{zz}$	1	−2	−2	−2	1
$3I_0K_{zz}^{zz}$	1	1	4	−2	−2

Class 2

	ReAE^*	ReCD^*		ImAE^*	ImCD^*
$3I_0A_{xz}$	−3	−3	$3I_0A_y$	−2	2
$3I_0K_{xz}^{xx}$	6	−3	$3I_0K_y^{xx}$	4	2
$3I_0K_{xz}^{yy}$	−3	−3	$3I_0K_y^{yy}$	−2	2
$3I_0K_{xz}^{zz}$	−3	6	$3I_0K_y^{zz}$	−2	−4
	ReAD^*	ReCE^*		ImAD^*	ImCE^*
$3I_0P^{xz}$	−3	−3	$3I_0P^y$	2	−2
$3I_0K_{xx}^{xz}$	6	−3	$3I_0K_{xx}^y$	−4	−2
$3I_0K_{yy}^{xz}$	−3	−3	$3I_0K_{yy}^y$	2	−2
$3I_0K_{zz}^{xz}$	−3	6	$3I_0K_{zz}^y$	2	4

Class 3

	ReAC^*	ReDE^*		ImAC^*	ImDE^*
$3I_0K_y^y$	2	−2	$3I_0K_y^{xz}$	3	3
$3I_0K_{xz}^{xz}$	$\frac{9}{2}$	$\frac{9}{2}$	$3I_0K_{xz}^y$	−3	3

Class 4

	ReBC^*		ImBC^*		ReBD^*
$3I_0K_x^x$	2	$3I_0K_{yz}^x$	3	$3I_0K_z^x$	−2
$3I_0K_{yz}^{yz}$	$\frac{9}{2}$	$3I_0K_x^{yz}$	−3	$3I_0K_{xy}^{yz}$	$\frac{9}{2}$
	ImBD^*		ReAB^*		ImAB^*
$3I_0K_{xy}^x$	3	$3I_0K_z^z$	2	$3I_0K_{xy}^z$	3
$3I_0K_z^{yz}$	3	$3I_0K_{xy}^{xy}$	$\frac{9}{2}$	$3I_0K_z^{xy}$	−3
	ReBE^*		ImBE^*		
$3I_0K_x^z$	−2	$3I_0K_{yz}^z$	−3		
$3I_0K_{yz}^{xy}$	$\frac{9}{2}$	$3I_0K_x^{xy}$	−3		

TABLE II. $^4\mathrm{He}(d,d)^4\mathrm{He}$ cross section and polarization for a polarized beam.

Outgoing particle \ Beam	1	p_x	p_y	p_z	p_{xy}	p_{xz}	p_{yz}	p_{xx}	p_{yy}	p_{zz}
$I(\theta)$	$\lvert A\rvert^2+\lvert B\rvert^2+\lvert C\rvert^2$ $+\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0$		$2\,\mathrm{Im}(CD^*-AE^*)$ $3I_0A_y$			$-3\,\mathrm{Re}(AE^*+CD^*)$ $3I_0A_{xz}$		$-2\lvert A\rvert^2+\lvert B\rvert^2+\lvert C\rvert^2$ $-2\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0A_{xx}$	$\lvert A\rvert^2-2\lvert B\rvert^2+\lvert C\rvert^2$ $+\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0A_{yy}$	$\lvert A\rvert^2+\lvert B\rvert^2-2\lvert C\rvert^2$ $+\lvert D\rvert^2-2\lvert E\rvert^2$ $3I_0A_{zz}$
$p_{x'}I(\theta)$		$2\,\mathrm{Re}BC^*$ $3I_0K_x^x$		$-2\,\mathrm{Re}BD^*$ $3I_0K_z^x$	$3\,\mathrm{Im}BD^*$ $3I_0K_{xy}^x$		$3\,\mathrm{Im}BC^*$ $3I_0K_{yz}^x$			
$p_{y'}I(\theta)$	$2\,\mathrm{Im}(AD^*-CE^*)$ $3I_0P^y$		$2\,\mathrm{Re}(AC^*-DE^*)$ $3I_0K_y^y$			$3\,\mathrm{Im}(DE^*-AC^*)$ $3I_0K_{xz}^y$		$-4\,\mathrm{Im}AD^*-2\,\mathrm{Im}CE^*$ $3I_0K_{xx}^y$	$2\,\mathrm{Im}AD^*-\mathrm{Im}CE^*$ $3I_0K_{yy}^y$	$2\,\mathrm{Im}AD^*+4\,\mathrm{Im}CE^*$ $3I_0K_{zz}^y$
$p_{z'}I(\theta)$		$-2\,\mathrm{Re}BE^*$ $3I_0K_x^z$		$2\,\mathrm{Re}AB^*$ $3I_0K_z^z$	$3\,\mathrm{Im}AB^*$ $3I_0K_{xy}^z$		$-3\,\mathrm{Im}BE^*$ $3I_0K_{yz}^z$			
$p_{x'y'}I(\theta)$		$-3\,\mathrm{Im}BE^*$ $3I_0K_x^{xy}$		$-3\,\mathrm{Im}AB^*$ $3I_0K_z^{xy}$	$\frac{3}{2}\mathrm{Re}AB^*$ $3I_0K_{xy}^{xy}$		$\frac{3}{2}\,\mathrm{Re}BE^*$ $3I_0K_{yz}^{xy}$			
$p_{x'z'}I(\theta)$	$-3\,\mathrm{Re}(AD^*+CE^*)$ $3I_0P^{xz}$		$3\,\mathrm{Im}(AC^*+DE^*)$ $3I_0K_y^{xz}$			$\frac{3}{2}\,\mathrm{Re}(AC^*+DE^*)$ $3I_0K_{xz}^{xz}$		$6\,\mathrm{Re}AD^*-3\,\mathrm{Re}CE^*$ $3I_0K_{xx}^{xz}$	$-3\,\mathrm{Re}AD^*-3\,\mathrm{Re}CE^*$ $3I_0K_{yy}^{xz}$	$-3\,\mathrm{Re}AD^*+6\,\mathrm{Re}CE^*$ $3I_0K_{zz}^{xz}$
$p_{y'z'}I(\theta)$		$-3\,\mathrm{Im}BC^*$ $3I_0K_x^{yz}$		$3\,\mathrm{Im}BD^*$ $3I_0K_z^{yz}$	$\frac{3}{2}\mathrm{Re}BD^*$ $3I_0K_{xy}^{yz}$		$\frac{3}{2}\,\mathrm{Re}BC^*$ $3I_0K_{yz}^{yz}$			
$p_{x'x'}I(\theta)$	$-2\lvert A\rvert^2+\lvert B\rvert^2+\lvert C\rvert^2$ $+\lvert D\rvert^2-2\lvert E\rvert^3$ $3I_0P^{xx}$		$4\,\mathrm{Im}AE^*$ $2\,\mathrm{Im}CD^*$ $3I_0K_y^{xx}$			$6\,\mathrm{Re}AE^*-3\,\mathrm{Re}CD^*$ $3I_0K_{xz}^{xx}$		$4\lvert A\rvert^2+\lvert B\rvert^2+\lvert C\rvert^2$ $-2\lvert D\rvert^2-2\lvert E\rvert^2$ $3I_0K_{xx}^{xx}$	$-2\lvert A\rvert^2-2\lvert B\rvert^2$ $+\lvert C\rvert^2+\lvert D\rvert^2-2\lvert E\rvert^2$ $3I_0K_{yy}^{xx}$	$-2\lvert A\rvert^2+\lvert B\rvert^2$ $-2\lvert C\rvert^2+\lvert D\rvert^2+4\lvert E\rvert^2$ $3I_0K_{zz}^{xx}$
$p_{y'y'}I(\theta)$	$\lvert A\rvert^2-2\lvert B\rvert^2+\lvert C\rvert^2$ $+\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0P^{yy}$		$-2\,\mathrm{Im}AE^*$ $+2\,\mathrm{Im}CD^*$ $3I_0K_y^{yy}$			$-3\,\mathrm{Re}AE^*-3\,\mathrm{Re}CD^*$ $3I_0K_{xz}^{yy}$		$-2\lvert A\rvert^2-2\lvert B\rvert^2+\lvert C\rvert^2$ $-2\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0K_{xx}^{yy}$	$\lvert A\rvert^2+4\lvert B\rvert^2+\lvert C\rvert^2$ $+\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0K_{yy}^{yy}$	$\lvert A\rvert^2-2\lvert B\rvert^2-2\lvert C\rvert^2$ $+\lvert D\rvert^2-2\lvert E\rvert^2$ $3I_0K_{zz}^{yy}$
$p_{z'z'}I(\theta)$	$\lvert A\rvert^2+\lvert B\rvert^2-2\lvert C\rvert^2$ $-2\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0P^{zz}$		$-2\,\mathrm{Im}AE^*$ $-4\,\mathrm{Im}CD^*$ $3I_0K_y^{zz}$			$-3\,\mathrm{Re}AE^*+6\,\mathrm{Re}CD^*$ $3I_0K_{xz}^{zz}$		$-2\lvert A\rvert^2+\lvert B\rvert^2-2\lvert C\rvert^2$ $+4\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0K_{xx}^{zz}$	$\lvert A\rvert^2-2\lvert B\rvert^2-2\lvert C\rvert^2$ $-2\lvert D\rvert^2+\lvert E\rvert^2$ $3I_0K_{yy}^{zz}$	$\lvert A\rvert^2+\lvert B\rvert^2+4\lvert C\rvert^2$ $-2\lvert D\rvert^2-2\lvert E\rvert^2$ $3I_0K_{zz}^{zz}$
Factor	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$\mathrm{Tr}M\Omega_i M^\dagger \Omega_f$. M has the form of Eq. (28),

$$M = A\vec{\chi}\cdot\hat{q}'(\vec{\chi}\cdot\hat{q})^\dagger + B\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{n})^\dagger + C\vec{\chi}\cdot\hat{k}'(\vec{\chi}\cdot\hat{k})^\dagger + D\vec{\chi}\cdot\hat{k}'(\vec{\chi}\cdot\hat{q})^\dagger + E\vec{\chi}\cdot\hat{q}'(\vec{\chi}\cdot\hat{k})^\dagger, \qquad (37)$$

where the unprimed unit vectors $(\hat{q}, \hat{k})$ are x and z axes in terms of which the initial particle spin is described; the primed unit vectors $(\hat{q}', \hat{k}')$ are the x' and z' axes in terms of which the final particle spin is described, and $\hat{n}$ is the y axis, along $\vec{k}_{\mathrm{in}} \times \vec{k}_{\mathrm{out}}$, which is common to both systems. The Hermitian conjugate of M is given by

$$M^\dagger = A^*\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{q}')^\dagger + B^*\vec{\chi}\cdot\hat{n}(\vec{\chi}\cdot\hat{n})^\dagger + C^*\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{k}')^\dagger + D^*\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{k}')^\dagger + E^*\vec{\chi}\cdot\hat{k}(\vec{\chi}\cdot\hat{q}')^\dagger, \qquad (38)$$

where we have used the fact that $[\vec{\chi}\cdot\vec{R}(\vec{\chi}\cdot\vec{Q})^\dagger]^\dagger = \vec{\chi}\cdot\vec{Q}(\vec{\chi}\cdot\vec{R})^\dagger$ (see Appendix II). Thus the trace consists of 25 terms, most of which will vanish when Ω_i and Ω_f are chosen from any reasonably defined basis set. A typical term would be $AE^* \times \mathrm{Tr}\vec{\chi}\cdot\hat{q}'(\vec{\chi}\cdot\hat{q})^\dagger\Omega_i\vec{\chi}\cdot\hat{q}(\vec{\chi}\cdot\hat{k}')^\dagger\Omega_f$. We can write the general trace of interest, then, as $\mathrm{Tr}\vec{\chi}\cdot\vec{R}'(\vec{\chi}\cdot\vec{R})^\dagger \times\Omega_i\vec{\chi}\cdot\vec{Q}(\vec{\chi}\cdot\vec{Q}')^\dagger\Omega_f$, where $\vec{R}$ and $\vec{Q}$ are any of the initial system unit vectors $\hat{q}$, $\hat{n}$, $\hat{k}$; where $\vec{R}'$ and $\vec{Q}'$ are any of the final system unit vectors, $\hat{q}'$, $\hat{n}$, $\hat{k}'$; and where Ω_i and Ω_f are any of the operators from the set I, $\mathcal{P}_i(i=x,y,z)$, $\mathcal{P}_{ij}(i,j=x,y,z)$. We can apply the cyclic property of traces to rewrite this as $\mathrm{Tr}(\vec{\chi}\cdot\vec{Q}')^\dagger\Omega_f\vec{\chi}\cdot\vec{R}'(\vec{\chi}\cdot\vec{Q})^\dagger\Omega_i\vec{\chi}\cdot\vec{R}$. Notice that the trace permutation rule is applicable even though nonsquare matrices are involved (see Appendix II). This new form can be written

$$\mathrm{Tr}\left[(Q_+'\;Q_0'\;Q_-')(\Omega_f)\begin{pmatrix}R_+'^*\\R_0'^*\\R_-'^*\end{pmatrix}(Q_+\;Q_0\;Q_-)(\Omega_i)\begin{pmatrix}R_+^*\\R_0^*\\R_-^*\end{pmatrix}\right], \qquad (39)$$

which is seen to be a product of two 1×1 matrices so that the trace operation actually becomes unnecessary. Notice also that the vectors $\vec{R}$ and $\vec{Q}$ are associated with the projectile frame, while the vectors $\vec{R}'$ and $\vec{Q}'$ are associated with the outgoing particle frame, so that we are, in fact, free in this formulation to choose these coordinate systems differently.

Each of the traces may be easily evaluated according to the rules presented in Eq. (AII5) of Appendix II. The observables are given in Table I in a form which facilitates discussion of the relationships between the various observables. The same results are displayed in Table II in a different format. In this second format each of the beam polarization components corresponds to a column and each outgoing polarization component corresponds to a row. For convenience the factors which multiply each term of the cross section or outgoing polarization component of interest [see Eq. (4)] are given in the last row.

It is important to reemphasize that the initial particle and final particle may be described in individual frames, with only the restriction that the y axis is chosen along $\vec{k}_{\mathrm{in}}\times\vec{k}_{\mathrm{out}}$ for both systems. In particular, the forms apply if the initial particle is described in a frame with z along $\vec{k}_{\mathrm{in}}$ and the final particle with z along $\vec{k}_{\mathrm{out}}$(lab), so that the relations between observables implied in Tables I and II apply in the experimentally convenient laboratory helicity coordinate systems.

As noted in Sec. IV, time reversal implies a certain relation bwtween A, B, C, D, and E. In the c.m. helicity frames, or in the natural $(\hat{k}_-, \hat{n}, \hat{k}_+)$ system where both particle z axes are along $\hat{k}_+$, this relation assumes the simple form $D=-E$. Applying this rule to Tables I or II, we note that each of the transfer coefficients indeed obeys the general elastic scattering time-reversal rule quoted earlier (rule 3).

VI. RELATIONS BETWEEN THE TRANSFER COEFFICIENTS

As mentioned earlier, there are only nine independent quantities involved in the general $1+0 \rightarrow 1'+0'$ matrix, that is, the real and imaginary parts of A, B, C, D, and E, where one relative phase may be chosen arbitrarily. For elastic scattering, time-reversal invariance reduces this number to seven. Since there are 52 observables altogether, there evidently must exist many relations between them. We discuss in this section both linear relations and quadratic relations between the observables.

A. Linear Relations

We will discuss the linear relations one class at a time, where we define four classes which are distinguished by the set of products of A, B, C, D, and E that are involved. (The classes defined here are for convenience in the present discussion and do not correspond to the classes defined by Csonka, Moravcsik, and Scadron.[11]) We will always discuss the general $1+0\rightarrow 1'+0'$ (even-parity) situation, and then specialize to elastic scattering by setting $D=-E$. Thus, the results quoted for elastic scattering that go beyond the relations generally true for $1+0\rightarrow 1'+0'$ will be valid only in one of the (two) coordinate systems for which the time-reversal condition is $D=-E$; the coordinate-system restrictions will not be explicitly stated each time we specialize to this case. Notice that since there are 52 observables (41 plus 11 due to "overcompleteness") and 25 amplitude products, there must exist 27 linear relations.

Class 1. Products $|A|^2$, $|B|^2$, $|C|^2$, $|D|^2$, and $|E|^2$. References to Table I show that there are 16 observables which involve the products $|A|^2$, $|B|^2$, $|C|^2$, $|D|^2$, and $|E|^2$. There are eight trivial relations, of which seven are independent, which follow from the overcompleteness of our basis set:

$$
\begin{aligned}
&A_{xx}+A_{yy}+A_{zz}=0\,,\\
&P^{xx}+P^{yy}+P^{zz}=0\,,\\
&K_{xx}^{xx}+K_{xx}^{yy}+K_{xx}^{zz}=0\,,\\
&K_{yy}^{xx}+K_{yy}^{yy}+K_{yy}^{zz}=0\,,\\
&K_{zz}^{xx}+K_{zz}^{yy}+K_{zz}^{zz}=0\,,\\
&K_{xx}^{xx}+K_{yy}^{xx}+K_{zz}^{xx}=0\,,\\
&K_{xx}^{yy}+K_{yy}^{yy}+K_{zz}^{yy}=0\,,\\
&K_{xx}^{zz}+K_{yy}^{zz}+K_{zz}^{zz}=0\,.
\end{aligned}
\tag{40}
$$

These particular eight relations hold for any process in which a spin-1 particle is incident and a spin-1 particle emerges, regardless of the spins of the other particles.

We will choose five observables from class 1 which we will call the basic set: I_0, A_{xx}, A_{yy}, P^{xx}, and K_{xx}^{xx}. This choice is somewhat arbitrary, of course; we have been guided by requiring as many of the coefficients as possible to be related to the simpler experiments. We have arbitrarily chosen to eliminate observables involving zz components. For elastic scattering, this set of five reduces to four through the relation $P^{xx}=A_{xx}$. As can be easily verified with the aid of Table I, at least one observable of the polarization-transfer type must be included in the basic set.

We give the specific expressions for each of the 11 quantities not included in our basic set in terms of the basic set. When the relation simplifies for elastic scattering, this is indicated by the additional expression to the right of the arrow. We have

$$
\begin{aligned}
&A_{zz}=-A_{xx}-A_{yy}\,,\\
&P^{yy}=A_{yy}\,,\\
&P^{zz}=-P^{xx}-A_{yy}\to -A_{xx}-A_{yy}\,,\\
&K_{xx}^{yy}=-1+A_{xx}+A_{yy}\,,\\
&K_{xx}^{zz}=1-A_{xx}-A_{yy}-K_{xx}^{xx}\,,\\
&K_{yy}^{xx}=-1+A_{yy}+P^{xx}\to -1+A_{yy}+A_{xx}\,,\\
&K_{yy}^{yy}=2-A_{yy}\,,\\
&K_{yy}^{zz}=-1-P^{xx}\to -1-A_{xx}\,,\\
&K_{zz}^{xx}=1-A_{yy}-P^{xx}-K_{xx}^{xx}\to 1-A_{yy}-A_{xx}-K_{xx}^{xx}\,,\\
&K_{zz}^{yy}=-1-A_{xx}\,,\\
&K_{zz}^{zz}=A_{xx}+A_{yy}+P^{xx}+K_{xx}^{xx}\to 2A_{xx}+A_{yy}+K_{xx}^{xx}\,.
\end{aligned}
\tag{41}
$$

Finally, the products $|A|^2$, $|B|^2$, $|C|^2$, $|D|^2$, and $|E|^2$ can be expressed in terms of the five observables of our basic set:

$$
\begin{aligned}
|A|^2&=\tfrac{1}{3}I_0(1-A_{xx}-P^{xx}+K_{xx}^{xx})\to\tfrac{1}{3}I_0(1-2A_{xx}+K_{xx}^{xx})\,,\\
|B|^2&=\tfrac{1}{3}I_0(3-3A_{yy})\,,\\
|C|^2&=\tfrac{1}{3}I_0(1+2A_{xx}+2P^{xx}+3A_{yy}+K_{xx}^{xx})\\
&\quad\to\tfrac{1}{3}I_0(1+4A_{xx}+3A_{yy}+K_{xx}^{xx})\,,\\
|D|^2&=\tfrac{1}{3}I_0(2-2A_{xx}+P^{xx}-K_{xx}^{xx})\to\tfrac{1}{3}I_0(2-A_{xx}-K_{xx}^{xx})\,,\\
|E|^2&=\tfrac{1}{3}I_0(2+A_{xx}-2P^{xx}-K_{xx}^{xx})\to\tfrac{1}{3}I_0(2-A_{xx}-K_{xx}^{xx})\,,
\end{aligned}
\tag{42}
$$

where again the elastic scattering specialization is to the right of the arrows.

Class 2. Real and imaginary parts of AE, AD, CD, CE. In this case there are four trivial relations which follow from the overcompleteness of our basis set:

$$
\begin{aligned}
&K_{xy}^{xx}+K_{xy}^{yy}+K_{xy}^{zz}=0\,,\\
&K_{xx}^{xz}+K_{yy}^{xz}+K_{zz}^{xz}=0\,,\\
&K_{y}^{xx}+K_{y}^{yy}+K_{y}^{zz}=0\,,\\
&K_{xx}^{y}+K_{yy}^{y}+K_{zz}^{y}=0\,.
\end{aligned}
\tag{43}
$$

These particular relations, like those of Eq. (40), are true whether or not the target and recoil particle have spin 0.

As may be seen from Table I, four observables correspond to each set of two amplitude products. In each case, the four observables consist of (1) a polarization or an analyzing power, and (2) three polarization-transfer coefficients. We choose our basic set of eight observables, and find the relations to the remaining observables, as follows:

$$
\begin{array}{ll}
\text{Basic set} & \text{Relations}\\
A_{xz} & K_{xz}^{yy}=A_{xz}\\
K_{xz}^{xx} & K_{xz}^{zz}=-A_{xz}-K_{xz}^{xx}\\
A_{y} & K_{y}^{yy}=A_{y}\\
K_{y}^{xx} & K_{y}^{zz}=-A_{y}-K_{y}^{xx}\\
P^{xz}\to -A_{xz} & K_{yy}^{xz}=P^{xz}\to -A_{xz}\\
K_{xx}^{xz}\to -K_{xz}^{xx} & K_{zz}^{xz}=-K_{xx}^{xz}-P^{xz}\to K_{xz}^{xx}+A_{xz}\\
P^{y}\to A_{y} & K_{yy}^{y}=P^{y}\to A_{y}\\
K_{xx}^{y}\to K_{y}^{xx} & K_{zz}^{y}=-K_{xx}^{y}-P^{y}\to -K_{y}^{xx}-A_{y}\,.
\end{array}
\tag{44}
$$

In the case of elastic scattering, only four of these are independent; in that case, we choose A_{xz}, A_y, K_y^{xx}, and K_{xz}^{xx} for the basic set. [We are guided in this choice by our planned experimental measurements of outgoing second-rank moments via the reaction ^{3}He$(\vec{d},p)^4$He.] The expressions to the right of the arrows again indicate the specialization which takes place for this case.

We can express each of the amplitude products in terms of the basic set:

$$\begin{aligned}
&\mathrm{Re}AE^* = \tfrac{1}{3}I_0(A_{xz} - K_{xz}^{xx}),\\
&\mathrm{Re}AD^* = \tfrac{1}{3}I_0(P^{xz} - K_{xx}^{xz}) \to \tfrac{1}{3}I_0(-A_{xz} + K_{xz}^{xx}),\\
&\mathrm{Im}AE^* = \tfrac{1}{3}I_0(-\tfrac{3}{2}A_y + \tfrac{3}{2}K_y^{xx}),\\
&\mathrm{Im}AD^* = \tfrac{1}{3}I_0(\tfrac{3}{2}P^y - \tfrac{3}{2}K_{xx}^y) \to \tfrac{1}{3}I_0(\tfrac{3}{2}A_y - \tfrac{3}{2}K_y^{xx}),\\
&\mathrm{Re}CE^* = \tfrac{1}{3}I_0(-2P^{xz} - K_{xx}^{xz}) \to \tfrac{1}{3}I_0(2A_{xz} + K_{xz}^{xx}),\\
&\mathrm{Re}CD^* = \tfrac{1}{3}I_0(-2A_{xz} - K_{xz}^{xx}),\\
&\mathrm{Im}CE^* = \tfrac{1}{3}I_0(3P^y + \tfrac{3}{2}K_{xx}^y) \to \tfrac{1}{3}I_0(3A_y + \tfrac{3}{2}K_y^{xx}),\\
&\mathrm{Im}CD^* = \tfrac{1}{3}I_0(3A_y + \tfrac{3}{2}K_y^{xx}).
\end{aligned} \tag{45}$$

Class 3. Real and imaginary parts of AC, DE. For these observables, there is no redundancy in the general case; that is, there are four observables and four amplitude products. We can write the relations:

$$\begin{aligned}
&\mathrm{Re}AC^* = \tfrac{1}{3}I_0(\tfrac{9}{4}K_y^y + K_{xz}^{xz}),\\
&\mathrm{Re}DE^* = \tfrac{1}{3}I_0(-\tfrac{9}{4}K_y^y + K_{xz}^{xz}),\\
&\mathrm{Im}AC^* = \tfrac{1}{3}I_0(\tfrac{3}{2}K_y^{xz} - K_{xz}^y),\\
&\mathrm{Im}DE^* = \tfrac{1}{3}I_0(\tfrac{3}{2}K_y^{xz} + \tfrac{3}{2}K_{xz}^y).
\end{aligned} \tag{46}$$

If the time-reversal condition, $D = -E$, is imposed, it follows that

$$\begin{aligned}
&\mathrm{Im}DE^* = 0 = \tfrac{3}{2}K_y^{xz} + \tfrac{3}{2}K_{xz}^y,\\
&\mathrm{Re}DE^* = -|D|^2 = \tfrac{1}{3}I_0(-\tfrac{9}{4}K_y^y + K_{xz}^{xz}).
\end{aligned} \tag{47}$$

The first of these conditions yields the usual type of result, which follows from general arguments (rule 3), but the second result is of a new type. Using the expression for $|D|^2$ from Eq. (42), we have the relation

$$\tfrac{9}{4}K_y^y - K_{xz}^{xz} = 2 - A_{xx} - K_{xx}^{xx}. \tag{48}$$

Thus, K_y^y need not be measured, in principle, if time reversal is assumed. This is an interesting point because it turns out that all of the amplitude products in classes 1, 2, and 4 can be determined from the experimentally favorable measurement of outgoing-particle second-rank tensors, even without the time-reversal constraint. With the present result, we can extend this to say that, for elastic scattering, *all* of the amplitude products may be determined with only second-rank outgoing-particle measurements.

Class 4. Real and imaginary parts of AB, BC, BD, BE. In this case there are always precisely two observables which determine the same amplitude product. Again we will prefer the quantities which refer to second-rank outgoing polarizations. We have

$$\begin{aligned}
&\mathrm{Re}AB^* = \tfrac{1}{3}I_0(2K_{xy}^{xy}) = \tfrac{1}{3}I_0(\tfrac{9}{2}K_z^z),\\
&\mathrm{Im}AB^* = \tfrac{1}{3}I_0(-3K_z^{xy}) = \tfrac{1}{3}I_0(3K_{xy}^z),\\
&\mathrm{Re}BC^* = \tfrac{1}{3}I_0(2K_{yz}^{yz}) = \tfrac{1}{3}I_0(\tfrac{9}{2}K_x^x),\\
&\mathrm{Im}BC^* = \tfrac{1}{3}I_0(-3K_x^{yz}) = \tfrac{1}{3}I_0(3K_{yz}^x),\\
&\mathrm{Re}BD^* = \tfrac{1}{3}I_0(2K_{xy}^{yz}) = \tfrac{1}{3}I_0(-\tfrac{9}{2}K_z^x),\\
&\mathrm{Im}BD^* = \tfrac{1}{3}I_0(3K_z^{yz}) = \tfrac{1}{3}I_0(3K_{xy}^x),\\
&\mathrm{Re}BE^* = \tfrac{1}{3}I_0(2K_{yz}^{xy}) = \tfrac{1}{3}I_0(-\tfrac{9}{2}K_x^z),\\
&\mathrm{Im}BE^* = \tfrac{1}{3}I_0(-3K_x^{xy}) = \tfrac{1}{3}I_0(-3K_{yz}^z).
\end{aligned} \tag{49}$$

If the time-reversal condition $D = -E$ is applied, one new type of relation results:

$$K_z^{yz} = K_x^{xy}. \tag{50}$$

In addition, of course, all of the usual (time-reversal) rules about interchanging superscripts and subscripts hold.

B. Quadratic Relations

Relations of a quadratic nature between the observables are also simple to derive. Consider the identity,

$$(\mathrm{Re}AB^*)^2 + (\mathrm{Im}AB^*)^2 = |A|^2|B|^2, \tag{51}$$

which, using Eqs. (42) and (49), can be rewritten in terms of observables,

$$(2K_{xy}^{xy})^2 + (3K_z^{xy})^2 = (1 - A_{xx} - P^{xx} + K_{xx}^{xx})(3 - 3A_{yy}). \tag{52}$$

A similar relation can be rewritten for each pair selected from A, B, C, D, E, so that there are ten relations of this type. The number reduces to six for elastic scattering.

A second type of relation relates three of the amplitudes. For example, consider the identity,

$$\mathrm{Re}AB^*\,\mathrm{Re}AC^* + \mathrm{Im}AB^*\,\mathrm{Im}AC^* = |A|^2\,\mathrm{Re}BC^*, \tag{53}$$

which becomes, using Eqs. (42), (46), and (49),

$$(2K_{xy}^{xy})(\tfrac{9}{4}K_y^y + K_{xz}^{xz}) + (-3K_z^{xy})(\tfrac{3}{2}K_y^{xz} - \tfrac{3}{2}K_{xz}^y) = (1 - A_{xx} - P^{xx} + K_{xx}^{xx})(2K_{yz}^{yz}). \tag{54}$$

There are 30 relations of this type, reducing to 12 for elastic scattering.

The third type of relation, which also relates three amplitudes, is illustrated by the example,

$$\mathrm{Re}AB^*\,\mathrm{Im}AC^* - \mathrm{Im}AB^*\,\mathrm{Re}AC^* = |A|^2\,\mathrm{Im}BC^*, \tag{55}$$

which becomes, again using Eqs. (42), (46), and (49),

$$(2K_{xy}^{xy})(\tfrac{3}{2}K_y^{xz} - \tfrac{3}{2}K_{xz}^{y}) - (-3K_z^{xy})(\tfrac{9}{4}K_y^y + K_{xz}^{xz}) = (1 - A_{xx} - P^{xx} + K_{xx}^{xx})(-3K_z^{yz}) . \quad (56)$$

Again, there are 30 relations of this type, reducing to 12 for elastic scattering.

C. Applications

The most important application of the above discussion is to the problem of the selection of independent, or relatively independent, observables for measurement. However, certain other interesting observations can be made with the aid of Tables I and II. For example, the $p_{y'y'}$ polarization of the outgoing spin-1 particle is completely determined by the analyzing tensors. That is, from Eqs. (4) and (AI4), we can write

$$I = I_0[1 + \tfrac{3}{2}p_y A_y + \tfrac{2}{3}p_{xz}A_{xz} + \tfrac{1}{6}(p_{xx} - p_{yy})(A_{xx} - A_{yy}) + \tfrac{1}{2}p_{zz}A_{zz}] , \quad (57)$$

and

$$p_{y'y'}I = I_0[P^{y'y'} + \tfrac{3}{2}p_y K_y^{y'y'} + \tfrac{2}{3}p_{xz}K_{xz}^{y'y'} + \tfrac{1}{6}(p_{xx} - p_{yy})(K_{xx}^{y'y'} - K_{yy}^{y'y'})] . \quad (58)$$

Equations (41) and (43) give expressions for each of the polarization-transfer coefficients which occur in Eq. (58) in terms of analyzing tensors

$$\begin{aligned} &P^{y'y'} = A_{yy}, \\ &K_y^{y'y'} = A_y, \\ &K_{xz}^{y'y'} = A_{xz}, \\ &K_{xx}^{y'y'} - K_{zz}^{y'y'} = A_{xx} - A_{zz}, \\ &K_{yy}^{y'y'} = 2 - A_{yy} . \end{aligned} \quad (59)$$

From Eqs. (57)–(59) one obtains

$$I(1 - p_{y'y'}) = I_0(1 - A_{yy})(1 - p_{yy}) . \quad (60)$$

Equation (60) is a nondynamical result, independent of nuclear models, and may be used to connect polarizations of different particles. For example, measurements of the analyzing tensors for the reaction $^{14}C(\vec{d}, \alpha)^{12}B$ would completely specify the $p_{y'y'}$ component of the ^{12}B polarization in the $^{14}C(d, \alpha)^{12}\vec{B}$ experiment; notice that this component would be unity if p_{yy} for the incident deuteron is unity.

One can also show with the aid of Eq. (60) that one can "pump up"[13] the tensor polarization to nearly unity if A_{yy} is sufficiently large. For example, for $^4He(\vec{d}, d)^4He$ elastic scattering at 12 MeV and $\theta_{lab} = 38°$,[14] $A_{yy} \approx 0.97$. For scattering to the right ($\phi = 180°$), $I/I_0 = 1.46$, so that a polarized beam with $p_y = p_{yy} = 0.85$ would, in that case, lead to the value $p_{y'y'} = 0.997$ for the scattered deuteron.

VII. COMPLETE SET OF MEASUREMENTS

Since, for elastic scattering, one knows there are only seven parameters which characterize the M matrix at a given energy and angle (assuming time-reversal invariance), it is interesting to find the minimum set of experiments which could be used for a complete and unambiguous determination. We conclude in the arguments to be given that this number is eight. The number of experiments is larger than the number of parameters because of a discrete ambiguity which must be resolved. In practice one would prefer to measure more than eight observables because of finite errors, and because one might fortuitously have very low sensitivity to a particular quantity with a particular choice for the set of experiments.

We consider the M matrix in the laboratory systems; that is, the initial particle is described in the projectile helicity frame and the final particle is described in the outgoing-particle laboratory helicity frame. The discussions are presented in these frames because we want to count the number of actual laboratory measurements which are required for a unique determination of M. If, instead, we had elected to use the center-of-mass helicity frames, knowledge of a quantity such as $K_{xx}^{x'x'}$ would actually involve measurement of three laboratory transfer coefficients followed by a suitable rotation of the laboratory quantities. We wish to regard the determination of $K_{xx}^{x'x'}$ as a single measurement, and this is permissible only if we use the laboratory axes.

The M matrix is given in Eq. (28). We will use A'', B'', C'', D'', and E'' for the expansion coefficients in the present frame [these symbols had a different meaning in Eq. (33)]. The condition imposed by time reversal in this frame is easily shown to be

$$D'' + E'' = (C'' - A'')\tan\psi , \quad (61)$$

where $\psi = \theta_{c.m.} - \theta_{lab}$. It will be convenient not to invoke this condition immediately, but rather to keep A'', B'', C'', D'', and E'' as separate (but related) variables. There are various data sets which do what we require; here we will proceed by choosing the experiments in order of increasing difficulty.

We first measure the cross section, I_0, and the analyzing tensors, A_y, A_{xz}, A_{xx}, and A_{yy}. Referring to Eq. (42) we see that we need, in addition to these quantities, $(P^{x'x'})_{lab}$ and $(K_{xx}^{x'x'})_{lab}$ to completely determine the absolute magnitudes of A'', B'', C'',

D'', E''. Now $(P^{x'x'})_{lab}$ can be calculated from the values of A_{xx}, A_{zz} $(=-A_{xx}-A_{yy})$, and A_{xz} as follows. In the outgoing center-of-mass system, we have $P^{x'x'}=A_{xx}$, $P^{z'z'}=A_{zz}$, and $P^{x'z'}=-A_{xz}$. We have only to rotate these second-rank polarizations so that they are referred to the outgoing laboratory axis. Thus, if $\psi=\theta_{c.m.}-\theta_{lab}$ (a positive number), we have [see Appendix III, Eq. (AIII1)],

$$(P^{x'x'})_{lab}=A_{xx}\cos^2\psi+A_{zz}\sin^2\psi-A_{xz}\sin 2\psi\,. \quad (62)$$

We therefore need to measure only one polarization-transfer coefficient, $(K_{xx}^{x'x'})_{lab}$, to complete our knowledge of the absolute magnitudes. [Any of the coefficients from the set $(K_{xx}^{x'x'})_{lab}$, $(K_{zz}^{z'z'})_{lab}$, $(K_{xx}^{z'z'})_{lab}$, or $(K_{zz}^{x'x'})_{lab}$ would have sufficed here, but we have adopted $(K_{xx}^{x'x'})_{lab}$ as a "standard" one for the discussion.] Up to the present, then, we have used six measurements.

We now consider the phases between the amplitudes. From the quantities A_y and A_{xz} we can form

$$(-I_0A_{xz})=\mathrm{Re}(A''E''^*+D''C''^*)$$
$$=\mathrm{Re}[-A''D''^*+D''C''^*+(A''C''^*-|A''|^2)\tan\psi]\,, \quad (63)$$
$$(\tfrac{3}{2}I_0A_y)=\mathrm{Im}(A''E''^*+D''C''^*)$$
$$=\mathrm{Im}[-A''D''^*+D''C''^*+(A''C''^*-|A''|^2)\tan\psi]\,,$$

where in the second form E'' has been eliminated by means of Eq. (61). That is, both the real and imaginary part of a particular combination of quantities is known, and all of the absolute magnitudes are known. If we put $A''D''^*=|A''||D''|e^{i\theta_{AD}}$, where $\theta_{AD}=\theta_A-\theta_D$, and similarly treat the other quantities, we can represent the situation geometrically in the complex plane as shown in Fig. 3. Notice that the quantity $-|A|^2\tan\psi$ is a real number and does not contribute to the argument about

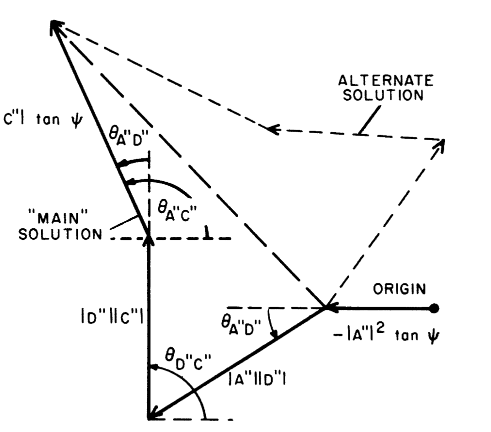

FIG. 3. Geometrical construction showing the twofold degeneracy of the phases of the amplitudes.

phase angles. In the geometrical construction there are actually only two independent angles, since $\theta_{AC}=\theta_{AD}+\theta_{DC}$. Since the lengths of the vectors are fixed, the angles are fixed uniquely except for the ambiguity shown in Fig. 3. To fix this ambiguity, we must turn to the measurement of another polarization-transfer coefficient. This can be either any additional one from class 2, with the exception of those proportional to A_y, A_{xz}, $(P^{y'})_{lab}$, or $(P^{x'z'})_{lab}$, or it could come from class 3. Favorable choices here, if (1) one prefers simple beam spin directions (i.e., if the beam quantization axis is to be along the x, y, or z direction), and (2) second-rank output moments are to be measured, are $(K_y^{x'x'})_{lab}$, $(K_y^{z'z'})_{lab}$, $(K_{xx}^{x'z'})_{lab}$, or $(K_{zz}^{x'z'})_{lab}$ from class 2, or $(K_y^{x'z'})_{lab}$ from class 3.

So far, we have used seven experimental quantities and have determined all of the parameters of the M matrix except the phase of B. We note that only observables in class 4 (and all observables in class 4) involve this phase. Therefore, the measurement of any of the class-4 variables would suffice for this determination, and there is no *a priori* reason for preferring one over another. A choice which would best meet the experimental requirements mentioned above is any one of the set $(K_x^{y'z'})_{lab}$, $(K_z^{y'z'})_{lab}$, $(K_z^{x'y'})_{lab}$, and $(K_x^{x'y'})_{lab}$.

Finally, then, we find that M is completely determined by eight measurements: The cross section, four analyzing tensors, and three polarization-transfer coefficients are required. Particularly convenient triads of transfer coefficients would be $(K_{xx}^{x'x'})_{lab}$, $(K_x^{y'z'})_{lab}$, and $(K_{xx}^{x'z'})_{lab}$, or $(K_{zz}^{z'z'})_{lab}$, $(K_{zz}^{x'z'})_{lab}$, and $(K_z^{y'z'})_{lab}$. To illustrate, if the beam quantization axis is along the x axis, the outgoing beam intensity and polarizations are:

$$\begin{aligned}
I&=I_0(1+\tfrac{1}{2}p_{xx}A_{xx})\,,\\
p_{x'}I&=I_0[\tfrac{3}{2}p_x(K_x^{x'})_{lab}]\,,\\
p_{y'}I&=I_0[(P^{y'})_{lab}+\tfrac{1}{2}p_{xx}(K_{xx}^{y'})_{lab}]\,,\\
p_{z'}I&=I_0[\tfrac{3}{2}p_x(K_x^{z'})_{lab}]\,,\\
p_{x'y'}I&=I_0[\tfrac{3}{2}p_x(K_x^{x'y'})_{lab}]\,,\\
p_{x'z'}I&=I_0[(P^{x'z'})_{lab}+\tfrac{1}{2}p_{xx}(K_{xx}^{x'z'})_{lab}]\,,\\
p_{y'z'}I&=I_0[\tfrac{3}{2}p_x(K_x^{y'z'})_{lab}]\,,\\
p_{x'x'}I&=I_0[(P^{x'x'})_{lab}+\tfrac{1}{2}p_{xx}(K_{xx}^{x'x'})_{lab}]\,,\\
p_{y'y'}I&=I_0[(P^{y'y'})_{lab}+\tfrac{1}{2}p_{xx}(K_{xx}^{y'y'})_{lab}]\,,\\
p_{z'z'}I&=I_0[(P^{z'z'})_{lab}+\tfrac{1}{2}p_{xx}(K_{xx}^{z'z'})_{lab}]\,.
\end{aligned} \quad (64)$$

We see that the sets $(K_{xx}^{x'x'})_{lab}$, $(K_x^{y'z'})_{lab}$, and $(K_{xx}^{x'z'})_{lab}$ are readily separable from the observable outgoing beam polarizations, provided that the polarization functions $(P^{x'z'})_{lab}$ and $(P^{x'x'})_{lab}$ have been previously determined.

VIII. TIME-REVERSAL TESTS

If the outgoing particle is referred to its center-of-mass direction, we have all of the time-reversal relations of the forms $K_x^{x'y'} = K_{xy}^{x'}$, $K_x^{z'z'} = -K_{zz}^{x'}$, etc. In addition, for the $1+0 \to 1+0$ elastic scattering problem only, we have the relations

$$\tfrac{9}{4}K_y^{y'} = 2 + K_{xz}^{x'z'} - A_{xx} - K_{xx}^{x'x'}\,,$$
$$K_z^{y'z'} = K_x^{x'y'}\,. \tag{65}$$

Neither of these relations appears to be particularly easy to check experimentally. A relation of the type,

$$K_{xx}^{x'z'} = -K_{xz}^{x'x'}\,, \tag{66}$$

offers more promise. Unfortunately, we have to contend with measurements in the laboratory system. The center-of-mass quantities expressed in terms of the laboratory quantities are

$$K_{xx}^{x'z'} = \tfrac{1}{2}(K_{xx}^{x'x'} - K_{xx}^{z'z'})_{\rm lab}\sin 2\psi + (K_{xx}^{x'z'})_{\rm lab}\cos 2\psi,$$
$$K_{xz}^{x'x'} = (K_{xz}^{x'x'})_{\rm lab}\cos^2\psi + (K_{xz}^{z'z'})_{\rm lab}\sin^2\psi \tag{67}$$
$$- (K_{xz}^{x'z'})_{\rm lab}\sin 2\psi\,,$$

where the rotation properties given in Appendix III have been used. (Although the relations given in Appendix III are written in terms of a beam of particles, they are, of course, applicable to any second-rank tensor quantities.) All of the outgoing polarization components would normally be measured simultaneously. Two different beam quantization directions could be used, although there exist certain choices for this direction which would simultaneously give a large p_{xz} and p_{xx} [see Eq. (2)]. Thus a check of Eq. (67) by a relatively accurate "ratio" technique may be possible. A similar test involving the zz components would be equally favorable.

IX. SUMMARY

We have developed a suitable description for $1+0 \to 1'+0'$ polarization-transfer phenomena. Emphasis has been placed on the use of different and convenient frames of reference for the initial and final particles. Linear relations between the observables have been listed, and a method for writing down the 70 quadratic relations has been given. The simplest set of experiments necessary to unambiguously determine the scattering matrix at a given energy and angle has been discussed, and the number of required experiments is found to be eight. A feasible comparison of polarization-transfer coefficients which would test time-reversal invariance was described.

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APPENDIX I: SPIN-1 MATRICES

The spin-1 matrices used in this paper are as follows:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{P}_x = S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{P}_y = S_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{P}_z = S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\mathcal{P}_{xy} = \tfrac{3}{2}(S_x S_y + S_y S_x) = \tfrac{3}{2} i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{P}_{xz} = \tfrac{3}{2}(S_x S_z + S_z S_x) = \frac{3}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{AI1}$$

$$\mathcal{P}_{yz} = \tfrac{3}{2}(S_y S_z + S_z S_y) = \frac{3i}{2\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\mathcal{P}_{xx} = 3(S_x)^2 - 2I = \tfrac{1}{2}\begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix},$$

$$\mathcal{P}_{yy} = 3(S_y)^2 - 2I = \tfrac{1}{2}\begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix},$$

$$\mathcal{P}_{zz} = 3(S_z)^2 - 2I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the S operators are the standard Cartesian spin-1 set given in, for example, Schiff's *Quantum Mechanics*.[15] Notice also that we have the operator identity,

$$\mathcal{P}_{xx} + \mathcal{P}_{yy} + \mathcal{P}_{zz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{AI2}$$

so that we need only two of these three matrices.

It is often convenient to work with the pairs $\frac{1}{2}(\mathcal{P}_{\alpha\alpha}-\mathcal{P}_{\beta\beta})$, $\mathcal{P}_{\gamma\gamma}$ where α, β, and γ are x, y, and z in any order. The explicit matrices are

$$\frac{1}{2}(\mathcal{P}_{xx}-\mathcal{P}_{yy})=\frac{3}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$\frac{1}{2}(\mathcal{P}_{xx}-\mathcal{P}_{zz})=\frac{3}{4}\begin{pmatrix}-1&0&1\\0&2&0\\1&0&-1\end{pmatrix}, \tag{AI3}$$

$$\frac{1}{2}(\mathcal{P}_{yy}-\mathcal{P}_{zz})=\frac{3}{4}\begin{pmatrix}-1&0&1\\0&2&0\\1&0&-1\end{pmatrix}.$$

This operator identity allows us to deduce the identity,

$$\frac{1}{3}p_{xx}B_{xx}+\frac{1}{3}p_{yy}B_{yy}+\frac{1}{3}p_{zz}B_{zz} =\frac{1}{6}(p_{\alpha\alpha}-p_{\beta\beta})(B_{\alpha\alpha}-B_{\beta\beta})+\frac{1}{2}p_{\gamma\gamma}B_{\gamma\gamma}, \tag{AI4}$$

where α, β, γ are again x, y, z in any order. This relation applies if both the p's and B's satisfy the identity of Eq. (AI2), and thus is true if the B's represent any of the polarization-transfer quantities or the analyzing tensors.

The density matrix, ρ, for an ensemble of polarized particles can be expanded in terms of these operators, in the form

$$\rho=\frac{1}{3}(I+\frac{3}{2}\langle\mathcal{P}_j\rangle\mathcal{P}_j+\frac{1}{3}\langle\mathcal{P}_{jk}\rangle\mathcal{P}_{jk}), \tag{AI5}$$

where repeated indices indicate a sum. We have $\mathrm{Tr}\rho=1$, $\mathrm{Tr}(\rho\mathcal{P}_j)=\langle\mathcal{P}_j\rangle$, and $\mathrm{Tr}(\rho\mathcal{P}_{jk})=\langle\mathcal{P}_{jk}\rangle$. In a scattering experiment, the initial density matrix, ρ_i, has the polarized beam components p_j and p_{jk} as expectation values $\langle\mathcal{P}_j\rangle_i$ and $\langle\mathcal{P}_{jk}\rangle_i$, respectively. The final density matrix, ρ_f, is related to ρ_i by $\rho_f=M\rho_iM^\dagger$, where M is the scattering matrix.

The cross section for an unpolarized beam is $I_0=\frac{1}{3}\mathrm{Tr}MM^\dagger$, since $\rho_i=\frac{1}{3}I$ for $\langle\mathcal{P}_j\rangle=\langle\mathcal{P}_{jk}\rangle=0$. For a polarized beam, the cross section is

$$\begin{aligned}I&=\frac{1}{3}\mathrm{Tr}M(1+\frac{3}{2}p_j\mathcal{P}_j+\frac{1}{3}p_{jk}\mathcal{P}_{jk})M^\dagger\\&=I_0(1+\frac{3}{2}p_jA_j+\frac{1}{3}p_{jk}A_{jk}),\end{aligned} \tag{AI6}$$

where

$$A_j=\frac{\mathrm{Tr}M\mathcal{P}_jM^\dagger}{\mathrm{Tr}MM^\dagger}$$

and

$$A_{jk}=\frac{\mathrm{Tr}M\mathcal{P}_{jk}M^\dagger}{\mathrm{Tr}MM^\dagger}.$$

After scattering, the polarizations are given by

$$\langle\mathcal{P}_j\rangle=\frac{\mathrm{Tr}\rho_f\mathcal{P}_j}{\mathrm{Tr}\rho_f};\quad \langle\mathcal{P}_{jk}\rangle=\frac{\mathrm{Tr}\rho_f\mathcal{P}_{jk}}{\mathrm{Tr}\rho_f}.$$

Therefore, the final polarizations are $p_{j'}=\langle\mathcal{P}_{j'}\rangle_f$ and $p_{j'k'}=\langle\mathcal{P}_{j'k'}\rangle_f$, so that

$$Ip_{j'}=I_0(p^{j'}+\frac{3}{2}p_jK_j^{j'}+\frac{1}{3}p_{jk}K_{jk}^{j'}),$$

$$Ip_{j'k'}=I_0(p^{j'k'}+\frac{3}{2}p_jK_j^{j'k'}+\frac{1}{3}p_{jk}K_{jk}^{j'k'}),$$

where $K_j^{j'}=\mathrm{Tr}(M\mathcal{P}_jM^\dagger\mathcal{P}_{j'})/\mathrm{Tr}MM^\dagger$, etc.

APPENDIX II: PROPERTIES OF THE χ REPRESENTATION

We define

$$\vec{\chi}\cdot\vec{R}=\begin{pmatrix}R_+^*\\R_0^*\\R_-^*\end{pmatrix}$$

and $(\vec{\chi}\cdot\vec{Q})^\dagger=(Q_+Q_0Q_-)$ [see Eqs. (18) and (20)].

(1) Proof that $[\vec{\chi}\cdot\vec{R}(\vec{\chi}\cdot\vec{Q})^\dagger]^\dagger=\vec{\chi}\cdot\vec{Q}(\vec{\chi}\cdot\vec{R})^\dagger$:

$$\vec{\chi}\cdot\vec{R}(\vec{\chi}\cdot\vec{Q})=\begin{pmatrix}R_+^*Q_+&R_+^*Q_0&R_+^*Q_-\\R_0^*Q_+&R_0^*Q_0&R_0^*Q_-\\R_-^*Q_+&R_-^*Q_0&R_-^*Q_-\end{pmatrix},$$

and (AII1)

$$\vec{\chi}\cdot\vec{Q}(\vec{\chi}\cdot\vec{R})=\begin{pmatrix}Q_+^*R_+&Q_+^*R_0&Q_+^*R_-\\Q_0^*R_+&Q_0^*R_0&Q_0^*R_-\\Q_-^*R_+&Q_-^*R_0&Q_0^*R_-\end{pmatrix}.$$

Since these matrices are Hermitian conjugates of each other, the statement is proved.

(2) Proof of the cyclic trace rule: $\mathrm{Tr}\Omega\vec{\chi}\cdot\vec{R}(\vec{\chi}\cdot\vec{Q})^\dagger=\mathrm{Tr}(\vec{\chi}\cdot\vec{Q})^\dagger\Omega\vec{\chi}\cdot\vec{R}$.

$$\begin{aligned}\mathrm{Tr}\Omega\vec{\chi}\cdot\vec{R}(\vec{\chi}\cdot\vec{Q})^\dagger&=\mathrm{Tr}\begin{pmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{pmatrix}\begin{pmatrix}R_+^*Q_+&R_+^*Q_0&R_+^*Q_-\\R_0^*Q_+&R_0^*Q_0&R_0^*Q_-\\R_-^*Q_+&R_-^*Q_0&R_-^*Q_-\end{pmatrix}\\&=\Omega_{11}R_+^*Q_++\Omega_{12}R_0^*Q_++\Omega_{13}R_-^*Q_+\\&\quad+\Omega_{21}R_+^*Q_0+\Omega_{22}R_0^*Q_0+\Omega_{23}R_-^*Q_0\\&\quad+\Omega_{31}R_+^*Q_-+\Omega_{32}R_0^*Q_-+\Omega_{33}R_-^*Q_-.\end{aligned} \tag{AII2}$$

Also,

$$\begin{aligned}\mathrm{Tr}(\vec{\chi}\cdot\vec{Q})^\dagger\Omega\vec{\chi}\cdot\vec{R}&=\mathrm{Tr}(Q_+Q_0Q_-)\begin{pmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{pmatrix}\begin{pmatrix}R_+^*\\R_0^*\\R_-^*\end{pmatrix}\\&=\mathrm{Tr}(Q_+Q_0Q_-)\begin{pmatrix}\Omega_{11}R_+^*+\Omega_{12}R_0^*+\Omega_{13}R_-^*\\\Omega_{21}R_+^*+\Omega_{22}R_0^*+\Omega_{23}R_-^*\\\Omega_{31}R_+^*+\Omega_{32}R_0^*+\Omega_{33}R_-^*\end{pmatrix},\end{aligned} \tag{AII3}$$

which is seen to be the trace of a 1×1 matrix with the value

$$\Omega_{11}R_+^*Q_+ + \Omega_{12}R_0^*Q_+ + \Omega_{13}R_-^*Q_+$$
$$+\Omega_{21}R_+^*Q_0 + \Omega_{22}R_0^*Q_0 + \Omega_{23}R_-^*Q_0$$
$$+\Omega_{31}R_+^*Q_- + \Omega_{32}R_0^*Q_- + \Omega_{33}R_-^*Q_- \,. \qquad (\text{AII4})$$

Equation (AII2) is identical to Eq. (AII4), which proves the statement.
(3) The trace rules are

$$\mathrm{Tr}(\chi\cdot\vec{\mathrm{R}})^\dagger\chi\cdot\vec{\mathrm{Q}} = \vec{\mathrm{R}}\cdot\vec{\mathrm{Q}}\,,$$
$$\mathrm{Tr}(\chi\cdot\vec{\mathrm{R}})^\dagger\mathcal{P}_j\,\chi\cdot\vec{\mathrm{Q}} = i\vec{R}\times\vec{\mathrm{Q}}\,,$$
$$\mathrm{Tr}(\chi\cdot\vec{\mathrm{R}})^\dagger\mathcal{P}_{jk}\,\chi\cdot\vec{\mathrm{Q}} = -\tfrac{3}{2}(R_jQ_k+Q_jR_k)\,, \qquad (\text{AII5})$$
$$\mathrm{Tr}(\chi\cdot\vec{\mathrm{R}})^\dagger\mathcal{P}_{jj}\,\chi\cdot\vec{\mathrm{Q}} = \vec{\mathrm{R}}\cdot\vec{\mathrm{Q}} - R_jQ_j\,.$$

These follow readily from explicit construction.

For example,

$$\mathrm{Tr}(\chi\cdot\vec{\mathrm{R}})^\dagger\mathcal{P}_{xy}(\chi\cdot\vec{\mathrm{Q}})$$
$$=(R_+R_0R_-)\begin{pmatrix}0&0&-\frac{3}{2}i\\0&0&0\\\frac{3}{2}i&0&0\end{pmatrix}\begin{pmatrix}Q_+^*\\Q_0^*\\Q_-^*\end{pmatrix}$$
$$=-\tfrac{3}{2}i(R_+Q_-^*-R_-Q_+^*)$$
$$=-\tfrac{3}{2}i\left[\frac{(-R_x-iR_y)(Q_x+iQ_y)}{2}-\frac{(R_x-iR_y)(-Q_x+iQ_y)}{2}\right]$$
$$=-\tfrac{3}{2}(R_xQ_y+R_yQ_x)\,. \qquad (\text{AII6})$$

We need these traces only when $\vec{\mathrm{R}}$ and $\vec{\mathrm{Q}}$ are real unit vectors along the x, y, z axes. The specific results are given in Table III, where α, $\beta = x$, y, or z, and Ω is one of the basis operators I, $\mathcal{P}_i$, or $\mathcal{P}_{ij}$ in the expression $\mathrm{Tr}\chi_\alpha^\dagger\Omega\chi_\beta$.

TABLE III. Traces of the Cartesian operators.

α	Ω	β	Trace	α	Ω	β	Trace	α	Ω	β	Trace
x	I	x	1	y	I	y	1	z	I	z	1
x	$\mathcal{P}_x$	x	0	y	$\mathcal{P}_x$	y	0	z	$\mathcal{P}_x$	z	0
x	$\mathcal{P}_y$	x	0	y	$\mathcal{P}_y$	y	0	z	$\mathcal{P}_y$	z	0
x	$\mathcal{P}_z$	x	0	y	$\mathcal{P}_z$	y	0	z	$\mathcal{P}_z$	z	0
x	$\mathcal{P}_{xy}$	x	0	y	$\mathcal{P}_{xy}$	y	0	z	$\mathcal{P}_{xy}$	z	0
x	$\mathcal{P}_{xz}$	x	0	y	$\mathcal{P}_{xz}$	y	0	z	$\mathcal{P}_{xz}$	z	0
x	$\mathcal{P}_{yz}$	x	0	y	$\mathcal{P}_{yz}$	y	0	z	$\mathcal{P}_{yz}$	z	0
x	$\mathcal{P}_{xx}$	x	-2	y	$\mathcal{P}_{xx}$	y	1	z	$\mathcal{P}_{xx}$	z	1
x	$\mathcal{P}_{yy}$	x	1	y	$\mathcal{P}_{yy}$	y	-2	z	$\mathcal{P}_{yy}$	z	1
x	$\mathcal{P}_{zz}$	x	1	y	$\mathcal{P}_{zz}$	y	1	z	$\mathcal{P}_{zz}$	z	-2
x	I	z	0	z	I	x	0	x	I	y	0
x	$\mathcal{P}_x$	z	0	z	$\mathcal{P}_x$	x	0	x	$\mathcal{P}_x$	y	0
x	$\mathcal{P}_y$	z	i	z	$\mathcal{P}_y$	x	$-i$	x	$\mathcal{P}_y$	y	0
x	$\mathcal{P}_z$	z	0	z	$\mathcal{P}_z$	x	0	x	$\mathcal{P}_z$	y	$-i$
x	$\mathcal{P}_{xy}$	z	0	z	$\mathcal{P}_{xy}$	x	0	x	$\mathcal{P}_{xy}$	y	$-\frac{3}{2}$
x	$\mathcal{P}_{xz}$	z	$-\frac{3}{2}$	z	$\mathcal{P}_{xz}$	x	$-\frac{3}{2}$	x	$\mathcal{P}_{xz}$	y	0
x	$\mathcal{P}_{yz}$	z	0	z	$\mathcal{P}_{yz}$	x	0	x	$\mathcal{P}_{yz}$	y	0
x	$\mathcal{P}_{xx}$	z	0	z	$\mathcal{P}_{xx}$	x	0	x	$\mathcal{P}_{xx}$	y	0
x	$\mathcal{P}_{yy}$	z	0	z	$\mathcal{P}_{yy}$	x	0	x	$\mathcal{P}_{yy}$	y	0
x	$\mathcal{P}_{zz}$	z	0	z	$\mathcal{P}_{zz}$	x	0	x	$\mathcal{P}_{zz}$	y	0
y	I	x	0	y	I	z	0	z	I	y	0
y	$\mathcal{P}_x$	x	0	y	$\mathcal{P}_x$	z	$-i$	z	$\mathcal{P}_x$	y	i
y	$\mathcal{P}_y$	x	0	y	$\mathcal{P}_y$	z	0	z	$\mathcal{P}_y$	y	0
y	$\mathcal{P}_z$	x	i	y	$\mathcal{P}_z$	z	0	z	$\mathcal{P}_z$	y	0
y	$\mathcal{P}_{xy}$	x	$-\frac{3}{2}$	y	$\mathcal{P}_{xy}$	z	0	z	$\mathcal{P}_{xy}$	y	0
y	$\mathcal{P}_{xz}$	x	0	y	$\mathcal{P}_{xz}$	z	0	z	$\mathcal{P}_{xz}$	y	0
y	$\mathcal{P}_{yz}$	x	0	y	$\mathcal{P}_{yz}$	z	$-\frac{3}{2}$	z	$\mathcal{P}_{yz}$	y	$-\frac{3}{2}$
y	$\mathcal{P}_{xx}$	x	0	y	$\mathcal{P}_{xx}$	z	0	z	$\mathcal{P}_{xx}$	y	0
y	$\mathcal{P}_{yy}$	x	0	y	$\mathcal{P}_{yy}$	z	0	z	$\mathcal{P}_{yy}$	y	0
y	$\mathcal{P}_{zz}$	x	0	y	$\mathcal{P}_{zz}$	z	0	z	$\mathcal{P}_{zz}$	y	0

APPENDIX III: ROTATION OF THE VECTOR AND TENSOR POLARIZATION COMPONENTS

We give for convenience the expressions needed for rotation of the vector and tensor components of polarization about the y axis. The sense of ψ is such that $\psi = \theta_{\mathrm{c.m.}} - \theta_{\mathrm{lab}}$, a positive number, should be used if the single-primed quantities are referred to the $\hat{k}_{\mathrm{out}}$(lab) and the double-primed quantities to $\hat{k}_{\mathrm{out}}$(c.m.):

$$
\begin{aligned}
&p_{x''} = p_{x'}\cos\psi - p_{z'}\sin\psi\,,\\
&p_{y''} = p_{y'}\,,\\
&p_{z''} = p_{x'}\sin\psi + p_{z'}\cos\psi\,,\\
&p_{x''x''} = p_{x'x'}\cos^2\psi + p_{z'z'}\sin^2\psi - p_{x'z'}\sin 2\psi\,,\\
&p_{y''y''} = p_{y'y'}\,,\\
&p_{z''z''} = p_{x'x'}\sin^2\psi + p_{z'z'}\cos^2\psi + p_{x'z'}\sin 2\psi\,,\\
&p_{x''z''} = \tfrac{1}{2}p_{x'x'}\sin 2\psi - \tfrac{1}{2}p_{z'z'}\sin 2\psi + p_{x'z'}\cos 2\psi\,,\\
&p_{x''y''} = p_{x'y'}\cos\psi - p_{y'z'}\sin\psi\,,\\
&p_{y''z''} = p_{x'y'}\sin\psi + p_{y'z'}\cos\psi\,,\\
&(p_{x''x''} - p_{y''y''}) = (p_{x'x'} - p_{y'y'})\times\tfrac{1}{2}(\cos^2\psi + 1) + p_{z'z'}(\tfrac{3}{2}\sin^2\psi) - p_{x'z'}\sin 2\psi\,.
\end{aligned}
\qquad \text{(AIII1)}
$$

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