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Off-Shell Two-Body t Matrix in the Nucleon-Nucleus Optical Potential*

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We investigate the effects of taking the two-nucleon scattering matrices off energy shell in the first term of the nucleon-nucleus optical potential. In order to avoid Coulomb interference, only neutron-nucleus elastic scattering is considered. We look both for the possibility of improving the fit to the elastic nucleon-nucleus scattering observables by varying off-shell parametrization, as well as the possibility of extracting off-shell information on the nucleon-nucleon force.

I. INTRODUCTION

In the Watson approach to the elastic scattering of nucleons from the ground state of a nucleus,[1] the optical-potential operator can be expanded to any order in terms of the Watson modified scattering operator,

$$\tau_i(E^0) = v_i + v_i G'(E^0)\tau_i(E^0), \tag{1}$$

where v_i is the free two-body interaction between the incident and ith target nucleon, and E^0 is the kinetic energy in the nucleon-nucleus center of mass. This scattering operator differs from the free two-body scattering matrix in that its Green's function,

$$G'(E^0) = Q_0(E^0 + \epsilon_0 + i\eta - \hat{E} - H_N)^{-1}, \tag{2}$$

contains the projection Q_0 off the ground state $|0\rangle$ (whose energy is ϵ_0) of the nuclear Hamiltonian H_N; these are all many-body operators defined in the center of mass of the target nucleus. In (2), and subsequently, a caret is used to denote the operator of a kinematic variable. In contrast, the free two-body scattering operator is

$$t_i(e_i) = v_i + v_i g(e_i) t_i(e_i), \tag{3}$$

whose Green's function,

$$g(e_i) = (e_i + i\eta - \hat{e}_i)^{-1}, \tag{4}$$

contains only the kinetic energy e_i of the interacting pair of nucleons in their center of mass.

The free operator, being translationally invariant, is intrinsically on its momentum shell; i.e., the initial and final total momenta are the same. However, it can be extended off its energy shell. This extension is necessary whenever the nucleon-nucleon force occurs in processes involving more than two bodies. In these processes, the energy of the initial and final momentum states of the two nucleons need not equal each other nor the value of the energy parameter e_i. Since only the on-shell matrix elements are experimentally determined in nucleon-nucleon scattering, there is considerable interest in the study of processes that are sensitive to differences between the off- and on-shell matrix elements. There is also interest in the possibility of selecting between elastically equivalent potentials[2] which lead to different off-shell extensions.

In this paper, elastic nucleon-nucleus scattering is examined in an extension of the impulse approximation in order to find if there is a range of nuclei and incident energies for which off-shell effects may be discerned from effects due to multiple scattering, target nucleon binding, etc. In Sec. II, our approach for going from a many-body to a two-body problem is presented. Section III develops the kinematic relations between the many-body and two-body relative momenta. The relation between the many-body and two-body matrix elements is obtained in Sec. IV. In Sec. V, the first-order potential is described, and our off-shell parametrization and numerical procedures are discussed in Sec. VI. Cross sections and polarizations for a variety of nuclei, energies, and off-shell parameters are illustrated and discussed in Sec. VII, and our conclusions are stated in Sec. VIII.

II. EXTENDED-IMPULSE-APPROXIMATION APPROACH

The basic problem is to relate matrix elements of the many-body modified scattering operators, which are defined and evaluated in the barycentric subspace of $A+1$ bodies and in terms of which the optical potential is expanded, to the free two-body scattering matrix elements, which are defined in the barycentric subspace of two nucleons and whose on-shell values have been determined via nucleon-nucleon scattering experiments. For the usual choices of a modified scattering operator this cannot be done unambiguously, because they conserve neither the total two-body momentum of the interacting pair nor the individual momenta of the other nucleons. In the Watson operator, this is due to both Q_0 and H_N in the Green's function, since they are diagonal in the nuclear states, for which internal momenta are not constant.

A direct relation between matrix elements is possible if the potential is expanded in terms of operators which do not contain Q_0 and which replace H_N by some model-dependent operator H_i which is diagonal in the many-body momentum space, the new operators being

$$\tilde{\tau}_i(E^0) = v_i + v_i \tilde{G}_i(E^0)\tilde{\tau}_i(E^0), \tag{5}$$

with

$$\tilde{G}_i(E^0) = (E^0 + \epsilon_0 + i\eta - \tilde{E} - \tilde{H}_i)^{-1}. \tag{6}$$

The many-body operators of (5), which are constructed so as to conserve two-body momentum, lead to an unambiguous extension of the impulse approximation (EIA) for which kinematic relations between the nucleon-nucleus and nucleon-nucleon variables are uniquely defined, as described in the next section.

To any order, the potential can be formally expanded in terms of the $\tilde{\tau}$ by using the operator identity

$$\tau_i(E^0) = \tilde{\tau}_i(E^0) + \tilde{\tau}_i(E^0)[G'(E^0) - \tilde{G}_i(E^0)]\tau_i(E^0) \tag{7}$$

in the Watson expansion to replace τ_i by $\tilde{\tau}_i$. Practically speaking, terms of third and higher order have not been calculated, while second-order terms have been only crudely evaluated using models of nuclear-correlation functions and making further approximations to avoid integrating over internal scattering angles and summing over intermediate nuclear states.[3] Even the first-order terms generally neglect exchange and internal momentum-distribution effects. In this paper, we consider only the first term, reserving for a later work binary expansions in the $\tilde{\tau}_i$.

To first order in these operators, the momentum-space matrix elements of the potential are

$$\langle \vec{K}| V(E^0)|\vec{K}'\rangle = \langle \vec{K}|\langle 0|\sum_{i=1}^{A}\tilde{\tau}_i(E^0)|0\rangle|\vec{K}'\rangle, \tag{8}$$

where the final and initial relative momenta between the incident nucleon and the nucleus obey the nonrelativistic kinematic relations

$$E = K^2/2\mu, \quad E' = K'^2/2\mu, \tag{9}$$

where $\mu = [A/(A+1)]m$ (m is the average nucleon mass) is the reduced mass. This nonlocal (because it is not just a function of $\vec{q} = \vec{K} - \vec{K}'$), complex potential is a matrix in spin space; for spin-zero nuclei, it is 2×2 and consists of a central and a spin-orbit part.

The potential matrix elements (8) are used in the momentum-space scattering equation for the reaction matrix to solve for the scattering amplitude of the neutron-nucleus elastic collision. The numerical procedure is described in detail elsewhere,[4,5] and is discussed further in Sec. VI. Here, it is important to stress that although the Born approximation fixes the magnitudes of $\vec{K}$ and $\vec{K}'$ at the value K^0 corresponding to the energy E^0, the magnitudes of these momenta range from 0 to ∞ in the exact partial-wave integral-scattering-equation calculation that is used in this work.

III. KINEMATICS OF THE EIA

As a consequence of conserving two-body momentum, the momentum transfer $\vec{q}$ from the incident nucleon to the nucleus is entirely transferred to the struck target nucleon, so that

$$\vec{K} - \vec{K}' = \vec{q} = \vec{k} - \vec{k}', \tag{10}$$

where $\vec{k}$ and $\vec{k}'$ are the final and initial relative momenta between the interacting nucleon pair, and the subscript i is dropped. The definitions of relative momenta imply

$$\vec{k} + \vec{k}' + \tfrac{1}{2}(\vec{l} + \vec{l}') = c(\vec{K} + \vec{K}'), \tag{11}$$

where $\vec{l}$ and $\vec{l}'$ are the final and initial internal momenta of the target nucleon relative to the nuclear center of mass, and $c = (A+1)/2A$ is the ratio between the energy e^0 for free scattering in the two-body center of mass and the over-all energy E^0.

If internal momentum is neglected, a simple calculation starting from (10) and (11) gives the relations[6]

$$\begin{aligned} k^2 &= \tfrac{1}{2}(c^2+c)K^2 + \tfrac{1}{2}(c^2-c)K'^2 + \tfrac{1}{4}(1-c^2)q^2, \\ k'^2 &= \tfrac{1}{2}(c^2-c)K^2 + \tfrac{1}{2}(c^2+c)K'^2 + \tfrac{1}{4}(1-c^2)q^2, \\ kk'\cos\theta &= \tfrac{1}{2}c^2(K^2+K'^2) - \tfrac{1}{4}(1+c^2)q^2, \end{aligned} \tag{12}$$

between the nucleon-nucleus kinematic variables on the right-hand side and the two-body kinematic

variables. For any fixed values of A, K, and K', the scattering angle θ in the nucleon-nucleon center of mass goes from 0 to π as the over-all center-of-mass scattering angle Θ goes from 0 to π. In the usual approach, free two-body kinematics is used with k and k' fixed at their on-shell value k^0, which is given by $e^0 = k^{0\,2}/m$. This is not consistant with (12) and causes θ to reach π before Θ does, so that an arbitrary extension of the matrix elements for q greater than $2k^0$ is necessary.[5]

The form of (12) illustrates two points. First, the definitions of k and k' are symmetric under the interchange of K and K'. This is consistent with reciprocity,[7] which requires that the elastic scattering potential be invariant under the transformation $\vec{K} \to -\vec{K}'$ and $\vec{K}' \to -\vec{K}$. Second, q rather than Θ is used as a nucleon-nucleus parameter, since local potentials are functions of $\vec{q}$ only, while potentials which depend on $\vec{K}$ and $\vec{K}'$, as well as $\vec{q}$, are nonlocal. The optical potential used in this work is nonlocal because it is constructed from two-body matrix elements whose kinematic variables, k, k', and θ, depend upon K and K', as well as q.

It can also be seen from (12) that fixing K and K' at their ON-shell value of K^0 (upper case denoting the nucleon-nucleus energy shell) does not fix the values of the two-nucleon momentum variables, since k and k' still depend on q. Thus, k and k', though equal to each other in this case, do not necessarily correspond to the appropriate nucleon-nucleon energy (derived in Sec. IV). This takes the two-body t matrix off shell in a local manner, and results in what we call the "local off-shell effect." When $K \neq K'$, so that the nucleon-nucleus matrix elements are OFF shell, it follows that $k \neq k'$, so that the two-body t matrix is off shell irrespective of the value of the appropriate nucleon-nucleon energy; and when $K = K' \neq K^0$, $k = k'$ is still off shell, in general. Both these situations shall be said to give rise to "nonlocal off-shell effects."

IV. MATRIX ELEMENTS OF THE EIA SCATTERING OPERATOR

The matrix elements of $\bar{\tau}_i(E^0)$ are now taken between the momentum states $|\vec{K}\vec{l}\cdots\vec{l}_A\rangle$ and $|\vec{K}'\vec{l}_1'\cdots\vec{l}_A'\rangle$, which are normalized to a δ function in all momenta. By the definition of internal momenta, their sum is zero in any nuclear state,

$$\sum_{j=1}^{A} \vec{l}_j = 0 = \sum_{j=1} \vec{l}_j', \tag{13}$$

and by the definition of $\bar{\tau}_i$, the final and initial values are related by (for $j \neq i$)

$$\vec{l}_j - \vec{l}_j' = -\vec{q}/A$$

and

$$\vec{l}_i - \vec{l}_i' = [(A-1)/A]\vec{q}\,. \tag{14}$$

In the nonrelativistic approximation, $\vec{l}_j$, $\vec{k}_j$, and $\vec{K}$ are Gallilean invariants,[8] as are the matrix elements of the scattering operators. This means that the two-body matrix elements are not affected by the dependence of the two-body center of mass on the internal momentum distribution of the struck particle.

The relation between the matrix elements of $\bar{\tau}_i$ and t_i is effected by inserting a complete intermediate set of states, $|\vec{K}''\vec{l}_1''\cdots\vec{l}_A''\rangle$, between the Green's function and the right-hand $\bar{\tau}_i$ in (5), which is being evaluated between final and initial many-body momentum states. In this way, a many-body Lippmann-Schwinger equation is written in a momentum representation. Integration over the complete set of states, appearing in the Green's function of $\bar{\tau}_i$, is restricted by relations (13) and (14) applied to the intermediate internal momentum states. Thus, integrating over the momentum space of $A+1$ nucleons is equivalent to making the integration $\int d^3p_0''$ over the momentum space of the incident particle, which is labeled with the index 0. In the over-all center of mass, this is equal to $\int d^3K''$, while in the two-body center of mass this equals $\int d^3k_i''$.

The Green's function of (5) is now evaluated in the intermediate momentum state. The model Hamiltonian $\tilde{H}_i$, which is diagonal in all momenta, is separated into a potential part $\tilde{U}$ and the exact kinetic part which together with $\hat{E}$ gives the term $\sum_{j=0}^{A}\hat{\mathcal{E}}_j$, where $\hat{\mathcal{E}}_j$ is the kinetic energy operator of the jth particle. Under the momentum-conserving assumption of the EIA, the single-particle energy, $\hat{\mathcal{E}}_j$, $j \neq 0, i$, and the total two-body energy of the incident and struck particle, $\hat{\mathcal{E}}_{0i}$, are c numbers, so that

$$\begin{aligned}
\sum_{j=0}^{A}\hat{\mathcal{E}}_j &= \hat{\mathcal{E}}_0 + \hat{\mathcal{E}}_i + \sum_{j\neq 0,i}\hat{\mathcal{E}}_j = \hat{e}_i + \mathcal{E}_{0i}' + \sum_{j\neq 0,i}\mathcal{E}_j' \\
&= \hat{e}_i - e_i' + \sum_{j=0}^{A}\mathcal{E}_j' = \hat{e}_i - e_i + \sum_{j=0}^{A}\mathcal{E}_j \\
&= \hat{e}_i - \tfrac{1}{2}(e_i + e_i') + \sum_{j=0}^{A}\tfrac{1}{2}(\mathcal{E}_j + \mathcal{E}_j') \\
&= \hat{e}_i - \bar{e}_i + \bar{E} + \sum_{j=1}^{A}\overline{l_j^2}/2m,
\end{aligned} \tag{15}$$

where the last two lines are the average of the two equal expressions of the second line, and the

following definitions have been made:

$$\bar{e}_i = \tfrac{1}{2}(k_i^2/m + k_i'^2/m),$$
$$\bar{E} = \tfrac{1}{2}(K^2/2\mu + K'^2/2\mu), \tag{16}$$
$$\overline{l_j^2} = \tfrac{1}{2}(l_j^2 + l_j'^2).$$

Thus, the energy denominator of $\tilde{G}(E^0)$ is

$$\bar{e}_i + (E^0 - \bar{E}) + \left(\epsilon_0 - \sum_{j=1}^{A} \overline{l_j^2}/2m - \tilde{U}_i\right) - \hat{e}_i . \tag{17}$$

The only kinematic operator explicitly appearing in (16) is $\hat{e}_i$, so that (16) is formally identical to the denominator of (4). To complete the identification, the form of $\tilde{U}_i$ must be specified. It is first noted that allowing $\tilde{U}_i$ to contain a function of $\hat{e}_i$ is equivalent to letting the target nucleon go off its mass shell.[9] Second, $\tilde{U}_i$ can be a function of momentum transfer and internal momenta, along with their scalar products. Finally, the actual choice should be made so as to minimize the size of terms containing $H_N - \tilde{H}_i$, which occur in second- and higher- order terms in the expansion of the optical potential. However, the generality in the formalism introduced by the operator $\tilde{U}_i$ will not be made use of in this paper, since here the primary interest is the examination of off-energy-shell effects in a first-order potential.

A simple form for $\tilde{U}_i$ can be inferred from the requirement of reciprocity, which dictates that the internal momentum distribution of the nuclear ground state be interpreted in a reciprocally invariant way. Thus, the ground-state wave function is written as $\langle \vec{\bar{l}}_1 \cdots \vec{\bar{l}}_A | 0 \rangle$, where $\vec{\bar{l}}_j$ is the vector average of final and initial momenta; moreover, the form factor is written

$$F(q) = \int d^3\bar{l}_1 \cdots d^3\bar{l}_A \delta\left(\sum_{j=1}^{A} \vec{\bar{l}}_j\right) \left\langle 0 \left| \vec{\bar{l}}_1 - \frac{q}{2A} \cdots \vec{\bar{l}}_i + \frac{A-1}{2A}\vec{q} \cdots \vec{\bar{l}}_A - \frac{\vec{q}}{2A} \right\rangle \left\langle \vec{\bar{l}}_1 + \frac{\vec{q}}{2A} \cdots \vec{\bar{l}}_i - \frac{A-1}{2A}\vec{q} \cdots \vec{\bar{l}}_A + \frac{\vec{q}}{2A} \right| 0 \right\rangle \tag{18}$$

instead of the equivalent but more commonly written

$$F(q) = \int d^3 l_1' \cdots d^3 l_A' \delta\left(\sum_{j=1}^{A} \vec{l}_j'\right) \left\langle 0 \left| \vec{l}_1' - \frac{\vec{q}}{A} \cdots \vec{l}_i' + \frac{A-1}{A}\vec{q} \cdots \vec{l}_A' - \frac{\vec{q}}{A} \right\rangle \langle \vec{l}_1' \cdots \vec{l}_i' \cdots \vec{l}_A' | 0 \rangle . \tag{19}$$

Since our model Hamiltonian conserves all momenta, the kinetic energy part of ϵ_0 is $\sum_{j=1}^{A} \bar{l}_j^2/2m$. An easy calculation shows that

$$\sum_{j=1}^{A} \frac{\overline{l_j^2}}{2m} - \sum_{j=1}^{A} \frac{\bar{l}_j^2}{2m} = \frac{A-1}{4A}\frac{q^2}{2m} . \tag{20}$$

This last expression can be interpreted as an excitation of the nucleus in the kinetic energy part of its Hamiltonian, since, for a given internal momentum state, the first term of (20) is the average internal kinetic energy during a collision with momentum transfer q and the second term of (20) is the expected internal kinetic energy in the ground state. Considering a harmonic-oscillator model of the nucleus, it is reasonable to assume that the potential energy part of the nuclear Hamiltonian is excited to the same order of magnitude as the kinetic energy part. Thus, the total excitation of the nucleus, $\epsilon_0 - \sum_{j=1}^{A} \overline{l_j^2}/2m - \tilde{U}_i$, is taken to be proportional to q^2. This total excitation is written as $-c'q^2/2m$, where a comparison with (20) indicates that c' is approximately $(A-1)/2A$.

The denominator of the many-body Green's function, $\tilde{G}_i$, is now completely identified with that of the free two-body Green's function, g. Thus, the unambiguous relation

$$\langle \vec{K}\vec{l}_1 \cdots \vec{l}_A | \tilde{\tau}_i(E^0) | \vec{K}'\vec{l}_1' \cdots \vec{l}_A' \rangle = \langle \vec{k}_i | t_i(w_i) | \vec{k}_i' \rangle \tag{21}$$

between the many-body and two-body matrix elements is at last obtained. In (21), the kinematic variables, k_i, k_i', and θ, are related to the optical-potential variables, K, K', and q, via (12), while the energy parameter of the two-body matrix is found from (17) and the discussion in the previous paragraph to be given by

$$w_i = \bar{e}_i + (E^0 - \bar{E}) - c'q^2/2m. \tag{22}$$

Using the relations (12) and the definitions (16) and $\overline{K^2} = \tfrac{1}{2}(K^2 + K'^2)$, w_i can be expressed as

$$w_i = c\frac{\overline{K^2}}{2\mu} + \frac{K^{0\,2} - \overline{K^2}}{2\mu} + (\tfrac{1}{2} - \tfrac{1}{2}c^2 - c')\frac{q^2}{2m}. \tag{23}$$

When $\overline{K^2} = K^{0\,2}$ and $q = 0$, w_i takes the value $e^0 = cE^0$ corresponding to scattering of two free particles. From (23), it can be seen that w_i decreases (increases) as $\overline{K^2}$ increases (decreases). On the other hand, it follows from (12) that $\overline{k_i^2} = \tfrac{1}{2}(k_i^2 + k_i'^2)$ equals

$$\overline{k_i^2} = c^2\overline{K^2} + \tfrac{1}{4}(1 - c^2)q^2 . \tag{24}$$

It is clear from (24) that $\overline{k_i^2}$ decreases (increases) as $\overline{K^2}$ decreases (increases). The result of this opposite behavior of w_i and $\overline{k_i^2}$ with respect to changes in $\overline{K^2}$ is that the distance off energy shell in the t matrix (21) is enlarged by a factor of $1/c$ over the value that would be obtained if w_i were fixed at e^0.

V. FIRST-ORDER OPTICAL POTENTIAL

Inserting complete sets of internal momentum states into (8) results in the integration $\int d^3l_1 \cdots d^3l_A \times d^3l_1' \cdots d^3l_A'$, which is equal to $\int d^3\bar{l}_1 \cdots d^3\bar{l}_A \times d^3\lambda_1 \cdots d^3\lambda_A$, with the definition $\vec{\lambda}_j = \vec{l}_j - \vec{l}_j'$. From (14), the integrals over the $\vec{\lambda}_j$ are removed by δ functions so that the first-order EIA potential is found to be

$$\langle \vec{K}|V|\vec{K}'\rangle = \int d^3\bar{l}_1 \cdots d^3\bar{l}_A \delta\left(\sum_{j=1}^{A} \bar{\bar{l}}_j\right) \langle 0|\vec{l}_1 \cdots \vec{l}_A\rangle \left\langle \vec{K}\vec{l}_1 \cdots \vec{l}_A \middle| \sum_{i=1}^{A} \bar{\tau}_i(E^0) \middle| \vec{K}'\vec{l}_1' \cdots \vec{l}_A' \right\rangle \langle \vec{l}_1' \cdots \vec{l}_A'|0\rangle . \tag{25}$$

We have already neglected the internal momentum distribution in deriving k_i, k_i', θ, and w_i, so that in this approximation the two-body matrix elements can be written as an average over the spin-isospin part of the nuclear ground-state wave function, denoted as $|0\rangle$. Thus, using (21) in (25) and moving the matrix outside of the integration, gives

$$\langle \vec{K}|V|\vec{K}'\rangle = \left\langle 0 \middle| \sum_{i=1}^{A} \langle \vec{k}_i|t_i(w_i)|\vec{k}_i'\rangle \middle| 0 \right\rangle F(q), \tag{26}$$

where $F(q)$ is the elastic form factor obtained in electron-nucleus scattering and defined by (18). Note that both (25) and (26) are reciprocally invariant.

When the averaging over spin and isospin is performed, (26) takes the form (particle index is dropped)

$$\langle \vec{K}|V|\vec{K}'\rangle = A\,\bar{t}\,(k, k', \theta; w)F(q), \tag{27}$$

where $\bar{t}$ is a linear combination of the two-body partial-wave matrix elements, which include the singlet, t_l, uncoupled triplet, t_{ll}, and the coupled triplet, the last written as a 2×2 matrix:

$$\begin{pmatrix} t_{j+1,j} & t_j^+ \\ t_j^- & t_{j-1,j} \end{pmatrix}. \tag{28}$$

Here, l and j are the orbital and total angular momenta of the two nucleons in their center of mass. The coefficients[10] of the linear combination depend only on the properties of the nuclear state and two-nucleon interaction under rotational symmetry; these coefficients are the same off as well as on shell. On shell, $t_j^+ = t_j^-$, while off shell, for $k \neq k'$, the matrix (28) need not be symmetric. For this reason it is noted that for the spin-zero–isospin-zero nuclei considered in this work, the nondiagonal matrix elements t_j^+ and t_j^- do not appear in the first-order potential, because the coefficient of each is separately equal to zero.[11]

The extension to off the energy shell can be made in several ways. For the problem considered here, we have found it convenient to use Fredholm reduction in the partial-wave matrix elements, which for the uncoupled case (with angular momentum indices omitted) gives[12]:

$$t(k, k'; w) = h(k, w)h(k', w)t(w) + r(k, k'; w), \tag{29}$$

where h is the half-off-shell factor defined by

$$h(k, w) = \frac{v(k, \kappa)}{v(\kappa, \kappa)} + \int_0^\infty \frac{k'^2 dk'}{\kappa^2 - k'^2} \Lambda(k, k', w)h(k', w), \tag{30}$$

with κ being the on-shell momentum value corresponding to w, i.e., $w = \kappa^2/m$, and where $\Lambda(k, k')$ is given by

$$\Lambda(k, k', w) = v(k, k') - \frac{v(k, \kappa)v(\kappa, k')}{v(\kappa, \kappa)}. \tag{31}$$

The so-called residual term, r, is found from

$$r(k, k'; w) = \Lambda(k, k', w) + \int_0^\infty \frac{k''^2 dk''}{\kappa^2 - k''^2} \Lambda(k, k'', w)r(k'', k'; w) \tag{32}$$

so that it is real, symmetric in k and k', and vanishes when either k or k' equals κ. In (29), $t(w)$ is proportional to the experimentally determined on-shell scattering amplitude at the two-body energy w. On the other hand, the functions h and r depend on the choice of nucleon-nucleon potential $v(k, k')$. These potentials are restricted to fit the elastic nucleon-nucleon phase parameters in addition to certain assumptions on analytic, assymptotic, and bound-state behavior.[13] For different models, there can be considerably different behavior in the half-off-shell factors for a given partial wave, although Mongan[14] has shown that the essential shape of these factors for the singlet case is similar for a variety of separable, as well as local potentials. He has also derived the more involved expresssions for the coupled case.[15]

VI. NUMERICAL ASPECTS AND PROCEDURE

Our purpose is to estimate the size and determine the direction of the changes in calculated values of the cross section and polarization in elastic nucleon-nucleus scattering due to the fact that the two-body matrices go off their energy shell when correct kinematics are used. Thus, we make rather crude approximations which we believe are justified at this time for this problem. All residual terms are ignored; this is consistent with the models used here and with estimates[12] of this term based on the fact that it vanishes when either momentum variable is on shell. Coupling between the spin-triplet partial waves with $l=j\pm1$ is also ignored, since the coupling parameters are usually small compared to the phase shifts.[16] Furthermore, the half-off-shell factors are approximated as

$$h(k,w)\approx 1+h'(\kappa)(k-\kappa)+h''(\kappa)(k-\kappa)^2+h^*(\kappa)(k-\kappa)^3 \tag{33}$$

$$\approx 1+h'(k^0)(k-\kappa)+h''(k^0)(k-\kappa)^2+h^*(k^0)(k-\kappa)^3, \tag{34}$$

where $h'(k^0)=dh(k,w)/dk$ evaluated at $k=k^0$, and $h''(k^0)$ and $h^*(k^0)$ are determined by fitting the expression (34) to the values of $h(k,e^0)$ calculated from models for k equal to $\frac{1}{2}k^0$ and $2k^0$.

The approximations are justified by the numerical aspects of this problem at intermediate and higher energies. Reduced matrix elements $V_{LJ}(K,K')$ are computed by quadrature from $\langle\vec{K}|V|\vec{K}'\rangle$ in a partial-wave decomposition, with L and J being nucleon-nucleus orbital and total angular momenta. For spin-zero nuclei, there is no coupling between different L, and $J=L\pm\frac{1}{2}$. These reduced matrix elements are used in an integral scattering equation for the reaction matrix $R_{LJ}(K,K')$, whose half-OFF-shell vector $R_{LJ}(K)$ is obtained for a discrete set of values of $K\neq K^0$. The fully ON-shell matrix elements of the reaction and scattering matrices are then obtained by performing the principal-value integration

$$\Pi_{LJ}=\frac{2}{\pi K^0}\int_0^\infty \frac{dK\,K^2}{(K^0)^2-K^2}\,V_{LJ}(K)R_{LJ}(K), \tag{35}$$

as a sum for the above discrete set using a suitable quadrature rule,[4] and then substituting Π_{LJ} into the relations[17]

$$R_{LJ}=V_{LJ}+\Pi_{LJ} \tag{36}$$

and

$$T_{LJ}=R_{LJ}/(1+iR_{LJ}). \tag{37}$$

Because the form factor is very peaked in q, the $V_{LJ}(K,K')\to 0$ rapidly as K and/or $K'\to\infty$. Also, because these matrix elements are determined by an integration over Legendre functions, they approach zero as $(KK')^L$ as K and/or $K'\to 0$. Noting this behavior of the reduced matrix elements and noting the Jacobian weight of K^2 and the pole at $K=K^0$ in (36), it can be concluded that matrix elements $V_{LJ}(K,K')$ with both K and K' near K^0 make the most significant contributions to the values of Π_{LJ}, R_{LJ}, or T_{LJ}, and, hence, to the elastic scattering observables, which depend only on the fully ON-shell matrix element, R_{LJ} (or T_{LJ}). For energies above 142 MeV and $A\geq 12$, our calculations have shown that ignoring matrix elements of $V_{LJ}(K,K')$ with momentum variables greater than $2K^0$ or less than $\frac{1}{2}K^0$ changes the magnitude of Π_{LJ} by less than 2%. The effect on cross section and polarization is even smaller at lab angles less than 43°. Because the potential before reduction is also a very peaked function of q, the range of k/k^0 and k'/k^0 is also from about $\frac{1}{2}$ to 2 [see (12)], with the most significant contributions coming from a smaller range of nearly on-shell values of these momentum variables.

An examination of Mongan's case-I and case-IV separable potentials, which are fitted to the Livermore phase parameter set,[16] has shown that (34) is an excellent approximation for all phases, at positive energies not very near zeros of the phase shifts, when $\frac{1}{2}<k/k^0<2$, while a linear approximation is adequate for most phases when $0.8<k/k^0<1.2$. Even for the larger range, the residual term of (29) is generally less than 10% of $t(w)$ for two-term separable models, while it is identically zero for the single-term models. The approximation $h(k,w)\approx h(k,e^0)$, i.e., (34), is poor when e^0 is near a zero of the phase shift, in which case, the slope $h'(w)$, instead of varying slowing with w, rapidly passes from large negative to large positive values. Thus, near the zero of a phase shift, the contribution from the off-shell part of its partial wave is likely to be exaggerated and may possibly have the wrong sign. Since there are zeros in the $^1S_0$, $^3S_1$, and $^3P_0$ phases at about 250–300, 205, and 295 MeV, respectively, our calculations and conclusions at 210 MeV are less definite than calculations at other energies.

In our approach, there is a weak dependence of the energy parameter on the value of c'. For $\frac{1}{4}<c'<1$, the change in the observables is insignificant for the energies and angles considered in this paper. For convenience in programming, this constant was given the value $\frac{1}{2}-\frac{1}{2}c^2$, thereby eliminating the term of (23) that depends on q and, hence, making the energy-parameter variation a

wholly nonlocal effect, i.e., dependent on K and K', but not on q.

VII. RESULTS

All of the results discussed in this section make use of the kinematic relations described by (12). Using this modification of the kinematics for the nucleon-nucleon momentum and angle variables instead of the previously used approximation $k = k' = k^0$ [3,18] changes the observables very little when the nucleon-nucleus angle is small. The change becomes noticeable on a log plot only after the cross section has dropped several decades; even at a lab energy of 95 MeV, the change in small-angle observables is less than 2%.

On the other hand, the effects of the energy variation (23) are more pronounced. In Table I, the zero-angle lab cross sections are presented for neutrons incident on C^{12} at lab energies of 45, 95, 137, 210, and 350 MeV. The first row neglects both the energy-parameter and energy-shell variations in the two-body t matrix, so that the approximation $t(k,k';w) = t(e^0)$ is made, while the third row neglects only the energy-shell variation, making the approximation $t(k,k';w) = t(w)$, where w varies according to (23) with $c' = \frac{1}{2} - \frac{1}{2}c^2$; at the above energies, respectively, the relative changes in cross section are about 14, 18, 3, 9, and 6%. The second row fixes w at e^0 in the t matrix so that $t(k,k';w) = h(k,e^0)h(k',e^0)t(e^0)$, where h is determined from Mongan's case-IV model and is approximated as described in Sec. VI. These three rows show that at every energy considered the effect of varying w is opposite that due to taking the two-body matrix off its energy shell. The partial cancellation when both variations are included (see the fourth row of Table I) can be explained by noting that the magnitude of the off-shell matrix in the uncoupled case is given by

$$|t(k,k',w)| \approx [1 + 2h'(\bar{k} - \kappa)]\,|\sin\delta(w)/\kappa|\,, \qquad (38)$$

where $\bar{k} = \frac{1}{2}(k + k')$. As noted in Sec. IV, $\bar{k}^2$ (and hence $\bar{k}$) increases as κ decreases. It has also been observed in our calculations that $h'(e^0)$ is usually negative or positive according to whether the magnitude of $\delta(e^0)$ is decreasing or increasing. In particular, the phase shifts of the S waves below 250 MeV rapidly decrease in magnitude as e^0 increases, while the slopes of their half-off-shell factors are positive; thus, as κ increases, the first factor in (38) increases while the second decreases.

At larger angles, the effect due to energy variation alone is such that the shape of the cross section is not significantly changed. The relative change in the polarization at small angles is nearly equal and opposite the relative change of the cross section. Since polarization is inversely proportional to cross section but directly proportional to the difference between spin-flip and nonflip cross sections, we conclude that this difference is not affected by (23) at small angles. This conclusion is maintained also at larger angles except near cross-section diffraction minima, where the polarization dips markedly. However, computations clearly show that, when its value is nearly zero, the calculated polarization is sensitive to small changes in the potential.

The remainder of the rows of Table I illustrate that the size (but not direction) of the net off-energy-shell effect is strongly model-dependent. Here and subsequently, w is varied according to (23) with $c' = \frac{1}{2} - \frac{1}{2}c^2$. In rows 4, 5, and 6, the t matrices are taken off shell by Mongan's case-IV half-off-shell factors; in these rows, respectively, all phases are taken off shell; only the S waves are taken off shell; and only $^1S_0$, $^3S_1$, and $^3P_0$ (whose phase shifts pass through zero in the intermediate-energy region and whose h' thus become large) are left on shell. A comparison of these rows indicates that there is some cancellation between the off-shell contributions of the S waves and the others – particularly at 137 MeV. Also, it can be seen that the off-shell contribution of the $^3P_0$ phase at 210 MeV dominates all others, strongly indicating that this contribution is exaggerated. The off-shell calculation with case-I parameters, illustrated in row 7, did not show the cancellation that occurred in case IV. For the two cases, the S-wave parameters are nearly identical, while some higher partial waves have h' that even differ in sign.

In Table II, zero-angle cross sections in the lab for He^4 and Ca^{40} are presented. These nuclei are

TABLE I. Zero-angle n-C^{12} elastic cross sections in millibarns at energies indicated in MeV by column headings. FIX and VAR refer to w fixed and varied according to (23). ON and OFF refer to leaving the t matrix on shell and taking it off shell, as described in Sec. VI, with case-I and case-IV potential models. For the case-IV model only, S-OFF takes only the S waves off shell, while −3-OFF leaves only $^1S_0$, $^3S_1$, and $^3P_0$ on shell.

	45	95	137	210	350
FIX, ON	976	1411	1286	945	791
FIX, OFF IV	1526	1668	1208	869	781
VAR, ON	838	1164	1317	1025	839
VAR, OFF IV	1239	1509	1253	702	778
VAR, S-OFF	971	1276	1250	928	803
VAR, −3-OFF	1090	1467	1388	1016	828
VAR, OFF I	1305	1600	1197	666	788

the smallest and largest spin-zero–isospin-zero nuclei with equal number of protons and neutrons. Assuming symmetry under charge exchange, the incident-proton potential differs from the incident-neutron potential only by inclusion of an extended-charge Coulomb potential. Thus, outside the Coulomb peak, observables calculated from the neutron potential could be compared with incident-proton data. It was hoped that low-energy helium and high-energy calcium calculations would indicate effects large enough to be discerned from those of multiple scattering, binding, exchange, etc. Table II clearly shows that these hopes are not realized for our models of off-energy-shell extrapolation. In particular, multiple scattering alone has been shown to lead to larger changes than those observed here.[3,18]

In Fig. 1, n-C^{12} cross sections are presented for lab energies of 95, 137, and 350 MeV, and they are compared, respectively, with experimental results of Salmon,[19] Harding,[20] and Ashmore, Mather, and Sen.[21] The experimental data are included only for the purpose of a rough comparison; the differences between different single scattering calculations, though large, are smaller than those due to binary terms in the potential. However, it is noted that the off-shell curves (dashed for case IV, dotted for case I) lie below their corresponding on-shell curves (solid lines) at large angles, so that inclusion of the off-shell parametrization used here does not improve agreement with experiment for large-angle cross sections, assuming (erroneously) that such fitting should be possible without binary (and higher?) terms.

In Fig. 2, the experimental polarization data for n-C^{12} at 137 MeV (Harding[20]) and at 350 MeV (Siegel[22]) and p-C^{12} at 210 MeV (Hafner[23]) are compared with the neutron calculations at 137, 350, and 210 MeV, respectively. The solid lines are on shell, the dashed are off-shell, case-IV, and the dotted are off-shell, case-I. The off-shell factors produce a definite dip in the polarization at 137 MeV, which is not observed in the on-shell curve and was previously considered due only to multiple scattering.[3] At 210 MeV, the dip is even more pronounced (but is possibly exaggerated because the $^3P_0$ phase is nearly zero and its h' is very large) and is even made negative. Also, there seems to be some improvement (i.e., a reduction) in the polarization calculated before the dip at 350 MeV.

TABLE II. Zero-angle n-He^4 and n-Ca^{40} elastic cross sections in millibarns at energies indicated in MeV by column headings. Notation is the same as in Table I.

	45	95	137	210	350
		Helium			
VAR, ON	189	239	206	143	115
VAR, OFF IV	259	256	186	94	110
VAR, OFF I	295	305	168	85	108
		Calcium			
VAR, ON	4355	6786	8470	7573	6558
VAR, OFF IV	6386	8989	8933	4306	6222
VAR, OFF 1	6690	9105	8060	4151	6106

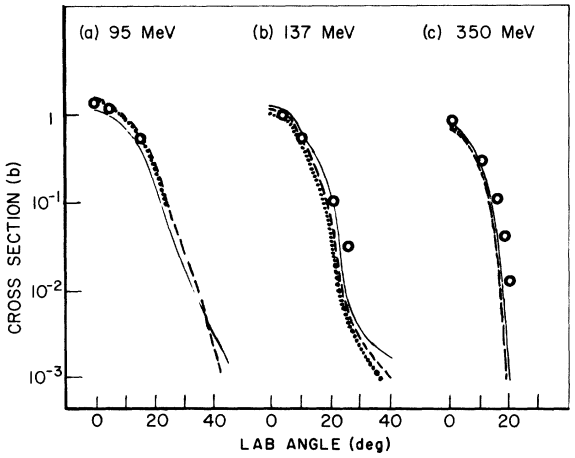

FIG. 1. Elastic n-C^{12} cross sections in barns. Solid curves are on shell; dashed curves use Mongan's case-IV models to go off shell; dotted curves are off shell via case-I models. All curves are calculated with w given by (23) and $c' = \frac{1}{2} - \frac{1}{2}c^2$. (a) The experimental 95-MeV data are from Ref. 19. (b) The 137-MeV data are from Ref. 20. (c) The 350-MeV data are from Ref. 21. All experimental data points are illustrated by circles.

VIII. CONCLUSIONS

Our most definite conclusion is that, for elastic scattering of nucleons off nuclei at intermediate

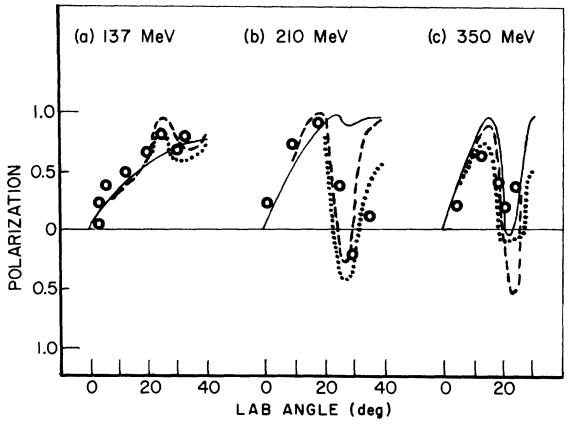

FIG. 2. Elastic n-C^{12} polarization. Solid, dashed, and dotted curves are used, experimental data illustrated, and w is varied as in Fig. 1. (a) The 137-MeV data are from Ref. 20. (b) Proton incident 210-MeV data from Ref. 23 are used. (c) The 350-MeV data are from Ref. 22.

energies, off-energy-shell effects are, like the energy-parameter effects, almost wholly nonlocal (in the sense of Sec. III). The ON-shell matrix elements of the potential, $\langle K \mid V \mid K' \rangle$, with $K = K^0 = K'$, change very little when the two-body t matrix elements are taken off their energy shell, unless q is very large. On the other hand, the half-OFF-shell angularly reduced vectors, $V_{LJ}(K, K^0)$ are so sensitive to going off the two-body shell that the principal values calculated from (31) change by over 50% for the lower values of L. Unfortunately, these Π_{LJ} are typically an order of magnitude smaller than the V_{LJ}, so that the Born approximation in the form $R_{LJ} = V_{LJ}$ is accurate for cross section as well as polarization. Thus, our situation is such that the potential – due to rapid variation of the form factor with q – is almost local, but the effects that we wish to investigate are essentially nonlocal. This situation for a nonlocal, real Gaussian potential was examined by Monahan and Thaler,[24] who concluded, as we do, that distinguishing off-shell effects, given reasonable nonlocality, would be extraordinarily difficult for the case of C^{12} at intermediate energies. However, we stress that purely local considerations[18] definitely underestimate the size of off-energy-shell contributions.

Our results are certainly model-dependent. Other models of the nucleon-nucleon interaction can lead to more rapidly varying off-shell behavior, i.e., larger h', as well as to fewer cancellations between energy-parameter and off-shell effects or amongst off-shell parts of different phases. Inclusion of coupling and residual terms might enlarge the net size of the effect of going off the energy shell. Finally, other means[25] of extending the two-body t matrices off their energy shell should be investigated.

It appears doubtful that intermediate-energy elastic nucleon-nucleus scattering can be used to extract information on the off-energy-shell character of the nucleon-nucleon interaction. There does not appear to be a range of A or E^0 for which off-shell effects in the first-order potential are likely to dominate the on-shell contribution of second-order terms. On the other hand, our work shows that off-shell effects are not negligible in the first-order term and leaves open the possibility that they significantly alter the second-order terms. It would seem that off-energy-shell considerations are a necessary ingredient in a multiple-scattering approach that attempts to fit the elastic nucleon-nucleus observables with precision.

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[17]The elastic scattering operator T evolves from the optical-potential operator V via the out-going-wave Green's function, $(E^0 + i\eta - \hat{E})^{-1}$, while the reaction operator R evolves via the stationary wave, $(E^0 - \hat{E})^{-1}$. The operator equations for T and R become integral equations in the momentum-space L, J decomposition, and these in turn become matrix equations when a discrete set of points is chosen to represent the momentum variable. Equations (35)–(37) follow directly (for spin-zero nuclei) from the matrix representations of R and T. Equation (37) is equivalent to the statements $T_{LJ} = -e^{i\delta_{LJ}} \times \sin\delta_{LJ}$ and $R_{LJ} = -\tan\delta_{LJ}$. Calculations have shown (see Ref. 4) that the quadrature (35) is accurate to about 1%, with as few as eight points chosen for the momentum variable, for intermediate energies.

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